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Finite and Infinite sums Involving Reciprocals of Products of Gibonacci Numbers

Kunle Adegoke [1,*], Robert Frontczak [2] and Karol Gryszka [3]

[1] Department of Physics and Engineering Physics, Obafemi Awolowo University, 220005 Ile-Ife, Nigeria
[2] Independent Researcher, 72764 Reutlingen, Germany
[3] Institute of Mathematics, University of the National Education Commission, Krakow, Podchorążych 2, 30-084 Kraków, Poland
* adegoke00@gmail.com

Abstract

We prove many identities involving sums with products of Gibonacci numbers in the denominator. Three of our results provide generalizations of problems published in The Fibonacci Quarterly. We also study Brousseau sums with Gibonacci entries.

Keywords: Fibonacci numbers; Gibonacci numbers; Gibonacci identities; reciprocal sums; convergent series

MSC: primary 11B39; secondary 97I30

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1. Motivation and Preliminaries

Sums and series of reciprocals of recurrence sequences is a topic of great interest in recent research articles. In [13] the authors consider a very general sums of the form

$$\sum_{n=0}^{m-1} \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}},$$

where w_n is an arbitrary second order recurrence sequence (Horadam sequence), B is some parameter related to that sequence and Δ is the difference operator. In the paper from 2023, Adegoke et al. [3] study three-parameter sums and series of the form

$$\sum_{i=1}^{N} \frac{q^{m(i-k)}}{w_{m(i-k)+n} w_{m(i+k)+n}} \quad \text{and} \quad \sum_{i=1}^{N} \frac{q^{m(2i-k)}}{w_{m(2i-k)} w_{m(2i+k)}},$$

where q is a parameter, and m, n and k are integers. Similar sums are the subject of interest in [1,9,10,14]. Extraordinary work on reciprocal sums with three and more (generalized) Fibonacci factors has been done by Melham. In a series of papers starting in 2000 he gives closed forms for many types of these sums (see [15–22]). But their history can be tracked back further to André-Jeannin [4], Melham and Shannon [23], Rabinowitz [28] and even to Hoggart and Bicknell [11], Bruckman and Good [8] and Brother Brousseau back in 1967.

Brousseau in his articles [6,7] initiated a trend that is currently named after him "Brousseau sums" and is related to finding various sums with inverses of Fibonacci-like numbers and their (weighted) products. In particular, Brousseau applied a telescoping summation method (referenced in this article as (T1)) in [7] to obtain many Fibonacci and Lucas identities, which are, among others:

$$\sum_{k=1}^{\infty} \frac{F_k}{F_{k+1}F_{k+2}} = 1, \qquad \sum_{k=1}^{\infty} \frac{F_{k+1}}{F_k F_{k+3}} = \frac{5}{4},$$

$$\sum_{k=1}^{\infty} \frac{F_{4k+3}}{F_{2k}F_{2k+1}F_{2k+2}F_{2k+3}} = \frac{1}{2}, \qquad \sum_{k=1}^{\infty} \frac{F_{2k+5}}{F_k F_{k+1}F_{k+2}F_{k+3}F_{k+4}F_{k+5}} = \frac{1}{15}.$$

For example, the first identity on the right hand side is a consequence of the following relation:

$$\frac{1}{F_n} - \frac{1}{F_{n+3}} = \frac{F_{n+3} - F_n}{F_n F_{n+3}} = \frac{2F_{n+1}}{F_n F_{n+3}},$$

and thus applying the telescoping principle we arrive at

$$\sum_{k=1}^{\infty} \frac{F_{n+1}}{F_n F_{n+3}} = \frac{1}{2}\left(\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3}\right) = \frac{5}{4}.$$

Notice that the only place where initial values matter is at the very end of the computation, therefore the reasoning can be easily generalized to any Gibonacci sequence. This idea will be heavily used in our article.

