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Global Weak Solution in a *p*-Laplacian Attraction-Repulsion Chemotaxis System with Nonlinear Sensitivity and Signal Production

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Posted Date: 22 July 2025

doi: 10.20944/preprints202507.1776.v1

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 doi:10.20944/preprints202507.1776.v1

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Global weak solution in a p-Laplacian attraction-repulsion chemotaxis system with nonlinear sensitivity and signal production

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Abstract

We consider the following p-Laplacian attraction-repulsion chemotaxis system:

$$\begin{cases} u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u) - \chi\nabla\cdot(u^\alpha\nabla v) + \xi\nabla\cdot(u^\beta\nabla w) + f(u), & x\in\Omega,\ t>0,\\ v_t = \triangle v - \sigma v + \eta u^{k_1}, & x\in\Omega,\ t>0,\\ 0 = \triangle w - \delta w + \gamma u^{k_2}, & x\in\Omega,\ t>0, \end{cases}$$

where $\chi,\xi,\alpha,\beta,\eta,\sigma,\gamma,\delta,k_1,k_2>0, p\geq 2$, $\Omega\subset\mathbb{R}^n(n\geq 2)$ is a smoothly bounded domain. The logistic-type source $f(u)\leq\kappa-\mu u^m$ for $\kappa\in\mathbb{R},\mu>0$ and $m>1$. We obtain the global boundedness of solutions if (i) $m>\max\{2k_1,\frac{p\alpha}{p-1}-1\}$, or (ii) $k_2>\max\{2k_1-\beta,\frac{p\alpha}{p-1}-\beta-1\}$ with $m>\max\{2,2\alpha,2\beta\}$.

Keywords: Global weak solutions; Attraction-repulsion; p-Laplacian; Nonlinear production

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1. Introduction

In this paper, we study the following p-Laplacian chemotaxis system with nonlinear sensitivity and signal production:

$$\begin{cases} u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u) - \chi\nabla\cdot(u^\alpha\nabla v) + \xi\nabla\cdot(u^\beta\nabla w) + f(u), & x\in\Omega,\ t>0,\\ v_t = \Delta v - \sigma v + g_1(u), & x\in\Omega,\ t>0,\\ 0 = \Delta w - \delta w + g_2(u), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0\\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x),\ \ w(x,0) = w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\chi,\xi,\alpha,\beta,\eta,\sigma,\gamma,\delta,k_1,k_2>0$ and $p\geq 2,\Omega\subset\mathbb{R}^n(n\geq 2)$ is a smoothly bounded domain. u is the density of the cells, v,w denote the concentration of the chemoattractant and chemorepellant. The nonlinear diffusion $\Delta_p u = \bigtriangledown\cdot(|\nabla u|^{p-2}\nabla u)$ is called the slow and fast $p-$Laplacian diffusion if $p>2$ and $1<p<2$, respectively. The smooth logistic-type source $f:\mathbb{R}\to\mathbb{R}$ satisfies for all s>0,

$$f(s)\leq\kappa-\mu s^m \text{ and } f(0)\geq 0 \tag{1.2}$$

with $\kappa\in\mathbb{R},\mu>0$ and $m>1$. The signal production $g_i(i=1,2)$ satisfies for all s>0,

$$g_1(s)=\eta s^{k_1},\ \ g_2(s)=\gamma s^{k_2}, \tag{1.3}$$

 doi:10.20944/preprints202507.1776.v1

where $\eta, \gamma, k_1, k_2 > 0$.

Considering the effect of attraction without repulsion, problem (1.1) becomes the following Keller-Segel chemotaxis system

$$\begin{cases} u_t = \triangle u - \chi \cdot (u\nabla v), & x \in \Omega, t > 0, \\ v_t = \triangle v - v + g(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.4}$$

When the signal production $g(u)$ is linear (i.e.$g(u) = u$), it was showed that the solutions of (1.4) are global bounded when $n = 1$, or $n = 2$ with $\int_\Omega u_0 < \frac{4\pi}{\chi}$, or $n \geq 3$ with small initial conditions(see [1,19,23,35]), whereas when $n = 2$ with $\int_\Omega u_0 > \frac{4\pi}{\chi}$, or $n \geq 3$, the finite-time blow up may occur for a large class of initial date [2,34]. When the signal production $g(u)$ is nonlinear (i.e.$0 < g(u) \leq u^k$), the boundedness of (1.4) was studied in [17] if $k \in (0, \frac{2}{n})$ with $n \geq 2$. Considering the solution of (1.4) with nonlinear production $g(u) = u(u+1)^{k-1}$ under the condition $f(u) = u - \mu u^m$, it was proved in [20,21] that the weak solution of (1.4) is bounded globally if $\frac{2(n+4)}{n+6} < m \leq 2$, $0 < k < \frac{(n+6)(m-1)}{2(n+2)}$ and $n = 2, 3$. Recently, Zhuang et al. [39] got the globally bounded classical solution of (1.4) under $k < m - 1$, or $k = m - 1$ with $\mu > 0$ sufficiently large.

Now, recall a more general attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi\nabla \cdot (u\nabla v) + \xi\nabla \cdot (u\nabla w) + f(u), & x \in \Omega,\ t > 0, \\ \tau_1 v_t = \triangle v - \sigma v + g_1(u), & x \in \Omega,\ t > 0, \\ \tau_2 w_t = \triangle w - \delta w + g_2(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x),\ \ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where $\tau_i = \{0, 1\}(i = 1, 2)$. For the case $f(u) \equiv 0, g_1(u) = \eta u, g_2(u) = \gamma u$. If $\tau_1 = 1$, $\tau_2 = 0$ and $D(u)$ satisfying $D(u) = D_0 u^\theta$,there exists a global weak solution under the case $\theta > 1 - \frac{2}{n}$ [14]. Later, [8] optimized the condition of θ into $\theta > 1 - \frac{4}{n+2}$ but added $\xi\gamma - \chi\eta \geq 0$. If $\tau_1 = \tau_2 = 0$, and considering $p-$Laplacian diffusion (i.e. $D(u) = |\nabla u|^{p-2}$), it was showed in [12] that the system (1.5) exists a global bounded weak solution under the case (i) $\xi\gamma - \chi\eta \leq 0$, or case (ii) $\xi\gamma - \chi\eta > 0$ with $p > \frac{3n}{n+1}$, or $1 < p \leq \frac{3n}{n+1}$ and $\|u_0\|_{L^{\frac{(3-p)n}{p}}}$ is small. More results on attraction-repulsion chemotaxis system can be found in [1,6,7,9–11,24,26,29,31,36,38]. For the case $f(u) = \kappa - \mu u^m, g_1(u) = \eta u^{k_1}, g_2(u) = \gamma u^{k_2}$. If $\tau_1 = 1$, $\tau_2 = 0$, it was shown that when (i) $m > \max\{2k_1, \frac{2k_1 n}{2+n} + \frac{1}{p-1}\}$, or case (ii) $k_2 > \max\{2k_1 - 1, \frac{2k_1 n}{2+n} + \frac{2-p}{p-1}\}$ with $m > 2$, the problem (1.5) possesses a global bounded weak solution [32]. If $\tau_1 = \tau_2 = 1$, Jia [3] proved that when $\max\{k_1, k_2\} < m - 1$ or $\max\{k_1, k_2\} = m - 1$ with large $\mu > 0$, there exists a global bounded weak solution. More results on p-Laplacian chemotaxis models can be found in [4,12,15,16,18,27,28,37,40].

