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On 2-nil Primary Ideals of Commutative Rings

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Abstract: The present article addresses the concept of 2-nil primary ideals in commutative rings, expanding the comprehension of ideal categories such as 2-nil, 2-absorbing and quasi primary ideals. The study explores the characteristics and connections of 2-nil primary ideals, offering a comprehensive framework for understanding their significance in ring theory. The paper presents examples and arguments that illustrate the relationship between 2-nil primary ideals and other well-known classes of ideals, such as prime, primary and n-ideals, while highlighting their differences.Furthermore, we investigate how the 2-nil primary ideal behaves concerning images and inverse images of homomorphism, quotients, localization, products, and idealizations.

Keywords: n-ideal; 2-absorbing ideal; primary ideal; 2-nil ideal

MSC: 13A15;13A70; 13A99

Introduction

Throughout this paper, we consider all rings R as commutative rings with unity $1 \neq 0$. The radical of an ideal I in R is traditionally denoted by $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$, in particular, the set of nilpotent elements of R is represented by $\sqrt{0}$. The set of all zero divisors of the ring R is denoted by $Z(R)$.

The concepts of prime and primary ideals have been well-established in ring theory, serving as fundamental tools for understanding the structure and properties of commutative rings. These ideals have been extended to more complex concepts such as 2-absorbing and 2-absorbing primary ideals. The concept of a 2-absorbing and 2-absorbing primary ideals, introduced by [4,5], defines a proper ideal I such that for any elements $a, b, c \in R$,

$$abc \in I \implies ab \in I \text{ or } ac \in I \text{ or } bc \in I.$$

Similarly, a 2-absorbing primary ideal is defined by the condition

$$abc \in I \implies ab \in I \text{ or } ac \in \sqrt{I} \text{ or } bc \in \sqrt{I}.$$

These generalizations have paved the way for further exploration into more nuanced types of ideals. In the continuation of extending these concepts, the notion of n-ideals was introduced in [24] . For a proper ideal I and elements $a, b \in R$,

$$ab \in I \implies a \in \sqrt{0} \text{ or } b \in I.$$

In [23],this concept has been further expanded to $(2, n)$-ideals, where a proper ideal I is considered a $(2, n)$-ideal if for any $a, b, c \in R$,

$$abc \in I \implies ab \in I \text{ or } ac \in \sqrt{0} \text{ or } bc \in \sqrt{0}.$$

Recently, in [6], a new class of ideals which is called 2-nil ideal have been introduced. A proper ideal I of a ring R is said to be 2-nil ideal if whenever $a, b, c \in R$ then

$$abc \in I \Rightarrow ab \in \sqrt{0} \text{ or } ac \in I \text{ or } bc \in I. \quad (1)$$

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Despite these advancements, there remains a need to explore other potential classes of ideals that can offer deeper insights into the structure of commutative rings. In this paper, we introduce a new class of ideals known as 2-nil primary ideals. This new class seeks to bridge the gap between existing concepts and provide a broader framework for understanding ideal theory in commutative rings. In this study, a proper ideal I is defined as a 2-nil primary ideal if for any elements $a, b, c \in R$,

$$abc \in I \implies ab \in \sqrt{0} \text{ or } ac \in \sqrt{I} \text{ or } bc \in \sqrt{I}.$$

This definition presents a distinct viewpoint by including the collection of nilpotent elements of R, denoted as $\sqrt{0}$, in the criteria for an ideal to be 2-nil primary. This work seeks to clarify the relevance and importance of 2-nil primary ideals within the broader field of ring theory by analysing their characteristics and relationships.

The paper is organized as follows: Section 1 introduces the definition of a 2-nil primary ideal (see Definition 1). From this definition, it immediately follows that every 2-nil ideal is a 2-nil primary ideal. However, the converse is not true (see Example 1). Diagram 1 summarises the relationships of 2-nil primary ideals with some well-known classes of ideals in the literature. Proposition 1 defines a 2-nil primary ideal as an ideal whose radical is a 2-nil ideal. Proposition 2 demonstrates that in a Dedekind domain, which is not a field, every 2-nil primary ideal is quasi-primary, though the converse is not true (see Example 2). Theorem 3 describes the 2-nil primary ideals of the ring of integers $\mathbb{Z}$, while Proposition 3 provides a characterization of fields in terms of 2-nil primary ideals. At the end of this section, we present the relationships between different classes of ideals in which all elements are nilpotent (see Diagram 2). Section two examines the image and inverse images of 2-nil primary ideals under homomorphisms (refer to Theorem 4). Theorem 5 explores transferring 2-nil primary ideals to the trivial ring extension. Theorem 7 discusses the product of 2-nil primary ideals.

In summary, this paper seeks to advance the theoretical foundation of ideal structures in commutative rings by introducing and investigating the concept of 2-nil primary ideals. Through this exploration, we aim to contribute to a deeper understanding of ring theory and its applications.