The idea for writing this article has a second source of motivation. In the section Advanced Problems and Solutions of the journal The Fibonacci Quarterly Ohtsuka [25] challenges the readers to show that

$$\sum_{k=1}^{\infty} \frac{1}{F_k F_{k+4}}\left(\frac{1}{F_{k+1}} + \frac{1}{F_{k+2}} - \frac{1}{F_{k+3}}\right) = \frac{1}{3}, \tag{1}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{F_k F_{k+4}}\left(\frac{1}{F_{k+1}} + \frac{1}{F_{k+2}} - \frac{1}{F_{k+3}}\right) = -\frac{1}{6}, \tag{2}$$

$$\sum_{k=1}^{\infty} \frac{1}{F_k F_{k+1}F_{k+2}F_{k+3}F_{k+4}}\left(\frac{1}{F_{k+1}} + \frac{1}{F_{k+2}} - \frac{1}{F_{k+3}}\right) = \frac{1}{24}. \tag{3}$$

The problem was solved by Bataille [5] (among others). A Lucas version of the problem also exists [27]:

$$\sum_{k=1}^{\infty} \frac{3}{L_k L_{k+4}}\left(\frac{1}{L_{k+1}} + \frac{1}{L_{k+2}} - \frac{1}{L_{k+3}}\right) = \frac{1}{7}, \tag{4}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k 3}{L_k L_{k+4}}\left(\frac{1}{L_{k+1}} + \frac{1}{L_{k+2}} - \frac{1}{L_{k+3}}\right) = -\frac{3}{28}, \tag{5}$$

$$\sum_{k=1}^{\infty} \frac{3}{L_k L_{k+1}L_{k+2}L_{k+3}L_{k+4}}\left(\frac{1}{L_{k+1}} + \frac{1}{L_{k+2}} - \frac{1}{L_{k+3}}\right) = \frac{1}{672}. \tag{6}$$

As it turns out, these sums can be easily generalized to the Gibonacci sequence as well and furthermore, these identities have given rise to a search for similar identities. In addition, we have found other problem proposals that allow for such generalizations.

In the next sections, we find the aforementioned identities, but we also go back to some classic problems, such as summation of

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+m}}$$

considered by André-Jeannin [4], and later generalized by Melham and Shannon [23] and Melham [18,19,22].

Throughout the article we use the standard notation for the Fibonacci numbers F_n, the Lucas numbers L_n and the Gibonacci numbers G_n. Recall that the Gibonacci sequences has the same recurrence relation as the Fibonacci sequence but starts with arbitrary initial values, i.e.,

$$G_k = G_{k-1} + G_{k-2}, \qquad (k \geq 2),$$

with G_0 and G_1 arbitrary numbers (usually integers) not both zero. When $G_0 = 0$ and $G_1 = 1$ then $G_n = F_n$, and when $G_0 = 2$ and $G_1 = 1$ then $G_n = L_n$, respectively. The sequence obeys the generalized Binet formula

$$G_n = A\alpha^n + B\beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 = x + 1$, and $A = \frac{G_1 - G_0\beta}{\alpha-\beta}$ and $B = \frac{G_0\alpha - G_1}{\alpha-\beta}$. To avoid indeterminates, we assume that $G_n \neq 0$ for all n.

We recall the following following classic telescoping summation formulas:

$$\sum_{j=1}^{n} (f_{j+1} - f_j) = f_{n+1} - f_n, \tag{T1}$$

$$\sum_{j=1}^{n} (-1)^{j-1}(f_{j+1} + f_j) = (-1)^{n+1} f_{n+1} + f_1, \tag{T2}$$

$$\sum_{j=1}^{n} d^{n-j} c^{j-1}(c f_{j+1} - d f_j) = c^n f_{n+1} - d^n f_1, \tag{T3}$$

where we assume that f_j, d and c are real numbers with $cd \neq 0$. Throughout the article, we will mainly use (T1) and a variation of (T3).

We note that the range of summation in our article is from $k = 1$. This is just a cosmetic choice and any formula obtained for the Gibonacci sequence in range from $k = 1$ can be adjusted accordingly to any other range from $k = \ell$ where ℓ is any non-negative integer.