We now formulate the principal result .

 doi:10.20944/preprints202507.1776.v1

Theorem 1.1. Let $p \geq 2$ and $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain with smooth boundary. Assume that (1.2), (1.3) hold with $k_1 > 0$, $0 < k_2 \leq 1$ and $m > 1$. Then for any non-negative initial data $u_0, v_0, w_0 \in W^{1,\infty}(\Omega)$, problem (1.1) has a globally bounded weak solution $(u, v, w) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$ under the assumption (i) $m > \max\{2k_1, \frac{p\alpha}{p-1} - 1\}$, or (ii) $k_2 > \max\{2k_1 - \beta, \frac{p\alpha}{p-1} - \beta - 1\}$ with $m > \max\{2, 2\alpha, 2\beta\}$.

Remark 1.1. When $\alpha = \beta = 1$ for (1.1), we optimized the assumption (i) $m > \max\{2k_1, \frac{2k_1 n}{2+n} + \frac{1}{p-1}\}$, or (ii) $k_2 > \max\{2k_1 - 1, \frac{2k_1 n}{2+n} + \frac{2-p}{p-1}\}$ with $m > 2$ of reference [32] into (i) $m > \max\{2k_1, \frac{1}{p-1}\}$ or (ii) $k_2 > \max\{2k_1 - 1, \frac{2-p}{p-1}\}$ with $m > 2$.

This paper is organized as follows. In Sec. 2, we will introduce a regularized problem (2.1) and preliminary lemmas. In Sec. 3, we prove the boundedness of weak solutions of (2.1). Finally, we prove Theorem 1.1 by an approximation procedure in Sec. 4.

2. Preliminaries

Due to the p-Laplacian diffusion, we begin with the definition of weak solution to (1.1).

Definition 2.1. Let

$$0 \leq u \in L^2_{loc}([0,T); L^2(\Omega)),$$
$$0 \leq v, w \in L^2_{loc}([0,T); W^{1,2}(\Omega))$$

We call (u, v, w) a weak solution of (1.1) if the following equalities hold:

$$\begin{aligned} -\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0(x)\varphi(x,0) = &-\int_0^T \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \chi \int_0^T \int_\Omega u^\alpha \nabla v \cdot \nabla\varphi \\ &-\xi \int_0^T \int_\Omega u^\beta \nabla w \cdot \nabla\varphi + \int_0^T \int_\Omega f(u)\varphi, \end{aligned}$$

$$-\int_0^T \int_\Omega v\varphi_t - \int_\Omega v_0(x)\varphi(x,0) = -\int_0^T \int_\Omega \nabla v \cdot \nabla\varphi - \sigma \int_0^T \int_\Omega v\varphi + \int_0^T \int_\Omega g_1(u)\varphi$$

and

$$0 = -\int_0^T \int_\Omega \nabla w \cdot \nabla\varphi - \delta \int_0^T \int_\Omega w\varphi + \int_0^T \int_\Omega g_2(u)\varphi$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,T))$.

In order to construct weak solutions of (1.1), we consider the regularized problem

$$\begin{cases} u_{\varepsilon t} = \nabla \cdot ((|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon) - \chi \nabla \cdot (u_\varepsilon^\alpha \nabla v_\varepsilon) + \xi \nabla \cdot (u_\varepsilon^\beta \nabla w_\varepsilon) + f(u_\varepsilon), & \\ v_{\varepsilon t} = \triangle v_\varepsilon - \sigma v_\varepsilon + g_1(u_\varepsilon), & x \in \Omega,\ t > 0, \\ 0 = \triangle w_\varepsilon - \delta w_\varepsilon + g_2(u_\varepsilon), & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = \frac{\partial w_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0 \\ u_\varepsilon(x,0) = u_0(x),\ \ v_\varepsilon(x,0) = v_0(x), \quad w_\varepsilon(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

 doi:10.20944/preprints202507.1776.v1

for each $\varepsilon \in (0,1)$. System (2.1) is locally solvable in the classical sense by using a fixed point theorem similar to [27,29].

Lemma 2.1. Let $p \geq 2$ and $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a smooth bounded domain, and that the initial data value $u_0, v_0, w_0 \in W^{1,\infty}(\Omega)$. Then for any $\varepsilon \in (0,1)$, there exists $T_{\max,\varepsilon} \in (0,\infty]$ and a local-in-time classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon) \in C(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times [0, T_{\max,\varepsilon}))$ that satisfies (2.1). Moreover, if $T_{\max,\varepsilon} < \infty$, then

$$\lim_{t \to T_{\max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Lemma 2.2. Assume (1.2) holds, then there exists a positive constant $M := M(u_0, |\Omega|)$ such that

$$\int_\Omega u_\varepsilon(x,t)dx \leq M \quad \text{for } t \in (0, T_{max,\varepsilon}). \tag{2.2}$$

Proof. Integrating the first functions of (2.1), we have by using (1.2) that

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx \leq \kappa|\Omega| - \mu|\Omega|^{1-m}\left(\int_\Omega u_\varepsilon dx\right)^m, \quad t \in (0, T_{max,\varepsilon}) \tag{2.3}$$

which implies (2.2). □

Lemma 2.3. For any $\varepsilon_1, \varepsilon_2 > 0$ and $q > 0$, there exists constants $C_1, C_2 > 0$ such that the solution z of the problem

$$-\triangle z + \delta z = \gamma u^{k_2} \text{ in } \Omega, \ \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{2.4}$$

satisfies

$$\int_\Omega z^{q+1} \leq \varepsilon_1 \int_\Omega u^{k_2(q+1)} + C_1 \leq \varepsilon_2 \int_\Omega u^{q+1} + C_2. \tag{2.5}$$

for $u \in L^1(\Omega)$ and $0 < k_2 \leq 1$.

Proof. The proof closely resembles that of Lemma 2.2 in [30]; therefore, we will omit it. □

Next, we present the Gagliardo-Nirenberg inequality, which will be frequently utilized in the subsequent analysis [13,22].