1. Some Properties of 2-nil Primary Ideals

In this section, a new class of ideals, the 2-nil primary ideal, will be introduced. We then delve into the properties and characterizations of 2-nil primary ideals. We aim to thoroughly understand their defining characteristics by presenting various theorems and proofs. Examples are included to illustrate the practical implications of these theoretical results.

Definition 1. *Let R be a ring and I a proper ideal of R. We call I a 2-nil primary ideal if whenever $abc \in I$ then either $ab \in \sqrt{0}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.*

From the definition 1 one can immediately infer that if I is 2-nil ideal, then I is 2-nil primary ideal. The converse of this implication is not true in general, as demonstrated by the following example. However, if I is a radical ideal, then I is a 2-nil primary if and only if I is a 2-nil ideal.

Example 1. *Consider $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$. Then I is not 2-nil ideal as $2 \cdot 2 \cdot 1 \in I$ but $2.2 \notin \sqrt{0}$ and $2.1 \notin I$. However, I is 2-nil primary ideal. To verify this claim, consider $abc \in 4\mathbb{Z}$. Then $abc = 4k$, where $k \in \mathbb{Z}$. It is clear that at least one of a, b, and c should be even. If $ab \notin \sqrt{0}$, then $ac \in \sqrt{I} = 2\mathbb{Z}$ or $bc \in \sqrt{I} = 2\mathbb{Z}$.*

The following diagram illustrates the relationship of 2-nil primary ideal with some well-known ideal classes in the literature.

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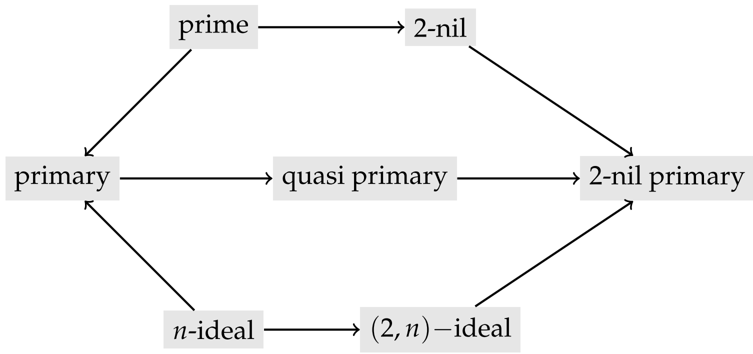

However, as the reader can see in the following, there is a strict relationship between 2-nil primary ideal and 2-nil ideals. By the following proposition, we can define 2-nil primary ideal as an ideal whose radical is 2-nil, that is; a proper ideal I of a ring R is called a 2-nil primary ideal if $\sqrt{I}$ is a 2-nil ideal of R.

Proposition 1. *Let I be a proper ideal of a ring R. Then I is a 2-nil primary ideal if and only if $\sqrt{I}$ is a 2-nil ideal.*

Proof. Let $a, b, c \in R$ such that $abc \in \sqrt{I}$ and $ab \notin \sqrt{0}$. Then $(abc)^n = a^n b^n c^n \in I$ for some $n \in \mathbb{N}$. Since I is 2-nil primary, we have either $a^n c^n \in \sqrt{I}$ or $b^n c^n \in \sqrt{I}$. This means that either $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence, $\sqrt{I}$ is a 2-nil ideal of R. Conversely, suppose that $\sqrt{I}$ is a 2-nil ideal and $a, b, c \in R$ such that $abc \in I$. Since $I \subseteq \sqrt{I}$, we have either $ab \in \sqrt{0}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Thus I is a 2-nil primary ideal of R. □

Next, we discuss the connection between the concepts of 2-nil primary ideal and quasi primary ideal. The next example indicates that a 2-nil primary ideal need not be a quasi primary ideal in general.

Example 2. *Let $R = \mathbb{Z}_{12}$ and consider the ideal $I = 6\mathbb{Z}_{12}$. I is not a quasi primary ideal since $2.3 \in I$ but neither $2 \in \sqrt{I}$ nor $3 \in \sqrt{I} = 6\mathbb{Z}_{12}$. However, I is a 2-nil primary ideal of R. To show that let $abc \in I$ for some $a, b, c \in R$. Then $2|abc$ and $3|abc$ which implies that at least one of the elements ab, ac or bc is a multiple of 6. Hence, either $ab \in \sqrt{0} = 6\mathbb{Z}_{12}$ or $ac \in \sqrt{I} = 6\mathbb{Z}_{12}$ or $bc \in \sqrt{I} = 6\mathbb{Z}_{12}$.*

We conclude some conditions for a 2-nil primary ideal to be quasi primary.

Proposition 2. *In any ring R with $dim(R) = 1$ (in particular, in Dedekind domains that are not fields; for example, in discrete valuation rings) every 2-nil primary ideals are quasi primary.*

Proof. Let R be a ring with Krull dimension 1, and I be a 2-nil primary ideal of R. Assume on the contrary that there are some $a, b \in R - \sqrt{I}$ such that $ab \in I$. Then $a.b.1 \in I$. Since $a, b \notin \sqrt{0}$ and $\sqrt{0}$ is prime, $ab \notin \sqrt{0}$. Hence, $a = a.1 \in \sqrt{I}$ or $b = b.1 \in \sqrt{I}$, a contradiction. Thus, I is a quasi primary ideal of R. □

In the following, we determine 2-nil primary ideals of the ring of integer $\mathbb{Z}$.