1.1. Gibonacci identities

We conclude with a collection of Gibonacci identities which will be used later in various forms. Two classical identities for Gibonacci numbers are

$$\begin{aligned} G_{k+m} &= F_m G_{k+1} + F_{m-1} G_k, \\ G_{k-m} &= (-1)^m (F_{m+1} G_k - F_m G_{k+1}). \end{aligned} \tag{7}$$

The first identity is Equation (8) in Vajda's book [29]. Two other general identities from Vajda's book are

$$\begin{aligned} G_{k+m} + (-1)^m G_{k-m} &= L_m G_k, \\ G_{k+m} - (-1)^m G_{k-m} &= F_m (G_{k-1} + G_{k+1}), \end{aligned} \tag{8}$$

which are Equations (10a) and (10b), respectively. These can be turned into identities involving fractions:

$$\frac{(-1)^r}{G_{k+2r}} + \frac{1}{G_k} = \frac{L_r G_{k+r}}{G_k G_{k+2r}}, \tag{9}$$

$$\frac{1}{G_k} - \frac{F_{r-1}}{G_{k+r}} = \frac{F_r G_{k+1}}{G_k G_{k+r}}. \tag{10}$$

The first one is a restatement of Equation (10a) in Vajda's book, which we include for further reference:

$$G_{k+2m} + (-1)^m G_k = L_m G_{k+m}, \tag{11}$$

while the second one is a restatement of (7). We also have a special case of (8): for odd m we have

$$G_{k+2m} = F_m (G_{k+m-1} + G_{k+m+1)}) - G_k. \tag{12}$$

If we now change indices in (11) and rearrange, we get

$$G_{nk} = L_n G_{nk-n} + (-1)^{n+1} G_{nk-2n}. \tag{13}$$

An identity of Howard [12] can be written as

$$\frac{F_{m-r}}{G_{k+r}} - (-1)^r \frac{F_m}{G_k} = (-1)^{r-1} \frac{F_r G_{k+m}}{G_k G_{k+r}}. \tag{14}$$

Still another classical identity, the Catalan identity, reads as

$$G_{k-r} G_{k+r} - G_k^2 = (-1)^{k-r} (G_0 G_2 - G_1^2) F_r^2, \qquad r \geq 0,$$

of which the Cassini identity

$$G_{k-1} G_{k+1} - G_k^2 = (-1)^{k+1} (G_0 G_2 - G_1^2) \tag{15}$$

is a special case.

Finally, we note the following limit properties.

Lemma 1. *If m is a non-negative integer, then*

$$\lim_{k \to \infty} \frac{G_k}{G_{k+m}} = \frac{1}{\alpha^m} = (-1)^m \beta^m = (-1)^m (\beta F_m + F_{m-1})$$

and

$$\lim_{k \to \infty} \frac{F_k}{G_{k+m}} = \frac{1}{(G_1 - G_0 \beta) \alpha^m} = \frac{(-1)^{m-1}}{\varepsilon_G} (G_{m+1} - \alpha G_m), \tag{16}$$

where here and throughout this paper $\varepsilon_G = G_0^2 - G_1^2 + G_1 G_0$.

Proof. These results are consequences of the Binet formula. Note that in obtaining the second equality in (16), we used

$$\frac{1}{\alpha^m} = (-1)^m \beta^m = (-1)^m (\beta F_m + F_{m-1})$$

and (7). □

For convenience and to shorten the formulas, we will commonly write $G_k \cdots G_{k+n}$ instead of $G_k \cdot G_{k+1} \cdots G_{k+n}$.

2. The Generalization of (1)–(6) and Other Identities

Our first set of results is the direct generalization of the identities (1)–(6) to the Gibonacci case. Thus, we shall utilize elementary identities. A general approach or the structure of some identities will be discussed in the next section.

Theorem 1. *We have*

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4}} \left(\frac{1}{G_{k+1}} + \frac{1}{G_{k+2}} - \frac{1}{G_{k+3}} \right) = \frac{1}{G_1 G_2 G_4}. \tag{17}$$

Proof. Notice that

$$G_{k+3}(G_{k+3} + G_k) = G_{k+3} \cdot 2G_{k+2} = G_{k+2}(G_{k+1} + G_{k+4}),$$

so

$$G_{k+3}^2 - G_{k+1}G_{k+2} = G_{k+2}G_{k+4} - G_k G_{k+3} \tag{18}$$

and thus

$$\begin{aligned}
\frac{1}{G_k G_{k+4}} \left(\frac{1}{G_{k+1}} + \frac{1}{G_{k+2}} - \frac{1}{G_{k+3}} \right) &= \frac{1}{G_k G_{k+4}} \cdot \frac{G_{k+3}^2 - G_{k+1}G_{k+2}}{G_{k+1}G_{k+2}G_{k+3}} \\
&= \frac{G_{k+2}G_{k+4} - G_k G_{k+3}}{G_k \cdots G_{k+4}} \\
&= \frac{1}{G_k G_{k+1} G_{k+3}} - \frac{1}{G_{k+1}G_{k+2}G_{k+4}}.
\end{aligned}$$

Hence the desired sum telescopes and we have via (T1)

$$\lim_{n\to\infty} \left(\frac{1}{G_1 G_2 G_4} - \frac{1}{G_{n+1}G_{n+2}G_{n+4}} \right) = \frac{1}{G_1 G_2 G_4},$$

which gives (17). □

The next result is the following alternating version of (17).