Lemma 2.4. Assume that $1 \leq p, q \leq \infty$ satisfying $(n-q)p \leq nq$. Let $s > 0$ and $0 < r \leq p \leq \infty$. Then for any $\phi \in W^{1,q}(\Omega) \cap L^r(\Omega)$, there exists a constant $C_{GN} > 0$ such that

$$\|\phi\|_{L^p(\Omega)} \leq C_{GN}(\|\nabla\phi\|^{\lambda^*}_{L^q(\Omega)}\|\phi\|^{1-\lambda^*}_{L^r(\Omega)} + \|\phi\|_{L^s(\Omega)}), \tag{2.6}$$

where

$$\lambda^* = \frac{\frac{n}{r} - \frac{n}{p}}{1 - \frac{n}{q} + \frac{n}{r}} \in (0,1). \tag{2.7}$$

 doi:10.20944/preprints202507.1776.v1

3. Regularity Estimates to Regularized Problem

In the section, we study the global boundedness to the regularized problem (2.1), we pay attention to the following lemma.

Lemma 3.1. Assume that (1.2), (1.3) hold with $k_1 > 0, 0 < k_2 \leq 1$ and $m > 1$. If $m > 2k_1$ or $k_2 > 2k_1 - \beta$ with $m > \max\{2, 2\alpha, 2\beta\}$, there exists a constant $C_1 > 0$ such that

$$\int_\Omega |\nabla v_\varepsilon|^2 dx \leq C_1 \quad \text{on} \quad t \in (0, T_{\max}). \tag{3.1}$$

for any $\varepsilon \in (0,1)$.

Proof. Case (i): $m > 2k_1$**.** Multiplying $(2.1)_2$ by $-2\triangle v_\varepsilon$, integrating by parts, we obtain

$$\begin{aligned}\tfrac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 &= -2\int_\Omega |\triangle v_\varepsilon|^2 - 2\sigma\int_\Omega |\nabla v_\varepsilon|^2 - 2\eta\int_\Omega u_\varepsilon^{k_1}\triangle v_\varepsilon \\ &\leq -2\int_\Omega |\triangle v_\varepsilon|^2 - 2\sigma\int_\Omega |\nabla v_\varepsilon|^2 + 2\int_\Omega |\triangle v_\varepsilon|^2 + \tfrac{\eta^2}{2}\int_\Omega u_\varepsilon^{2k_1} \\ &= -2\sigma\int_\Omega |\nabla v_\varepsilon|^2 + \tfrac{\eta^2}{2}\int_\Omega u_\varepsilon^{2k_1}\end{aligned} \tag{3.2}$$

Together with (3.2) and the first equation of (2.1), there exists a constant $C_2 > 0$ such that

$$\begin{aligned}\tfrac{d}{dt}\left(\int_\Omega u_\varepsilon + \int_\Omega |\nabla v_\varepsilon|^2\right) &\leq \kappa|\Omega| - \mu\int_\Omega u_\varepsilon^m - 2\sigma\int_\Omega |\nabla v_\varepsilon|^2 + \tfrac{\eta^2}{2}\int_\Omega u_\varepsilon^{2k_1} \\ &\leq -2\sigma\int_\Omega |\nabla v_\varepsilon|^2 + C_2,\end{aligned} \tag{3.3}$$

which used $m > 2k_1$ and Young's inequality. Collecting Lemma 2.2 and (3.3), we have

$$\tfrac{d}{dt}\left(\int_\Omega u_\varepsilon + \int_\Omega |\nabla v_\varepsilon|^2\right) + 2\sigma\left(\int_\Omega u_\varepsilon + \int_\Omega |\nabla v_\varepsilon|^2\right) \leq 2\sigma\int_\Omega u_\varepsilon + C_2 \leq C_3, \tag{3.4}$$

Case (ii): $k_2 > 2k_1 - \beta$ with $m > \max\{2, 2\alpha, 2\beta\}$. Multiplying the first equation of (2.1) by $1 + \ln u_\varepsilon$ and using Young's inequality, we get

$$\begin{aligned}\tfrac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon &\leq -\left(\tfrac{p}{p-1}\right)^p \int_\Omega |\nabla u_\varepsilon^{\frac{p-1}{p}}|^p + \chi\int_\Omega u_\varepsilon^{\alpha-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\ &\quad -\xi\int_\Omega u_\varepsilon^{\beta-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon + \kappa|\Omega| + \kappa\int_\Omega \ln u_\varepsilon \\ &\quad -\mu\int_\Omega u_\varepsilon^m - \mu\int_\Omega u_\varepsilon^m \ln u_\varepsilon \\ &\leq \chi\int_\Omega u_\varepsilon^{\alpha-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon - \xi\int_\Omega u_\varepsilon^{\beta-1}\nabla u_\varepsilon\cdot\nabla w_\varepsilon - \tfrac{\mu}{2}\int_\Omega u_\varepsilon^m + C_4,\end{aligned} \tag{3.5}$$

which used $\int_\Omega \ln u_\varepsilon \leq \int_\Omega u_\varepsilon$ and $-u_\varepsilon \ln u_\varepsilon \leq e^{-1}$.

Multiplying $(2.1)_2$ by $-\triangle v_\varepsilon$, integrating by part, we have

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 &= -\int_\Omega |\triangle v_\varepsilon|^2 - \sigma\int_\Omega |\nabla v_\varepsilon|^2 - \eta\int_\Omega u_\varepsilon^{k_1}\triangle v_\varepsilon \\ &\leq -\int_\Omega |\triangle v_\varepsilon|^2 - \sigma\int_\Omega |\nabla v_\varepsilon|^2 + \eta^2\int_\Omega u_\varepsilon^{2k_1} + \tfrac{1}{4}\int_\Omega |\triangle v_\varepsilon|^2 \\ &\leq -\tfrac{3}{4}\int_\Omega |\triangle v_\varepsilon|^2 - \sigma\int_\Omega |\nabla v_\varepsilon|^2 + \eta^2\int_\Omega u_\varepsilon^{2k_1}\end{aligned} \tag{3.6}$$

 doi:10.20944/preprints202507.1776.v1

for all $t \in (0, T_{\max})$. According to the Young's inequality, we estimate

$$\chi \int_\Omega u_\varepsilon^{\alpha-1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon = -\frac{\chi}{\alpha} \int_\Omega u_\varepsilon^\alpha \triangle v_\varepsilon \leq \frac{\chi^2}{\alpha^2} \int_\Omega u_\varepsilon^{2\alpha} + \frac{1}{4} \int_\Omega |\triangle v_\varepsilon|^2, \tag{3.7}$$