Theorem 1. *In $\mathbb{Z}$, the following statements are equivalent:*

(1) I is a 2-nil primary ideal of $\mathbb{Z}$.
(2) $I = 0$ or $I = (p^n)$ for some prime integer p and $n \geq 1$.
(3) I is a primary ideal of $\mathbb{Z}$.

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Proof. (1) $\Rightarrow$ (2) Let I be a nonzero 2-nil primary ideal of $\mathbb{Z}$. Then $I = (p_1^{k_1} p_2^{k_2} ... p_m^{k_m})$ for some prime integers $p_1, p_2, ..., p_m$. Assume that $m \geq 2$. Then $p_1^{k_1}(p_2^{k_2}...p_m^{k_m})1 \in I$ but $p_1^{k_1}(p_2^{k_2}...p_m^{k_m}) \notin \sqrt{0} = 0$, $p_1^{k_1}.1 \notin \sqrt{I}$, $(p_2^{k_2}...p_m^{k_m})1 \notin \sqrt{I}$, a contradiction. Thus $m = 1$ and we are done.
(2) $\Rightarrow$ (3) is well-known.
(3) $\Rightarrow$ (1) It is straightforward. □

By the proposition below, we give a characterization for fields in terms of 2-nil primary ideals.

Proposition 3. *For any commutative ring R, the following are equivalent:*

(1) R is a field.
(2) The zero ideal is the only 2-nil primary ideal of R.

Proof. (1) $\Rightarrow$ (2) is straightforward. (2) $\Rightarrow$ (1) Let R be a commutative ring. Suppose that I is a proper ideal of R. Then there exists a prime ideal P containing I. Hence P is 2-nil primary, so $I \subseteq P = 0$ by our assumption. Thus $I = 0$ and R is a field. □

In the following, we present a characterization of 2-nil primary ideal in terms of ideal of R.

Theorem 2. *Let $R = \mathbb{Z}_n$ where $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$ for some prime integers $p_1, ..., p_m$. Every proper ideal of R is 2-nil primary if and only if $m \leq 2$.*

Proof. Let $n = p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$. Then an ideal of R is of the form $I = p_1^{t_1} p_2^{t_2} ... p_m^{t_m}$ where $t_i \leq k_i$ for all $i = 1, ..., m$. Assume that $m \geq 3$. Then $p_1^{t_1} p_2^{t_2}(p_3^{t_3}...p_m^{t_m}) \in I$ but neither $p_1^{t_1} p_2^{t_2} \in \sqrt{0}$ nor $p_2^{t_2}(p_3^{t_3}...p_m^{t_m}) \in \sqrt{I} = (p_1 p_2 ... p_m)$ nor $p_1^{t_1}(p_3^{t_3}...p_m^{t_m}) \in \sqrt{I}$, a contradiction. Thus $m \leq 2$. Conversely, let $m = 1$. Then $R = \mathbb{Z}_{p^k}$ and every ideal of R is of the form $I = (p^t)$ for some $1 \leq t \leq k$. Since any primary ideal is 2-nil primary, we are done. Now, let $m = 2$. Then $R = \mathbb{Z}_{{p_i}^{k_i}{p_j}^{k_j}}$ $I = ({p_i}^{t_i}{p_j}^{t_j})$ for some prime integers p_i, p_j and $t_i \leq k_i$, $t_j \leq k_j$. The claim follows from the fact that $I \subseteq \sqrt{0}$, Remark 1 and Corollary 2.12 in [5]. □

We recall from [?] that a proper ideal I of a ring R is said to be 2-absorbing quasi-primary ideal if its radical is a 2-absorbing ideal of R. Here, we have the following observations.

Remark 1. *Any 2-absorbing quasi primary ideal contained in $\sqrt{0}$ is 2-nil primary. To see that, let $I \subseteq \sqrt{0}$ be an ideal of a ring R and $abc \in I$ such that $ab \notin \sqrt{0}$. Then $ab \notin I$ and since I is 2-absorbing primary, we have either $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence, I is a 2-nil primary ideal of R.*

Note that if $\sqrt{0} = Z(R)$, then $\sqrt{0}$ is a 2-nil primary ideal. Moreover, the following statements are equivalent in general:

(1) $\sqrt{0}$ is a 2-absorbing ideal.
(2) $\sqrt{0}$ is a 2-absorbing primary ideal.
(3) $\sqrt{0}$ is a 2-absorbing quasi primary ideal.
(4) $\sqrt{0}$ is a 2-nil ideal.
(5) $\sqrt{0}$ is a 2-nil primary ideal.
(6) $\sqrt{0}$ is a $(2, n)$-ideal.