Theorem 2. *We have*

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{G_k G_{k+4}} \left(\frac{1}{G_{k+1}} + \frac{1}{G_{k+2}} - \frac{1}{G_{k+3}} \right) = \frac{1}{G_1^2 - G_0 G_2} \left(\frac{G_3}{G_1 G_4} - \frac{1}{G_3} - \frac{1}{G_4} \right). \tag{19}$$

Proof. We use the computation used to prove (17) and the Cassini identity for the Gibonacci numbers (15). This implies

$$\begin{aligned}
\frac{(-1)^k}{G_kG_{k+4}}\left(\frac{1}{G_{k+1}}+\frac{1}{G_{k+2}}-\frac{1}{G_{k+3}}\right) &= \frac{(-1)^k}{G_kG_{k+1}G_{k+3}}+\frac{(-1)^{k+1}}{G_{k+1}G_{k+2}G_{k+4}}\\
&= \frac{1}{G_1^2-G_0G_2}\left(\frac{G_{k+1}G_{k+3}-G_{k+2}^2}{G_kG_{k+1}G_{k+3}}+\frac{G_{k+2}G_{k+4}-G_{k+3}^2}{G_{k+1}G_{k+2}G_{k+4}}\right)\\
&= \frac{1}{G_1^2-G_0G_2}\left(\frac{1}{G_k}-\frac{G_{k+2}^2}{G_kG_{k+1}G_{k+3}}+\frac{1}{G_{k+1}}-\frac{G_{k+3}^2}{G_{k+1}G_{k+2}G_{k+4}}\right)\\
&= \frac{1}{G_1^2-G_0G_2}\left(\frac{G_{k+2}}{G_kG_{k+1}}-\frac{G_{k+2}^2}{G_kG_{k+1}G_{k+3}}-\frac{G_{k+3}^2}{G_{k+1}G_{k+2}G_{k+4}}\right)\\
&= \frac{1}{G_1^2-G_0G_2}\left(\frac{G_{k+2}G_{k+3}-G_{k+2}^2}{G_kG_{k+1}G_{k+3}}-\frac{G_{k+3}G_{k+2}+G_{k+3}G_{k+1}}{G_{k+1}G_{k+2}G_{k+4}}\right)\\
&= \frac{1}{G_1^2-G_0G_2}\left(\frac{G_{k+2}}{G_kG_{k+3}}-\frac{G_{k+3}}{G_{k+1}G_{k+4}}-\frac{G_{k+3}}{G_{k+2}G_{k+4}}\right)\\
&= \frac{1}{G_1^2-G_0G_2}\left(\frac{G_{k+2}}{G_kG_{k+3}}-\frac{G_{k+3}}{G_{k+1}G_{k+4}}-\left(\frac{1}{G_{k+2}}-\frac{1}{G_{k+4}}\right)\right).
\end{aligned}$$

Applying the telescoping formula (T1) we complete the proof of (19). □

The final generalized sum is as follows.

Theorem 3. *We have*

$$\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+1}G_{k+2}G_{k+3}G_{k+4}}\left(\frac{1}{G_{k+1}}+\frac{1}{G_{k+2}}-\frac{1}{G_{k+3}}\right)=\frac{1}{2G_1G_2^2G_3^3G_4}. \tag{20}$$

Proof. We use (18) so that

$$\begin{aligned}
\frac{1}{G_k\cdots G_{k+4}}\left(\frac{1}{G_{k+1}}+\frac{1}{G_{k+2}}-\frac{1}{G_{k+3}}\right) &= \frac{G_{k+2}G_{k+4}-G_kG_{k+3}}{G_kG_{k+1}^2G_{k+2}^2G_{k+3}^2G_{k+4}}\\
&= \frac{1}{G_kG_{k+1}^2G_{k+2}G_{k+3}^2}-\frac{1}{G_{k+1}^2G_{k+2}^2G_{k+3}G_{k+4}}.
\end{aligned}$$