as well as

$$\begin{aligned} -\xi \int_\Omega u_\varepsilon^{\beta-1} \nabla u_\varepsilon \cdot \nabla w_\varepsilon &= \frac{\xi}{\beta} \int_\Omega u_\varepsilon^\beta \triangle w_\varepsilon \\ &= \frac{\delta\xi}{\beta} \int_\Omega u_\varepsilon^\beta w_\varepsilon - \frac{\gamma\xi}{\beta} \int_\Omega u_\varepsilon^{k_2+\beta} \\ &\leq \int_\Omega u_\varepsilon^{2\beta} + \frac{\delta^2\xi^2}{4\beta^2} \int_\Omega w_\varepsilon^2 - \frac{\gamma\xi}{\beta} \int_\Omega u_\varepsilon^{k_2+\beta} \\ &\leq \int_\Omega u_\varepsilon^{2\beta} + \sigma \int_\Omega u^2 - \frac{\gamma\xi}{\beta} \int_\Omega u_\varepsilon^{k_2+\beta} + C_5 \end{aligned} \tag{3.8}$$

which used Lemma 2.3 and for all $t \in (0, T_{\max})$. Since $2\sigma \int_\Omega u_\varepsilon \ln u_\varepsilon \leq 2\sigma \int_\Omega u_\varepsilon^2$, and collecting (3.5)–(3.8),we have

$$\begin{aligned} &\frac{d}{dt} \int_\Omega (u_\varepsilon \ln u_\varepsilon + \frac{1}{2}|\nabla v_\varepsilon|^2) + 2\sigma \int_\Omega u_\varepsilon \ln u_\varepsilon + \sigma \int_\Omega |\nabla v_\varepsilon|^2 \\ &\leq 3\sigma \int_\Omega u_\varepsilon^2 + \frac{\chi^2}{\alpha^2} \int_\Omega u_\varepsilon^{2\alpha} + \int_\Omega u_\varepsilon^{2\beta} + \eta^2 \int_\Omega u_\varepsilon^{2k_1} - \frac{\gamma\xi}{\beta} \int_\Omega u_\varepsilon^{k_2+\beta} - \frac{\mu}{2} \int_\Omega u_\varepsilon^m + C_6. \end{aligned} \tag{3.9}$$

It follows from $k_2 > 2k_1 - \beta$, $m > \max\{2, 2\alpha, 2\beta\}$ and Young's inequality that

$$3\sigma \int_\Omega u_\varepsilon^2 + \frac{\chi^2}{\alpha^2} \int_\Omega u_\varepsilon^{2\alpha} + \int_\Omega u_\varepsilon^{2\beta} + \eta^2 \int_\Omega u_\varepsilon^{2k_1} - \frac{\gamma\xi}{\beta} \int_\Omega u_\varepsilon^{k_2+\beta} - \frac{\mu}{2} \int_\Omega u_\varepsilon^m \leq C_7. \tag{3.10}$$

Let $y(t) = \int_\Omega (u_\varepsilon \ln u_\varepsilon + \frac{1}{2}|\nabla v_\varepsilon|^2)$, together with (3.9) and (3.10), this indicates

$$y'(t) + 2\sigma y(t) \leq C_8. \tag{3.11}$$

So we have for all $t \in (0, T_{\max})$ that

$$\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2} \int_\Omega |\nabla v_\varepsilon|^2 \leq C_9. \tag{3.12}$$

Then we derive

$$\frac{1}{2} \int_\Omega |\nabla v_\varepsilon|^2 \leq -\int_\Omega u_\varepsilon \ln u_\varepsilon + C_9 \leq e^{-1}|\Omega| + C_9 \tag{3.13}$$

for all $t \in (0, T_{\max})$ and the desired results are proved. □

Lemma 3.2. Let $p \geq 2$ and assume that (1.2)(1.3) hold with $m > 1, k_1 > 0$ and $0 < k_2 \leq 1$. If $m > \max\{2k_1, \frac{p\alpha}{p-1} - 1\}$ or $k_2 > \max\{2k_1 - \beta, \frac{p\alpha}{p-1} - \beta - 1\}$ with $m > \max\{2, 2\alpha, 2\beta\}$, then for any $q > 1$, one can find a constant $C_{10} > 0$ such that for any $t \in (0, T_{\max})$

$$\int_\Omega u_\varepsilon^q(x,t)dx \leq C_{10} \tag{3.14}$$

for any $\varepsilon \in (0, 1)$.

 doi:10.20944/preprints202507.1776.v1

Proof. Multiply $(2.1)_1$ by qu_ε^{q-1} integrate over Ω ,we can find that

$$\begin{aligned}&\tfrac{d}{dt}\int_\Omega u_\varepsilon^q+q(q-1)\int_\Omega u_\varepsilon^{q-2}(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\\&\le\chi q(q-1)\int_\Omega u_\varepsilon^{q+\alpha-2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon+\tfrac{\xi q(q-1)}{q+\beta-1}\int_\Omega u_\varepsilon^{q+\beta-1}\triangle w_\varepsilon\\&+\kappa q\int_\Omega u_\varepsilon^{q-1}-\mu q\int_\Omega u_\varepsilon^{q+m-1}\end{aligned}\tag{3.15}$$

for all $t\in(0,T_{\max})$. Note that for $p\ge 2$ and $t\in(0,T_{\max})$,

$$(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\ge|\nabla u_\varepsilon|^p.\tag{3.16}$$

Together with (3.15) (3.16), and Young's inequality, it follows that for $p\ge 2$

$$\begin{aligned}&\tfrac{d}{dt}\int_\Omega u_\varepsilon^q+q(q-1)\int_\Omega u_\varepsilon^{q-2}|\nabla u_\varepsilon|^p\\&\le\chi q(q-1)\int_\Omega u_\varepsilon^{q+\alpha-2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon+\tfrac{\xi q(q-1)}{q+\beta-1}\int_\Omega u_\varepsilon^{q+\beta-1}\triangle w_\varepsilon\\&+\kappa q\int_\Omega u_\varepsilon^{q-1}-\mu q\int_\Omega u_\varepsilon^{q+m-1}\\&\le\chi q(q-1)\int_\Omega u_\varepsilon^{q+\alpha-2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon+\tfrac{\xi q(q-1)}{q+\beta-1}\int_\Omega u_\varepsilon^{q+\beta-1}\triangle w_\varepsilon\\&-\tfrac{\mu q}{2}\int_\Omega u_\varepsilon^{q+m-1}+C_{11}\end{aligned}\tag{3.17}$$

for all $t\in(0,T_{\max})$, where C_{11} is a positive constant. As a consequence of Young's inequality, we have

$$\begin{aligned}&\chi q(q-1)\int_\Omega u_\varepsilon^{q+\alpha-2}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\\&\le\tfrac{q(q-1)}{2}\int_\Omega u_\varepsilon^{q-2}|\nabla u_\varepsilon|^p+C_{12}\int_\Omega u_\varepsilon^{\frac{(p-1)(q-2)+p\alpha}{p-1}}|\nabla v_\varepsilon|^{\frac{p}{p-1}},\end{aligned}\tag{3.18}$$