We enclosed the above relations between different classes of ideals, in which all elements are nilpotent, in the following diagram. Let R be a commutative ring with unity, and let I be an ideal of R such that $I \subset \sqrt{0}$.

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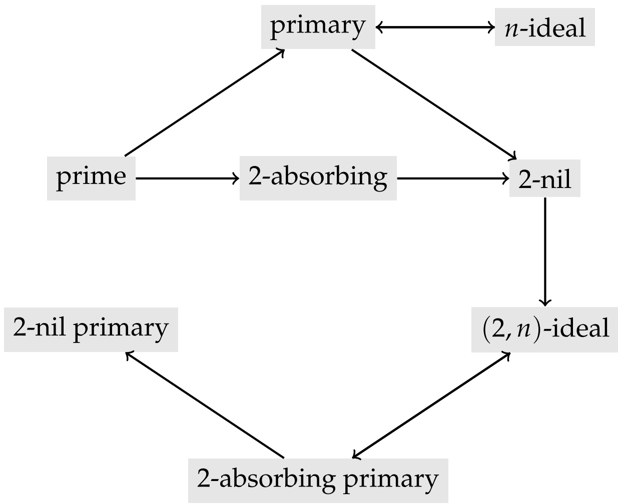

The following example shows that the condition $I \subset \sqrt{0}$ is an essential for a 2-absorbing primary ideal to be 2-nil primary ideal.

Example 3. *Consider the ideal $I = \langle pq \rangle$ in $\mathbb{Z}$ where p and q are prime numbers, then $p.q.1 \in I$. However, neither $pq \in \sqrt{0}$ nor $p \cdot 1 \in \sqrt{\langle pq \rangle}$ nor $q \cdot 1 \in \sqrt{\langle pq \rangle}$. Hence, it is not a 2-nil primary ideal. But I is 2-absorbing primary ideal. To show this, let $abc \in I$. Then $p|abc$ and $q|abc$, thus at least one of ab, ac or bc is a multiple of pq, which implies that either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.*

From the above example, it is evident that the intersection of a family of 2-nil primary ideals is not necessarily a 2-nil primary ideal. However, through the next definition, we can conclude a condition for this case.

Definition 2. *Let R be a ring and I be a 2-nil primary ideal of R. By Proposition 1, $Q = \sqrt{I}$ is a 2-nil ideal. In this case, we call I a Q-2-nil ideal of R.*

Following this definition we have:

(1) Let R be a Noetherian ring. If I is a Q-2-nil primary ideal of R, then $Q^k \subseteq I \subseteq Q$ for some $k \geq 1$.

(2) Let I be an ideal of a ring R providing $Q^k \subseteq I \subseteq Q$. Then I is a 2-nil primary ideal if and only if Q is a 2-nil primary ideal.

(3) Let J and K be Q-2-nil primary ideals of a ring R with the order $J \subseteq I \subseteq K$. Then I is a Q-2-nil primary ideal.
(4) If J and K are Q-2-nil primary and Q'-2-nil primary ideals of a ring R with $Q \subseteq Q'$, then $I = JK$ is a Q-2-nil primary ideal of R.
(5) If $I_1, I_2, ..., I_n$ are P-2-nil primary ideal of a ring R where P is a prime ideal, then $\bigcap_{j=1}^{n} I_j$ is a P-2-nil primary ideal of R.

In terms of ideals of a ring, a characterization of the class of 2-nil primary ideals is presented below.

Theorem 3. *Let I be a proper ideal of R. Then the following statements are equivalent:*

(1) I is a 2-nil primary ideal of R.
(2) For any $a, b \in R$ and $ab \notin \sqrt{0}$, then $(I : ab) \subseteq (\sqrt{I} : a) \cup (\sqrt{I} : b)$.
(3) For any $a, b \in R$ and $ab \notin \sqrt{0}$, then $(I : ab) \subseteq (\sqrt{I} : a)$ or $(I : ab) \subseteq (\sqrt{I} : b)$.
(4) If $abJ \subseteq I$ for some $a, b \in R$ and J is an ideal of R then either $ab \in \sqrt{0}$ or $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$.