Letting

$$A(k)=\frac{1}{G_kG_{k+1}^2G_{k+2}G_{k+3}^2},\qquad B(k)=\frac{1}{G_k^2G_{k+1}^2G_{k+2}G_{k+3}}$$

we have

$$\frac{1}{G_k\cdots G_{k+4}}\left(\frac{1}{G_{k+1}}+\frac{1}{G_{k+2}}-\frac{1}{G_{k+3}}\right)=\big(A(k)-B(k)\big)+\big(B(k)-B(k+1)\big).$$

Further notice that

$$G_{k+3}^2-G_k^2=(G_{k+3}-G_k)(G_{k+3}+G_k)=4G_{k+1}G_{k+2}=-2(G_k-G_{k+3})G_{k+2},$$

therefore

$$\begin{aligned} A(k) - B(k) &= \frac{G_k - G_{k+3}}{G_k^2 G_{k+1}^2 G_{k+2} G_{k+3}^2} \\ &= -\frac{1}{2} \cdot \frac{G_{k+3}^2 - G_k^2}{(G_k \cdots G_{k+3})^2} \\ &= -\frac{1}{2}\left(\frac{1}{G_k^2 G_{k+1}^2 G_{k+2}^2} - \frac{1}{G_{k+1}^2 G_{k+2}^2 G_{k+3}^2}\right). \end{aligned}$$

Finally, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+1} G_{k+2} G_{k+3} G_{k+4}} \left(\frac{1}{G_{k+1}} + \frac{1}{G_{k+2}} - \frac{1}{G_{k+3}}\right) \\ &\quad = \lim_{n\to\infty}\left(-\frac{1}{2}\left(\frac{1}{G_1^2 G_2^2 G_3^2} - \frac{1}{G_{n+1}^2 G_{n+2}^2 G_{n+3}^2}\right) + B(1) - B(n+1)\right) \\ &\quad = \frac{1}{G_1^2 G_2^2 G_3 G_4} - \frac{1}{2G_1^2 G_2^2 G_3^2} \\ &\quad = \frac{2G_3 G_4 - G_4^2}{2(G_1 \cdots G_4)^2} = \frac{G_1 G_4}{2(G_1 \cdots G_4)^2} = \frac{1}{2G_1 G_2^2 G_3^3 G_4} \end{aligned}$$

and we get (20). □

Continuing the current trend of generalizations, we shall recall the following identities obtained by Ohtsuka in 2018, proposed as Problem H-818 in The Fibonacci Quarterly, and solved in [24]:

$$S = \sum_{k=1}^{\infty} \frac{1}{F_k F_{k+1} F_{k+2} F_{k+4}} = \frac{38 - 15\sqrt{5}}{36}, \tag{21}$$

$$T = \sum_{k=1}^{\infty} \frac{1}{F_k F_{k+2} F_{k+3} F_{k+4}} = \frac{-32 + 15\sqrt{5}}{36}. \tag{22}$$

We now evaluate the Gibonacci case of these sums.

Theorem 4. *The following summation hold:*

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+1} G_{k+2} G_{k+3}} = \frac{1}{2(G_1^2 - G_0^2 - G_0 G_1)}\left(\frac{G_3 + G_5}{G_1 G_2 G_3} - \frac{5}{G_1^2 \alpha + G_0 G_1}\right).$$

Consequently, the following identities hold:

$$S_G = \sum_{k=1}^{\infty} \frac{1}{G_k G_{k+1} G_{k+2} G_{k+4}}$$
$$= \frac{1}{6(G_1^2 - G_0^2 - G_0 G_1)} \left(\frac{G_3 + G_5}{G_1 G_2 G_3} - \frac{5}{G_1^2 \alpha + G_0 G_1} \right) + \frac{1}{3 G_1 G_2 G_3 G_4},$$
$$T_G = \sum_{k=1}^{\infty} \frac{1}{G_k G_{k+2} G_{k+3} G_{k+4}}$$
$$= -\frac{1}{6(G_1^2 - G_0^2 - G_0 G_1)} \left(\frac{G_3 + G_5}{G_1 G_2 G_3} - \frac{5}{G_1^2 \alpha + G_0 G_1} \right) + \frac{2}{3 G_1 G_2 G_3 G_4}.$$