where C_{12} is a positive constant. Quoting Lemma 2.3 and applying Young's inequality to get constants $C_{13},C_{14}>0$ such that

$$\begin{aligned}&\tfrac{\xi q(q-1)}{q+\beta-1}\int_\Omega u_\varepsilon^{q+\beta-1}\triangle w_\varepsilon\\&=\tfrac{\xi\delta q(q-1)}{q+\beta-1}\int_\Omega u_\varepsilon^{q+\beta-1}w_\varepsilon-\tfrac{\xi\gamma q(q-1)}{q+\beta-1}\int_\Omega u_\varepsilon^{q+\beta+k_2-1}\\&\le\tfrac{\xi\gamma q(q-1)}{4(q+\beta-1)}\int_\Omega u_\varepsilon^{q+\beta+k_2-1}+C_{13}\int_\Omega w_\varepsilon^{\frac{q+\beta+k_2-1}{k_2}}-\tfrac{\xi\gamma q(q-1)}{q+\beta-1}\int_\Omega u_\varepsilon^{q+\beta+k_2-1}\\&=C_{13}\int_\Omega w_\varepsilon^{\frac{q+\beta+k_2-1}{k_2}}-\tfrac{3\xi\gamma q(q-1)}{4(q+\beta-1)}\int_\Omega u_\varepsilon^{q+\beta+k_2-1}\\&\le-\tfrac{\xi\gamma q(q-1)}{2(q+\beta-1)}\int_\Omega u_\varepsilon^{q+\beta+k_2-1}+C_{14}\end{aligned}\tag{3.19}$$

for all $t\in(0,T_{\max})$. Together with (3.17) (3.18) and (3.19), we have

$$\begin{aligned}&\tfrac{d}{dt}\int_\Omega u_\varepsilon^q+\tfrac{q(q-1)}{2}\int_\Omega u_\varepsilon^{q-2}|\nabla u_\varepsilon|^p\\&\le C_{12}\int_\Omega u_\varepsilon^{\frac{(p-1)(q-2)+p\alpha}{p-1}}|\nabla v_\varepsilon|^{\frac{p}{p-1}}-\tfrac{\xi\gamma q(q-1)}{2(q+\beta-1)}\int_\Omega u_\varepsilon^{q+\beta+k_2-1}\\&-\tfrac{\mu q}{2}\int_\Omega u_\varepsilon^{q+m-1}+C_{11}+C_{14}\end{aligned}\tag{3.20}$$

for all $t\in(0,T_{\max})$.

 doi:10.20944/preprints202507.1776.v1

Due to the fact that

$$\frac{q(q-1)}{2}\int_\Omega u_\varepsilon^{q-2}|\nabla u_\varepsilon|^p = \frac{q(q-1)p^p}{2(p+1-2)^p}\|\nabla u_\varepsilon^{\frac{p+q-2}{p}}\|^p_{L^p(\Omega)} \tag{3.21}$$

for all $t \in (0, T_{\max})$. From (3.20) and (3.21), we conclude that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_\varepsilon^q + \frac{q(q-1)p^p}{2(p+1-2)^p}\|\nabla u_\varepsilon^{\frac{p+q-2}{p}}\|^p_{L^p(\Omega)}\\ &\le C_{12}\int_\Omega u_\varepsilon^{\frac{(p-1)(q-2)+p\alpha}{p-1}}|\nabla v_\varepsilon|^{\frac{p}{p-1}} - \frac{\xi\gamma q(q-1)}{2(q+\beta-1)}\int_\Omega u_\varepsilon^{q+\beta+k_2-1}\\ &-\frac{\mu q}{2}\int_\Omega u_\varepsilon^{q+m-1} + C_{11} + C_{14}\end{aligned} \tag{3.22}$$

for all $t \in (0, T_{\max})$. Analogous to Lemma 3.11 in [25],we can choose a positive constant C_{15} for any $q' \ge 2$ such that the following inequality holds

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^{2q'} + \frac{2(q'-1)}{q'}\int_\Omega |\nabla|\nabla v_\varepsilon|^{q'}|^2 \le C_{15}\int_\Omega u_\varepsilon^{2k_1}|\nabla v_\varepsilon|^{2q'-2} + C_{15}. \tag{3.23}$$

From the combination of (3.22) and (3.23), it follows that

$$\begin{aligned}&\frac{d}{dt}\Big(\int_\Omega u_\varepsilon^q + \int_\Omega |\nabla_\varepsilon|^{2q'}\Big) + \frac{q(q-1)p^p}{2(p+1-2)^p}\|\nabla u_\varepsilon^{\frac{p+q-2}{p}}\|^p_{L^p(\Omega)} + \frac{2(q'-1)}{q'}\int_\Omega |\nabla|\nabla v_\varepsilon|^{q'}|^2\\ &\le C_{12}\int_\Omega u_\varepsilon^{\frac{(p-1)(q-2)+p\alpha}{p-1}}|\nabla v_\varepsilon|^{\frac{p}{p-1}} + C_{15}\int_\Omega u_\varepsilon^{2k_1}|\nabla v_\varepsilon|^{2q'-2}\\ &-\frac{\xi\gamma q(q-1)}{2(q+\beta-1)}\int_\Omega u_\varepsilon^{q+\beta+k_2-1} - \frac{\mu q}{2}\int_\Omega u_\varepsilon^{q+m-1} + C_{16}\end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max})$.

Case (i): $m > \max\{2k_1, \frac{p\alpha}{p-1} - 1\}$. Let $q' := \frac{(p-1)(q+m-1)}{(p-1)(m+1)-p\alpha}$ and take $q \ge m + 3 - \frac{2p\alpha}{p-1}$,and $m > \frac{p\alpha}{p-1} - 1$, we have $q' \ge 2$. Useing Hölder inequality, we obtain

$$C_{12}\int_\Omega u_\varepsilon^{\frac{(p-1)(q-2)+p\alpha}{p-1}}|\nabla v_\varepsilon|^{\frac{p}{p-1}} \le C_{12}\Big(\int_\Omega u_\varepsilon^{q+m-1}\Big)^{\frac{(p-1)(q-2)+p\alpha}{(p-1)(q+m-1)}}\Big(\int_\Omega |\nabla v_\varepsilon|^{\frac{pq'}{p-1}}\Big)^{\frac{1}{q'}} \tag{3.25}$$

for all $t \in (0, T_{\max})$. Similarly,

$$C_{15}\int_\Omega u_\varepsilon^{2k_1}|\nabla v_\varepsilon|^{2q'-2} \le C_{15}\Big(\int_\Omega u_\varepsilon^{q+m-1}\Big)^{\frac{2k_1}{q+m-1}}\Big(\int_\Omega |\nabla v_\varepsilon|^{(2q'-2)\lambda}\Big)^{\frac{1}{\lambda}}, \tag{3.26}$$