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(5) *If $JKL \subseteq I$ for some ideals J, K, L of R then either $JK \subseteq \sqrt{0}$ or $JL \subseteq \sqrt{I}$ or $KL \subseteq \sqrt{I}$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in (I : ab)$ and $ab \notin \sqrt{0}$. Then we have either $ax \in \sqrt{I}$ or $bx \in \sqrt{I}$. This implies that either $x \in (\sqrt{I} : a)$ or $x \in (\sqrt{I} : b)$. Hence, $x \in (\sqrt{I} : a) \cup (\sqrt{I} : b)$.
(2) $\Rightarrow$ (3) Since $(I : ab)$ is an ideal which is a subset of the union of two ideals $(\sqrt{I} : a), (\sqrt{I} : b)$. Then either $(I : ab) \subseteq (\sqrt{I} : a)$ or $(I : ab) \subseteq (\sqrt{I} : b)$.
(3) $\Rightarrow$ (4) Assume that $abJ \subseteq I$ but neither $ab \in \sqrt{0}$ nor $aJ \subseteq \sqrt{I}$ nor $bJ \subseteq \sqrt{I}$. Then there exist $c_1, c_2 \in J$ such that $ac_1 \notin \sqrt{I}$ and $bc_2 \notin \sqrt{I}$. But $abc_1, abc_2 \in I$ and $ac_1 \notin \sqrt{I}$ and $bc_2 \notin \sqrt{I}$ this means that $(I : ab) \nsubseteq (\sqrt{I} : a)$ and $(I : ab) \nsubseteq (\sqrt{I} : b)$ which is contradicts to (3).
(4) $\Rightarrow$ (5) Suppose that $JKL \subseteq I$ but neither $JK \subseteq \sqrt{0}$ nor $JL \subseteq \sqrt{I}$ nor $KL \subseteq \sqrt{I}$. Then there exist $c_1, c_2 \in J$ and $d_1, d_2 \in K$ such that $c_1d_1 \notin \sqrt{0}$ and $d_2L \nsubseteq \sqrt{I}$ and $c_2L \nsubseteq \sqrt{I}$. Since $c_2d_2L \subseteq I$ and $d_2L \nsubseteq \sqrt{I}$ and $c_2L \nsubseteq \sqrt{I}$, we have $c_2d_2 \in \sqrt{0}$. From $c_1d_1L \subseteq I$ and $c_1d_1 \notin \sqrt{0}$, we have either $c_1L \subseteq \sqrt{I}$ or $d_1L \subseteq \sqrt{I}$. Hence, we have to consider three cases as follows:

Case 1. Suppose that $c_1L \subseteq \sqrt{I}$ but $d_1L \nsubseteq \sqrt{I}$. Since $c_2d_1L \subseteq I$ and $c_2L \nsubseteq \sqrt{I}$ and $d_1L \nsubseteq \sqrt{I}$, we have $c_2d_1 \in \sqrt{0}$. But $c_1L \subseteq \sqrt{I}$ and $c_2L \nsubseteq \sqrt{I}$ imply $c_1L + c_2L \nsubseteq \sqrt{I}$. Now, from $(c_1 + c_2)d_1L \subseteq I$ and $d_1L \nsubseteq \sqrt{I}$, we conclude $(c_1 + c_2)d_1 \in \sqrt{0}$. Thus, $c_1d_1 \in \sqrt{0}$ which is a contradiction.

Case 2. If $d_1L \subseteq \sqrt{I}$ and $c_1L \nsubseteq \sqrt{I}$, then as in Case 1 we have a contradiction.

Case 3. Suppose that $c_1L \subseteq \sqrt{I}$ and $d_1L \subseteq \sqrt{I}$. Since $d_1L \subseteq \sqrt{I}$ and $d_2L \nsubseteq \sqrt{I}$ we have $(d_1 + d_2)L \nsubseteq \sqrt{I}$. Now, since $c_2(d_1 + d_2)L \subseteq I$ and $c_2L \nsubseteq \sqrt{I}$, then we have $c_2(d_1 + d_2) \in \sqrt{0}$. Since $c_2d_2 \in \sqrt{0}$, we have $c_2d_1 \in \sqrt{0}$. On the other hand, as $c_1L \subseteq \sqrt{I}$ and $c_2L \nsubseteq \sqrt{I}$, we have $(c_1 + c_2)L \nsubseteq \sqrt{I}$. Since $(c_1 + c_2)d_2L \subseteq I$, we conclude $(c_1 + c_2)d_2 \in \sqrt{0}$. Since $c_2d_2 \in \sqrt{0}$, we get $c_1d_2 \in \sqrt{0}$. Finally, since $(c_1 + c_2)(d_1 + d_2)L \subseteq I$, $(c_1 + c_2)L \nsubseteq \sqrt{I}$ and $(d_1 + d_2)L \nsubseteq \sqrt{I}$, we conclude $(c_1 + c_2)(d_1 + d_2) = c_1d_1 + c_1d_2 + c_2d_1 + c_2d_2 \in \sqrt{0}$. As $c_2d_2 \in \sqrt{0}$, $c_1d_2 \in \sqrt{0}$ and $c_2d_1 \in \sqrt{0}$, we obtain $c_1d_1 \in \sqrt{0}$, a contradiction.
(5) $\Rightarrow$ (1) Take $J = (a), K = (b)$ and $L = (c)$ in (5).

□

2. Extensions of 2-nil Primary Ideals

In this section, we are going to study quotients, subrings, localization, Cartesian products and idealizations of 2-nil primary ideals.