Proof. Adding the two series gives

$$S_G + T_G = \sum_{k=1}^{\infty} \frac{G_{k+1} + G_{k+3}}{G_k \cdots G_{k+4}} = \sum_{k=1}^{\infty} \frac{G_{k+4} - G_k}{G_k \cdots G_{k+4}}$$
$$= \sum_{k=1}^{\infty} \left(\frac{1}{G_k \cdots G_{k+3}} - \frac{1}{G_{k+1} \cdots G_{k+4}} \right)$$
$$= \frac{1}{G_1 G_2 G_3 G_4}.$$

Subtracting, on the other hand, gives,

$$S_G - T_G = \sum_{k=1}^{\infty} \frac{G_{k+3} - G_{k+1}}{G_k \cdots G_{k+4}} = \sum_{k=1}^{\infty} \frac{G_{k+2}}{G_k G_{k+1} G_{k+2} G_{k+3} G_{k+4}}$$
$$= \frac{1}{3} \cdot \sum_{k=1}^{\infty} \frac{G_{k+4} + G_k}{G_k \cdots G_{k+4}}$$
$$= \frac{1}{3} \cdot \sum_{k=1}^{\infty} \left(\frac{1}{G_{k+1} \cdots G_{k+4}} + \frac{1}{G_k \cdots G_{k+3}} \right).$$

We now apply identity (23) from [19] to obtain the following finite case of our sum:

$$2(G_1^2 - G_0^2 - G_0 G_1) \cdot G_1 \cdot \sum_{k=1}^{n-1} \frac{1}{G_k G_{k+1} G_{k+2} G_{k+3}} = \frac{G_3 + G_5}{G_2 G_3} - \left(\frac{F_{n-1}}{G_n} + \frac{3F_n}{G_{n+1}} + \frac{F_{n+1}}{G_{n+2}} \right).$$

Using (16) we obtain

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+1} G_{k+2} G_{k+3}} = \frac{1}{2(G_1^2 - G_0^2 - G_0 G_1)} \left(\frac{G_3 + G_5}{G_1 G_2 G_3} - \frac{5}{G_1^2 \alpha + G_0 G_1} \right). \tag{23}$$

This implies

$$S_G - T_G = \frac{1}{3(G_1^2 - G_0^2 - G_0 G_1)} \left(\frac{G_3 + G_5}{G_1 G_2 G_3} - \frac{5}{G_1^2 \alpha + G_0 G_1} \right) - \frac{1}{3 G_1 G_2 G_3 G_4}.$$

Since $S_G + T_G$ and $S_G - T_G$ are now explicitly given, solving for S_G and T_G gives the final result. □

We note that setting $G_n = F_n$ in Theorem 4 restores (21) and (22). If we now let $G_n = L_n$, then we obtain a Lucas version of the identities:

$$S_L = \sum_{k=1}^{\infty} \frac{1}{L_k L_{k+1} L_{k+2} L_{k+4}} = \frac{115 - 42\sqrt{5}}{2520},$$

$$T_L = \sum_{k=1}^{\infty} \frac{1}{L_k L_{k+2} L_{k+3} L_{k+4}} = \frac{-85 + 42\sqrt{5}}{2520}.$$

The next three corollaries are immediate consequences of Theorem 4.

Corollary 1. *We have*

$$\sum_{k=1}^{\infty} \frac{G_{k-1}}{G_k G_{k+1} G_{k+2} G_{k+3} G_{k+4}} = -\frac{2}{3(G_1^2 - G_0^2 - G_0 G_1)}\left(\frac{G_3 + G_5}{G_1 G_2 G_3} - \frac{5}{G_1^2 \alpha + G_0 G_1}\right) + \frac{5}{3 G_1 G_2 G_3 G_4}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{F_{k-1}}{F_k F_{k+1} F_{k+2} F_{k+3} F_{k+4}} = \frac{30\sqrt{5} - 67}{18}$$

and

$$\sum_{k=1}^{\infty} \frac{L_{k-1}}{L_k L_{k+1} L_{k+2} L_{k+3} L_{k+4}} = \frac{84\sqrt{5} - 185}{1260}.$$