where $\lambda = \frac{q+m-1}{q+m-2k_1-1} > 1$. Then by means of the Gagliardo–Nirenberg inequality, we deduce

$$\begin{aligned}&\Big(\int_\Omega |\nabla v_\varepsilon|^{\frac{pq'}{p-1}}\Big)^{\frac{1}{q'}} = \||\nabla v_\varepsilon|^{q'}\|^{\frac{p}{(p-1)q'}}_{L^{\frac{p}{p-1}}(\Omega)}\\ &\le \Big(C_{17}\|\nabla|\nabla v_\varepsilon|^{q'}\|^{\theta_1}_{L^2(\Omega)}\||\nabla v_\varepsilon|^{q'}\|^{1-\theta_1}_{L^{\frac{2}{q'}}(\Omega)} + C_{17}\||\nabla v_\varepsilon|^{q'}\|_{L^{\frac{2}{q'}}(\Omega)}\Big)^{\frac{p}{(p-1)q'}}\end{aligned} \tag{3.27}$$

 doi:10.20944/preprints202507.1776.v1

for all $t \in (0, T_{\max})$, where $\theta_1 = \frac{\frac{q'}{2} - \frac{p-1}{p}}{\frac{q'}{2} + \frac{1}{n} - \frac{1}{2}}$. Since $p \geq 2$, selecting $q > \frac{2[(p-1)(m+1)-p\alpha]}{p} - m + 1$, we have $\theta_1 \in (0,1)$. We invoke (3.27) and Lemma 3.1 with $m > 2k_1$ such that

$$(\int_\Omega |\nabla v_\varepsilon|^{\frac{pq'}{p-1}})^{\frac{1}{q'}} \leq C_{18}(\int_\Omega |\nabla|\nabla v_\varepsilon|^{q'}|^2)^{\frac{p\theta_1}{2(p-1)q'}} + C_{18} \tag{3.28}$$

with $C_{18} > 0$. Recalling the Gagliardo–Nirenberg inequality again we estimate

$$\begin{aligned}(\int_\Omega |\nabla v_\varepsilon|^{2(q'-1)\lambda})^{\frac{1}{\lambda}} &= \||\nabla v_\varepsilon|^{q'}\|_{L^{\frac{2\lambda(q'-1)}{q'}}(\Omega)}^{\frac{2(q'-1)}{q'}} \\ &\leq (C_{19}\|\nabla|\nabla v_\varepsilon|^{q'}\|_{L^2(\Omega)}^{\theta_2}\||\nabla v_\varepsilon|^{q'}\|_{L^{\frac{2}{q'}}(\Omega)}^{1-\theta_2} + C_{19}\||\nabla v_\varepsilon|^{q'}\|_{L^{\frac{2}{q'}}(\Omega)})^{\frac{2(q'-1)}{q'}}\end{aligned} \tag{3.29}$$

with $C_{19} > 0$, where $\theta_2 = \frac{\frac{q'}{2} - \frac{q'}{2\lambda(q'-1)}}{\frac{q'}{2} + \frac{1}{n} - \frac{1}{2}}$. Due to $m > \max\{1, 2k_1\}, p \geq 2$ and take $q > \max\{m + 3 - \frac{2p\alpha}{p-1}, \frac{(n-2)p\alpha}{2(p-1)} - \frac{n(m-2k_1)}{2} + 2 - \frac{n}{2}\}$, we get $\theta_2 \in (0,1)$. Then combining Lemma 3.1, we have

$$(\int_\Omega |\nabla v_\varepsilon|^{2(q'-1)\lambda})^{\frac{1}{\lambda}} \leq C_{20}(\int_\Omega |\nabla|\nabla v_\varepsilon|^{q'}|^2)^{\frac{(q'-1)\theta_2}{q'}} + C_{20}. \tag{3.30}$$

Together with (3.25),(3.26), (3.28) and (3.30), we obtain

$$\begin{aligned}&C_{12}\int_\Omega u_\varepsilon^{\frac{(p-1)(q-2)+p\alpha}{p-1}}|\nabla v_\varepsilon|^{\frac{p}{p-1}} + C_{15}\int_\Omega u_\varepsilon^{2k_1}|\nabla v_\varepsilon|^{2q'-2} \\ \leq\ & C_{12}C_{18}(\int_\Omega u_\varepsilon^{q+m-1})^{\frac{(p-1)(q-2)+p\alpha}{(p-1)(q+m-1)}}(\int_\Omega |\nabla|\nabla v_\varepsilon|^{q'}|^2)^{\frac{p\theta_1}{2(p-1)q'}} \\ &+C_{12}C_{18}(\int_\Omega u_\varepsilon^{q+m-1})^{\frac{(p-1)(q-2)+p\alpha}{(p-1)(q+m-1)}} \\ &+C_{15}C_{20}(\int_\Omega u_\varepsilon^{q+m-1})^{\frac{2k_1}{q+m-1}}(\int_\Omega |\nabla|\nabla v_\varepsilon|^{q'}|^2)^{\frac{(q'-1)\theta_2}{q'}} \\ &+C_{15}C_{20}(\int_\Omega u_\varepsilon^{q+m-1})^{\frac{2k_1}{q+m-1}}\end{aligned} \tag{3.31}$$

for $p \geq 2$ and $q > \max\{m + 3 - \frac{2p\alpha}{p-1}, \frac{(n-2)p\alpha}{2(p-1)} - \frac{n(m-2k_1)}{2} + 2 - \frac{n}{2}, \frac{2[(p-1)(m+1)-p\alpha]}{p} - m + 1\}$.

Next we prove $\frac{(p-1)(q-1)+1}{(p-1)(q+m-1)} + \frac{p\theta_1}{2(p-1)q'} < 1$.This is an obvious fact that holds true,otherwise, if $\frac{(p-1)(q-1)+1}{(p-1)(q+m-1)} + \frac{p\theta_1}{2(p-1)q'} \geq 1$,this leads to

$$\begin{aligned}\theta_1 &\geq \frac{2(p-1)q'}{p} \cdot \frac{(p-1)(m+1)-p\alpha}{(p-1)(q+m-1)}. \\ &= 2 - \frac{2}{p} \\ &\geq 1\end{aligned} \tag{3.32}$$

which used $q' = \frac{(p-1)(q+m-1)}{(p-1)(m+1)-p\alpha}$ and $p \geq 2$. This means that

$$\frac{(p-1)(q-1)+1}{(p-1)(q+m-1)} + \frac{p\theta_1}{2(p-1)q'} < 1. \tag{3.33}$$

 doi:10.20944/preprints202507.1776.v1

Subsequently,we prove $\frac{2k_1}{q+m-1}+\frac{(q'-1)\theta_2}{q'}<1$. Let

$$\frac{2k_1}{q+m-1}+\frac{(q'-1)\theta_2}{q'}=\frac{h_1(p,q)}{h_2(p,q)}. \tag{3.34}$$

where

$$\begin{aligned} h_1(p,q)=&2k_1[(p-1)(q+m-1)+\tfrac{2-2n}{n}((p-1)(m+1)-p\alpha)]\\ &-(q+m-1)((p-1)(q-2)+p\alpha)\\ &+(q+m-2k_1-1)((p-1)(q-2)+p\alpha) \end{aligned} \tag{3.35}$$

and

$$h_2(p,q)=(q+m-1)[(p-1)(q+m-1)+\tfrac{2-2n}{n}((p-1)(m+1)-p\alpha)]. \tag{3.36}$$