Theorem 4. *Let $\varphi : R_1 \to R_2$ be a homomorphism of a commutative rings. Then the following statements hold:*

(1)) *If φ is a monomorphism, then the inverse image $\varphi^{-1}(I_2)$ of a 2-nil primary ideal $I_2 \subset R_2$ is 2-nil primary ideal of R_1.*

(2) *If φ is a epimorphism, then the image $\varphi(I_1)$ of 2-nil primary ideal $I_1 \subset R_1$ is 2-nil primary in R_2 if $Ker(\varphi) \subseteq I_1 \cap \sqrt{0_{R_1}}$*

Proof. (1) Let $a, b, c \in R_1$ and assume that $abc \in \varphi^{-1}(I_2)$. Then $\varphi(abc) \in I_2$ which implies either $\varphi(a)\varphi(b) \in \sqrt{0_{R_2}}$ or $\varphi(a)\varphi(c) \in \sqrt{I_2}$ or $\varphi(b)\varphi(c) \in \sqrt{I_2}$. Since $Ker(\varphi) = 0$, we get $ab \in \varphi^{-1}(\sqrt{0_{R_2}}) \subseteq \sqrt{0_{R_1}}$ or $ac \in \varphi^{-1}(\sqrt{I_2}) \subseteq \sqrt{\varphi^{-1}(I_2)}$ or $bc \in \varphi^{-1}(\sqrt{I_2}) \subseteq \sqrt{\varphi^{-1}(I_2)}$, as needed.
(2) Let $a = \varphi(x)$, $b = \varphi(y)$, $c = \varphi(z) \in R_2$ such that $abc = \varphi(xyz) \in \varphi(I_1)$. It follows from $Ker(\varphi) \subseteq I_1$ that $xyz \in I_1$, which implies either $xy \in \sqrt{0_{R_1}}$ or $xz \in \sqrt{I_1}$ or $yz \in \sqrt{I_1}$. It is easy to verify that since $Ker(\varphi) \subseteq I_1 \cap \sqrt{0_{R_1}}$, we have $\varphi(\sqrt{0_{R_1}}) \subseteq \sqrt{0_{R_2}}$ and $\varphi(\sqrt{I_1}) \subseteq \sqrt{\varphi(I_1)}$. Hence, $\varphi(x)\varphi(y) \in \varphi(\sqrt{0_{R_1}}) \subseteq \sqrt{0_{R_2}}$ or $\varphi(x)\varphi(z) \in \varphi(\sqrt{I_1}) \subseteq \sqrt{\varphi(I_1)}$ or $\varphi(y)\varphi(z) \in \varphi(\sqrt{I_1}) \subseteq \sqrt{\varphi(I_1)}$. □

The following corollary is an immediate consequence of Theorem (4).

Corollary 1. *Let $J \subseteq I \subset R$ be ideals. Then*

(1) *If I is a 2-nil primary ideal of R, then I/J is a 2-nil primary of R/J.*
(2) *If A is a subring of R and I is a 2-nil primary ideal of R, then $I \cap A$ is a 2-nil primary ideal of A.*

(3) *If I/J is a 2-nil primary of R/J and $J \subseteq \sqrt{0}$, then I is a 2-nil primary of R.*

Let us define the set $Z_I(R) = \{r \in R : \ rs \in I, \ s \in R - I\}$

Theorem 5. *Let S be a multiplicatively closed subset of a ring R. Then the following assertions hold*

1. *If I is a 2-nil primary ideal of R and $I \cap S = \emptyset$, then $S^{-1}I$ is a 2-nil primary ideal of $S^{-1}R$.*
2. *If $S^{-1}I$ is a 2-nil primary ideal of $S^{-1}R$ and $S \cap Z(R) = S \cap Z_I(R) = \emptyset$, then I is a 2-nil primary ideal of R.*

Proof. (1) Suppose that $\frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} \in S^{-1}I$. Then $uabc \in I$ for some $u \in S$. Hence, $ab \in \sqrt{0}$ or $uac \in \sqrt{I}$ or $ubc \in \sqrt{I}$. This implies that $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}\sqrt{0} \subseteq \sqrt{0_{S^{-1}R}}$ or $\frac{a}{s_1}\frac{c}{s_3} = \frac{uac}{us_1s_3} \in S^{-1}\sqrt{I} \subseteq \sqrt{S^{-1}I}$ or $\frac{b}{s_2}\frac{c}{s_3} = \frac{ubc}{us_2s_3} \in S^{-1}\sqrt{I} \subseteq \sqrt{S^{-1}I}$. Therefore, $S^{-1}I$ is a 2-nil primary ideal of $S^{-1}R$.
(2) Let $abc \in I$ for some $a,b,c \in R$. Then $\frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{-1}I$ which yields $\frac{a}{1}\frac{b}{1} \in \sqrt{0_{S^{-1}R}}$ or $\frac{a}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$ or $\frac{b}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$. If $\frac{a}{1}\frac{b}{1} \in \sqrt{0_{S^{-1}R}}$, then there exist $u \in S$ and $n \in \mathbb{N}$ such that $ua^nb^n = 0$. But $S \cap Z(R) = \emptyset$ then $a^nb^n = 0$ and $ab \in \sqrt{0}$. If $\frac{a}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$ then there exist $v \in S$ such that $va^nc^n \in I$ but $Z_I(R) \cap S = \emptyset$, which implies that $(ac)^n \in I$. Hence, $ac \in \sqrt{I}$. Similarly, one can show that if $\frac{b}{1}\frac{c}{1} \in \sqrt{S^{-1}I}$, then $bc \in \sqrt{I}$. □