Proof. Calculate $3T_G - S_G$ and use

$$G_{k+3} - 3G_{k+1} = -G_{k-1}.$$

□

Corollary 2. *We have*

$$\sum_{k=1}^{\infty} \frac{G_{k-1}}{G_{k+1} G_{k+2} G_{k+3} G_{k+4} G_{k+5}} = \frac{7}{6(G_1^2 - G_0^2 - G_0 G_1)}\left(\frac{G_3 + G_5}{G_1 G_2 G_3} - \frac{5}{G_1^2 \alpha + G_0 G_1}\right) - \frac{8}{3 G_1 G_2 G_3 G_4} - \frac{G_{-1}}{G_1 G_2 G_3 G_4 G_5}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{F_{k-1}}{F_{k+1} F_{k+2} F_{k+3} F_{k+4} F_{k+5}} = \frac{1174 - 525\sqrt{5}}{180}$$

and

$$\sum_{k=1}^{\infty} \frac{L_{k-1}}{L_{k+1} L_{k+2} L_{k+3} L_{k+4} L_{k+5}} = \frac{7235 - 3234\sqrt{5}}{27720}.$$

Proof. Calculate $2S_G - 5T_G$ and use
$$2G_{k+3} - 5G_{k+1} = G_{k-2}.$$
□

Corollary 3. *We have*
$$\sum_{k=1}^{\infty} \frac{G_{k+1} + G_{k-1}}{G_k G_{k+1} G_{k+2} G_{k+3} G_{k+4}} = -\frac{5}{6(G_1^2 - G_0^2 - G_0 G_1)} \left(\frac{G_3 + G_5}{G_1 G_2 G_3} - \frac{5}{G_1^2 \alpha + G_0 G_1} \right) + \frac{7}{3G_1 G_2 G_3 G_4}.$$
In particular,
$$\sum_{k=1}^{\infty} \frac{L_k}{F_k F_{k+1} F_{k+2} F_{k+3} F_{k+4}} = \frac{75\sqrt{5} - 166}{36}$$
and
$$\sum_{k=1}^{\infty} \frac{F_k}{L_k L_{k+1} L_{k+2} L_{k+3} L_{k+4}} = \frac{6\sqrt{5} - 13}{360}.$$

Proof. Calculate $4T_G - S_G$. □

3. Some Mixed Sums of Type (1)

In this section we are interested in the following kind of sums:
$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(\frac{3}{G_{k+1}} - \frac{1}{G_{k+2}} - \frac{4}{G_{k+3}} \right) = \frac{1}{G_2 G_3 G_4 G_5},$$
$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(-\frac{1}{G_{k+1}} + \frac{2}{G_{k+2}} + \frac{3}{G_{k+3}} \right) = \frac{1}{G_1 G_3 G_4 G_5},$$
and similar ones (the two above will be justified later). Note that each term in the sum is the product of:

- the product of the inverse of certain Gibonacci numbers,
- the linear combination of the inverse of certain Gibonacci numbers,

and each Gibonacci number appears at most once in each term. In this section, we show several identities of that kind.

Theorem 5. *We have*
$$\sum_{k=1}^{\infty} \frac{1}{G_{k+1} G_{k+4}} \left(\frac{3}{G_k} - \frac{2}{G_{k+2}} - \frac{3}{G_{k+3}} \right) = \frac{2}{G_1 G_3 G_4}.$$

Proof. The proof follows the flow of the proof of Theorem 1. First, we have
$$\frac{3}{G_k} - \frac{2}{G_{k+2}} - \frac{3}{G_{k+3}} = \frac{3G_{k+2}G_{k+3} - 2G_k G_{k+3} - 3G_k G_{k+2}}{G_k G_{k+2} G_{k+3}} = \frac{2G_{k+1}G_{k+4} - 2G_k G_{k+2}}{G_k G_{k+2} G_{k+3}},$$

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where we used the identity

$$3G_{k+2}G_{k+3} - 2G_kG_{k+3} - 3G_kG_{k+2} = 2G_{k+1}G_{k+4} - 2G_kG_{k+2}.$$

This identity is justified by writing in the equivalent form

$$3G_{k+2}(G_{k+3} - G_k) = 2G_{k+1}G_{k+4} - 2G_k(G_{k+2} - G_{k+3})$$

and then noting that each of the sides equals $6G_{k+1}G_{k+2}$. Note that we have used $G_k + G_{k+4} = 3G_{k+2}$ and $G_{k+3} - G_k = 2G_{k+1}$. □