A direct calculation yields

$$\begin{aligned} h_2(p,q)-h_1(p,q)=&(q+m-2k_1-1)[(p-1)(q+m-1)\\ &+\tfrac{2-2n}{n}((p-1)(m+1)-p\alpha)]\\ &+(q+m-1)((p-1)(q-2)+p\alpha), \end{aligned} \tag{3.37}$$

Due to $q'\geq 2$,we have $(p-1)(q+m-1)\geq 2[(p-1)(m+1)-p\alpha]$.Combining $m>\max\{2k_1,\frac{p\alpha}{p-1}-1\}$, $p\geq 2$ and $q>1$,we have

$$\begin{aligned} h_2(p,q)-h_1(p,q)\geq&\tfrac{2}{n}(q+m-2k_1-1)((p-1)(m+1)-p\alpha)\\ &+(q+m-1)((p-1)(q-2)+p\alpha)\\ >&0. \end{aligned} \tag{3.38}$$

It means

$$\frac{2k_1}{q+m-1}+\frac{(q'-1)\theta_2}{q'}<1. \tag{3.39}$$

Together with (3.31), (3.33), (3.39) and by the Young's inequality twice, we deduce

$$\begin{aligned} &C_{12}\int_\Omega u_\varepsilon^{\frac{(p-1)(q-2)+p\alpha}{p-1}}|\nabla v_\varepsilon|^{\frac{p}{p-1}}+C_{15}\int_\Omega u_\varepsilon^{2k_1}|\nabla v_\varepsilon|^{2q'-2}\\ \leq&\epsilon(\int_\Omega u_\varepsilon^{q+m-1}+\int_\Omega|\nabla|\nabla v_\varepsilon|^{q'}|^2)+C_{21} \end{aligned} \tag{3.40}$$

for any $\epsilon>0$, $p\geq 2$ and $q>q_0:=\max\{1,m+3-\frac{2p\alpha}{p-1},\frac{n-2}{2(p-1)}-\frac{n(m-2k_1)}{2}+1,\frac{2[(p-1)m-1]}{p}-m+1\}$. Using the Gagliardo–Nirenberg inequality, there exists $C_{22}>0$ such that

$$\begin{aligned} \int_\Omega|\nabla v_\varepsilon|^{2q'}=&\||\nabla v_\varepsilon|^{q'}\|^2_{L^2(\Omega)}\\ \leq&C_{22}\|\nabla|\nabla v_\varepsilon|^{q'}\|^{2\theta_3}_{L^2(\Omega)}\||\nabla v_\varepsilon|^{q'}\|^{2(1-\theta_3)}_{L^{\frac{2}{q'}}(\Omega)}+C_{22}\||\nabla v_\varepsilon|^{q'}\|_{L^{\frac{2}{q'}}(\Omega)}, \end{aligned} \tag{3.41}$$

where $\theta_3=\frac{\frac{q'}{2}-\frac{1}{2}}{\frac{q'}{2}+\frac{1}{n}-\frac{1}{2}}\in(0,1)$. From (3.41) and Lemma 3.1, we conclude taht

$$\int_\Omega|\nabla v_\varepsilon|^{2q'}\leq C_{23}\|\nabla|\nabla v_\varepsilon|^{q'}\|^2_{L^2(\Omega)}\|+C_{23} \tag{3.42}$$

 doi:10.20944/preprints202507.1776.v1

Using Young's inequality again, we obtain

$$\int_\Omega u_\varepsilon^q \le C_{24} \int_\Omega u_\varepsilon^{q+m-1} + C_{24} \tag{3.43}$$

for all $t \in (0, T_{\max})$. Combine (3.24), (3.40), (3.42) and (3.43), we can see that

$$\frac{d}{dt}\left(\int_\Omega u_\varepsilon^q + \int_\Omega |\nabla_\varepsilon|^{2q'}\right) + \int_\Omega u_\varepsilon^q + \int_\Omega |\nabla_\varepsilon|^{2q'} \le C_{25} \tag{3.44}$$

for all $q > q_0 > 1$. By the Young's inequality again, we obtain for $q > 1$ that

$$\int_\Omega u_\varepsilon^q \le C_{26} \tag{3.45}$$

holds for all $t \in (0, T_{\max})$.

Case (ii): $k_2 > \max\{2k_1 - \beta, \frac{p\alpha}{p-1} - \beta - 1\}$ with $m > \max\{2, 2\alpha, 2\beta\}$. Following the approach used for Case (i), we reach corresponding outcomes. □

Lemma 3.3. Under assumption of Lemma 3.2, there exists $C_{27} > 0$ such that for any $\varepsilon \in (0, 1)$

$$\|v_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le C_{27}, \quad \|w_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le C_{27} \tag{3.46}$$

and

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le C_{27} \tag{3.47}$$

hold for all $t \in (0, T_{\max})$.

Proof. In analogy with Lemma 3.3 of [32] , we skip the proof. □

4. Proof of Theorem 1.1.

The existence of global weak solutions to system (1.1) can be shown by taking limits of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$. We start this analysis with a fundamental lemma from [12,16].

Lemma 4.1. Under assumption of Lemma 3.2, there exist constants $C_1, C_2 > 0$ such that for all $t > 0$

$$\int_0^t \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^p \le C_1 \tag{4.1}$$

and for all $T > 0$

$$\|\partial_t u_\varepsilon\|_{L^1((0,T);(W_0^{2,p}(\Omega))^*)} \le C_2. \tag{4.2}$$

Proof. Lemma 3.2's proof establishes that $\int_0^t \int_\Omega u_\varepsilon^{q-2} |\nabla u_\varepsilon|^p \le C_3$. Let $q = 2$, we deduce (4.1).