Let A be a ring and E an A-module. Then $A \ltimes E$, the trivial (ring) extension of A by E, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by

$$(a,e)(b,f) := (ab, af + be) \quad \text{for all } a,b \in A \text{ and all } e,f \in E.$$

The basic properties of trivial ring extensions are summarized in the celebrated books [?] and [?]. Recall from [?], that

$$\sqrt{0_{A\ltimes E}} = \sqrt{0_A} \ltimes E$$

We note that

$$\sqrt{I_{A\ltimes E}} = \sqrt{I} \ltimes E.$$

Now, we investigate the possible transfer of 2-nil primary ideals to the trivial ring extension.

Theorem 6. *Let A be a ring, E be an A-module and $R := A \ltimes E$. Then I is a 2-nil primary ideal of A, if and only if $I \ltimes E$ is a 2-nil primary ideal of R.*

Proof. Let I be a 2-nil primary ideal of A and assume that $(a,e)(b,f),(c,g) \in I \ltimes E$ for some elements of R. Thus, $abc \in I$ with a,b and c are elements of A. Furthermore, the fact that I is 2-nil primary ideal of A implies that either $ab \in \sqrt{0_A}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence, one has that either $(ab, af + be) = (a,e)(b,f) \in \sqrt{0_{A\ltimes E}}$ or $(ac, ag + ce) = (a,e)(c,g) \in \sqrt{I} \ltimes E$ or $(bc, bg + cf) = (b,f)(c,g) \in \sqrt{I} \ltimes E$. Therefore, $I \ltimes E$ is a 2-nil primary ideal of $A \ltimes E$.
Conversely, assume that I is an ideal of A such that $I \ltimes E$ is a 2-nil primary ideal of $A \ltimes E$. We are going to prove that I is a 2-nil primary ideal of A. Let $abc \in I$ for some elements a,b,c in A. Then $(a,0)(b,0)(c,0) \in I \ltimes E$ which yields $(a,0)(b,0) \in \sqrt{0_{A\ltimes E}} = \sqrt{0_A} \ltimes E$ or $(a,0)(c,0) \in \sqrt{I_{A\ltimes E}} = \sqrt{I} \ltimes E$ or $(b,0)(c,0) \in \sqrt{I_{A\ltimes E}} = \sqrt{I} \ltimes E$. Then one has that $ab \in \sqrt{0_A}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence, I is a 2-nil primary ideal of A. □

It is natural to ask the following question: Suppose that I is a 2-nil primary ideal of R_1. What can we say about the ideal $I \times R_2$ of $R_1 \times R_2$? Is this ideal also a 2-nil primary ideal? The following example will address this issue.

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Example 4. *Let R_1 and R_2 be commutative rings and let I be 2-nil primary ideal of R_1. Then $I \times R_2$ is not necessarily 2-nil primary of $R_1 \times R_2$. To show this, let $R_1 = R_2 = \mathbb{Z}_{12}$ and $I = 6\mathbb{Z}_{12}$. Then I is a 2-nil primary ideal by Theorem 2. However, $(2,1)(3,1)(1,1) \in 6\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ but neither $(2,1)(3,1) \in \sqrt{0_{\mathbb{Z}_{12}\times\mathbb{Z}_{12}}}$ nor $(2,1)(1,1) \in \sqrt{6\mathbb{Z}_{12} \times \mathbb{Z}_{12}}$ nor $(3,1)(1,1) \in \sqrt{6\mathbb{Z}_{12} \times \mathbb{Z}_{12}}$.*

Proposition 4. *Let R_1 and R_2 be commutative rings and I a proper ideal of R_1. Then the following assertions are equivalent:*

(1) $I \times R_2$ is a 2-nil primary ideal of $R_1 \times R_2$.
(2) I is a quasi primary ideal of R_1.
(3) $I \times R_2$ is a quasi primary ideal of $R_1 \times R_2$.

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R$ such that $ab \in I$. Then $(a,1)(b,1)(1,1) \in I \times R_2$. Since $(a,1)(b,1) \notin \sqrt{0_{R_1\times R_2}}$ then either $(a,1)(1,1) \in \sqrt{I \times R_2} \subseteq \sqrt{I} \times R_2$ or $(b,1)(1,1) \in \sqrt{I \times R_2} \subseteq \sqrt{I} \times R_2$. This implies that either $a \in \sqrt{I}$ or $b \in \sqrt{I}$ which proves that I is a quasi primary ideal of R_1.
(2) $\Rightarrow$ (3) From Lemma 2.2 in [?]
(3) $\Rightarrow$ (1) It is clear. □

Theorem 7. *Let I_1 and I_2 be proper ideals of R_1 and R_2, respectively. Then the following statements are equivalent:*

(1) $I_1 \times I_2$ is a 2-nil primary ideal of $R_1 \times R_2$.
(2) $I_1 \subseteq \sqrt{0_{R_1}}$ and $I_2 \subseteq \sqrt{0_{R_2}}$ are quasi primary ideals of R_1 and R_2, respectively.
(3) $I_1 \times I_2$ is a 2-absorbing quasi primary ideal of $R_1 \times R_2$ and $I_1 \times I_2 \subseteq \sqrt{0_{R_1\times R_2}}$.

Proof. (1) $\Rightarrow$ (2) Let $I_1 \nsubseteq \sqrt{0_{R_1}}$, then there exists $a \in I_1 - \sqrt{0_{R_1}}$. Then $(a,1)(1,0)(1,1) \in I_1 \times I_2$ and $(a,1)(1,0) \notin \sqrt{0_{R_1}} \times \sqrt{0_{R_2}}$. Hence, $(a,1)(1,1) \in \sqrt{I_1 \times I_2}$ or $(1,0)(1,1) \in \sqrt{I_1 \times I_2}$ which in both cases is a contradiction. Thus, $I_1 \subseteq \sqrt{0_{R_1}}$ and similarly, $I_2 \subseteq \sqrt{0_{R_2}}$. Now, if I_1 is not quasi primary, then there exist $a, b \in R_1 - I_1$ such that $ab \in I_1$, but neither $a \in \sqrt{I_1}$ nor $b \in \sqrt{I_1}$. Hence, $(a,1)(b,1)(1,0) \in I_1 \times I_2$ and since $(a,1)(b,1) \notin \sqrt{0_{R_1\times R_2}}$ and $(a,1)(1,0) \notin \sqrt{I_1 \times I_2}$ and $(b,1)(1,0) \notin \sqrt{I_1 \times I_2}$, a contradiction. Thus, I_1 is quasi primary in R_1. The same argument shows that I_2 is a quasi primary ideal of R_2.
(2) $\Rightarrow$ (3) Suppose that $I_1 \subseteq \sqrt{0_{R_1}} \subset R_1$ and $I_2 \subseteq \sqrt{0_{R_2}} \subset R_2$ are quasi primary ideals. Hence, by Proposition 4, one has that $I_1 \times R_2$ and $R_1 \times I_2$ are quasi primary ideals of $R_1 \times R_2$. By Theorem 2.17(ii) in [?], the intersection of two quasi primary ideals is 2-absorbing quasi primary ideal. So, we conclude that $I_1 \times I_2 = (I_1 \times R_2) \cap (R_1 \times I_2)$ is a 2-absorbing quasi primary ideal of $R_1 \times R_2$.
(3) $\Rightarrow$ (1) It follows from Remark 1. □

Theorem 8. *Let $R_1, R_2, ..., R_n$ $(n \geq 3)$ be commutative rings and $R = R_1 \times R_2 \times \cdots \times R_n$. The following assertions are equivalent:*

(1) $I = I_1 \times I_2 \times \cdots \times I_n$ is a 2-nil primary ideal of R.
(2) I_k is a quasi primary ideal of R_k for some $k \in \{1, 2, ..., n\}$ and $I_j = R_j$ for all $j \in \{1, 2, ..., n\} - \{k\}$.
(3) $I = I_1 \times I_2 \times \cdots \times I_n$ is a quasi primary ideal of R.

Proof. (1) $\Rightarrow$ (2) Suppose that $I = I_1 \times I_2 \times \cdots \times I_n$ $(n \geq 3)$ is a 2-nil primary ideal of R. Without loss of generality, assume on the contrary that I_1 and I_2 are proper ideals of R_1 and R_2, respectively. Since $(0,1,1,...,1)(1,0,1,1,...,1)(1,1,...,1) \in I$ and $(0,1,1,...,1)(1,0,1,1,...,1) \notin \sqrt{0}$, we have either $(0,1,1,...,1)(1,1,...,1) \in \sqrt{I}$ or $(1,0,1,1,...,1)(1,1,...,1) \in \sqrt{I}$. Thus, $I_1 = R_1$ or $I_2 = R_2$, a contradiction. Now, we may suppose that I_1 is proper and $I = I_1 \times R_2 \times \cdots \times R_n$. Let $ab \in I_1$. Since $(a,1,1)(b,1,1)(1,1,1) \in I$ but $(a,1,1)(b,1,1) \notin \sqrt{0}$, we have either $(a,1,1)(1,1,1) \in \sqrt{I}$ or $(b,1,1)(1,1,1) \in \sqrt{I}$ which means that I_1 is a 2-nil quasi primary ideal of R_1.
(2) $\Rightarrow$ (3) It is clear by Theorem 2.3 in [?].
(2) $\Rightarrow$ (3) Straightforward. □

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