Let $\psi \in C_0^\infty(\Omega)$ satisfies $\|\psi\|_{W^{2,p}(\Omega)} \le 1$. Multiplying $(2.1)_1$ by ψ, integrating by part, we obtain

$$\begin{aligned} \int_\Omega u_{\varepsilon t}\psi \le &- \int_\Omega (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \psi + \chi \int_\Omega u_\varepsilon^\alpha \nabla v_\varepsilon \cdot \nabla \psi \\ &- \xi \int_\Omega u_\varepsilon^\beta \nabla w_\varepsilon \cdot \nabla \psi + \kappa \int_\Omega \psi - \mu \int_\Omega u_\varepsilon^m \psi. \end{aligned} \tag{4.3}$$

 doi:10.20944/preprints202507.1776.v1

Since $p \geq 2$, using the Hölder inequality and Young's inequality, we can obtain

$$\begin{aligned} & |\int_\Omega (|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon \cdot \nabla \psi| \\ \leq\ & C_4 \int_\Omega (|\nabla u_\varepsilon|^{p-1}+1)\nabla|\psi| \\ \leq\ & C_5(\int_\Omega |\nabla u_\varepsilon|^p)^{\frac{p-1}{p}}(\int_\Omega |\nabla\psi|^p)^{\frac{1}{p}} + C_6(\int_\Omega |\nabla\psi|^p)^{\frac{1}{p}} \\ \leq\ & C_7 \int_\Omega |\nabla u_\varepsilon|^p + C_8 \end{aligned} \tag{4.4}$$

with $C_i > 0 (i = 4,5,6,7,8)$. Due to Lemma 3.3, we can find constants $C_9, C_{10} > 0$ such that

$$|\chi \int_\Omega u_\varepsilon^\alpha \nabla v_\varepsilon \cdot \nabla\psi| \leq \chi \|u_\varepsilon\|^\alpha_{L^\infty(\Omega)} \|\nabla v_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \|\nabla\psi\|_{L^p(\Omega)} \leq C_9 \tag{4.5}$$

and

$$|\xi \int_\Omega u_\varepsilon^\beta \nabla w_\varepsilon \cdot \nabla\psi| \leq \xi \|u_\varepsilon\|^\beta_{L^\infty(\Omega)} \|\nabla w_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega)} \|\nabla\psi\|_{L^p(\Omega)} \leq C_{10}. \tag{4.6}$$

Similarly, we can get

$$|\kappa \int_\Omega \psi - \mu \int_\Omega u_\varepsilon^l \psi| \leq C_{11}\|\psi\|_{L^1(\Omega)}(1+\|u_\varepsilon\|^m_{L^\infty(\Omega)}) \leq C_{12} \tag{4.7}$$

with $C_{11}, C_{12} > 0$. Combining (4.3)-(4.7) yields

$$\int_\Omega u_{\varepsilon t}\psi \leq C_7 \int_\Omega |\nabla u_\varepsilon|^p + C_{13} \tag{4.8}$$

Integrating (4.8) on $(0,T)$ and collecting (4.1) yields (4.2). □

Lemma 4.2. let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the classical solution of (2.1). There exists a nonnegative funtion (u,v,w) such that as $\varepsilon \to 0$,

$$u_\varepsilon \to u \ \text{ a.e. in } Q_T, \tag{4.9}$$

$$u_\varepsilon \to u \ \text{ in } L^p_{loc}(Q_T), \tag{4.10}$$

$$u_\varepsilon \rightharpoonup^* u \ \text{ in } L^\infty(Q_T), \tag{4.11}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \ \text{ in } L^p_{loc}(Q_T), \tag{4.12}$$

$$(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon \rightharpoonup |\nabla u|^{p-2}\nabla u \ \text{ in } L^{\frac{p}{p-1}}_{loc}(Q_T), \tag{4.13}$$

$$v_\varepsilon \to v, \ \ uniformly, \tag{4.14}$$

$$v_\varepsilon \rightharpoonup v \ \text{ in } W^{2,1}_r(Q_T), \ \ for\ any\ r > 1, \tag{4.15}$$

 doi:10.20944/preprints202507.1776.v1

$$w_\varepsilon \rightharpoonup w, \ \ in\ L^2_{loc}((0,T), W^{1,2}(\Omega)), \tag{4.16}$$

$$w_\varepsilon \rightharpoonup^* w, \ \ in\ L^\infty(\Omega \times (0,T)), \tag{4.17}$$

$$\bigtriangledown w_\varepsilon \rightharpoonup^* \bigtriangledown w, \ \ in\ L^\infty(\Omega \times (0,T)), \tag{4.18}$$

for any $T > 0$.

Proof. Through Lemmas 3.3 and 4.1 and together with Aubin-Lions lemma, we obtain a nonnegative function u satisfies as $\varepsilon \to 0$,

$$u_\varepsilon \to u \ \text{ in } L^p_{loc}(Q_T) \text{ and a.e. in } Q_T, \tag{4.19}$$

this yields (4.9) and (4.10).

Collecting Lemma 3.3 and (4.19),we deduce (4.11). Together with (4.1) and (4.19) that (4.12).

Collecting (4.11), (4.19) and Lemma 3.3, we obtain (4.13) by the method of the Lemma 4.5 in [28]. We may further conclude from Lemma 3.3 in conjunction with (1.3) that

$$\sup_{t\in(1,\infty)} \int_{t-1}^{t} \|g_1(u_\varepsilon)\|^r_{L^r} ds \le C_{14} \tag{4.20}$$

for any $r > 1$. Collecting Lemma 2.4 in [5] and (4.20), we have

$$\sup_{t\in(1,\infty)} \int_{t-1}^{t} (\|v_\varepsilon\|^r_{W^{2,r}} + \|v_{\varepsilon t}\|^r_{L^r}) ds \le C_{15}, \tag{4.21}$$

this implies

$$\|v_\varepsilon\|_{W^{2,1}_r(\Omega\times(t-1,t))} \le C_{16} \tag{4.22}$$

with $C_{16} > 0$. In view of (4.22), we obtain (4.15). Since $W^{2,1}_r(Q_T) \hookrightarrow C^{2-\frac{n+2}{r},1-\frac{n+2}{2r}}(Q_T)$ for $r > \frac{n+2}{2}$, one can from (4.22) gets (4.14). In addition, the estimate (3.52) implies (4.16)–(4.18). □

Proof of Theorem 1.1. We conclude from Lemma 4.2 and Definition 2.1 that (u, v, w) in Lemma 4.2 is a global weak solution of (1.1). □

Author Contributions: Conceptualization, Zhe Jia; methodology,Zhe Jia ; formal analysis, Hengyu Ren; investigation,Hengyu Ren ; writing—original draft preparation, Hengyu Ren; writing—review and editing, Zhe Jia;

Funding: Project Supported by the National Natural Science Foundation of China(Grant No.12301251); the Natural Science Foundation of Shandong Province, China (Grant No.ZR2021QA038) and the Scientific Research Foundation of Linyi University, China (Grant No.LYDX2020BS014

Conflicts of Interest: The authors declare no conflicts of interest.

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