Theorem 6. *We have*

$$\sum_{k=1}^{\infty} \frac{1}{G_{k+1}G_{k+4}} \left(\frac{3}{G_k} - \frac{2}{G_{k+2}} + \frac{1}{G_{k+3}} \right) = \frac{2}{G_1G_2G_4}.$$

Proof. This time we use

$$\frac{3}{G_k} - \frac{2}{G_{k+2}} + \frac{1}{G_{k+3}} = \frac{3G_{k+2}G_{k+3} - 2G_kG_{k+3} + G_kG_{k+2}}{G_kG_{k+2}G_{k+3}}$$

and apply the identity

$$3G_{k+2}G_{k+3} - 2G_kG_{k+3} + G_kG_{k+2} = G_{k+3}G_{k+4} - G_kG_{k+1},$$

which can be verified by simple manipulation based on the recurrence relation. This implies

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{G_{k+1}G_{k+4}} \left(\frac{3}{G_k} - \frac{2}{G_{k+2}} + \frac{1}{G_{k+3}} \right) &= \sum_{k=1}^{\infty} \left(\frac{1}{G_kG_{k+1}G_{k+2}} - \frac{1}{G_{k+2}G_{k+3}G_{k+4}} \right) \\ &= \frac{1}{G_1G_2G_3} + \frac{1}{G_2G_3G_4} \\ &= \frac{G_1 + G_4}{G_1 \cdots G_4} = \frac{2G_3}{G_1 \cdots G_4} = \frac{2}{G_1G_2G_4}. \end{aligned}$$

□

We have discovered many more identities of the form described at the beginning of this section. The following set shares the same indices in the corresponding places of the left-hand side of the identity, but the coefficients differ.

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Corollary 4. *We have the following identities:*

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(\frac{6}{G_{k+1}} + \frac{3}{G_{k+2}} - \frac{3}{G_{k+3}} \right) = \frac{1}{G_1 G_2 G_3 G_4} + \frac{1}{G_2 G_3 G_4 G_5},$$
$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(\frac{3}{G_{k+1}} - \frac{1}{G_{k+2}} - \frac{4}{G_{k+3}} \right) = \frac{1}{G_2 G_3 G_4 G_5},$$
$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(-\frac{1}{G_{k+1}} + \frac{2}{G_{k+2}} + \frac{3}{G_{k+3}} \right) = \frac{1}{G_1 G_3 G_4 G_5},$$
$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(\frac{2}{G_{k+1}} + \frac{1}{G_{k+2}} - \frac{1}{G_{k+3}} \right) = \frac{1}{G_1 G_2 G_4 G_5},$$
$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(\frac{1}{G_{k+1}} + \frac{3}{G_{k+2}} + \frac{2}{G_{k+3}} \right) = \frac{1}{G_1 G_2 G_3 G_5},$$
$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(\frac{3}{G_{k+1}} + \frac{4}{G_{k+2}} + \frac{1}{G_{k+3}} \right) = \frac{1}{G_1 G_2 G_3 G_4}.$$

Proof. All identities can be verified with simple Gibonacci identities and with the recurrence relation (similarly to other identities from this section). In particular, it can be verified that

$$\frac{6}{G_{k+1}} + \frac{3}{G_{k+2}} - \frac{3}{G_{k+3}} = \frac{G_{k+4}G_{k+5} - G_k G_{k+1}}{G_{k+1}G_{k+2}G_{k+3}},$$
$$\frac{3}{G_{k+1}} - \frac{1}{G_{k+2}} - \frac{4}{G_{k+3}} = \frac{G_{k+1}G_{k+5} - G_k G_{k+1}}{G_{k+1}G_{k+2}G_{k+3}},$$
$$-\frac{1}{G_{k+1}} + \frac{2}{G_{k+2}} + \frac{3}{G_{k+3}} = \frac{G_{k+1}G_{k+5} - G_k G_{k+2}}{G_{k+1}G_{k+2}G_{k+3}},$$
$$\frac{2}{G_{k+1}} + \frac{1}{G_{k+2}} - \frac{1}{G_{k+3}} = \frac{G_{k+2}G_{k+5} - G_k G_{k+3}}{G_{k+1}G_{k+2}G_{k+3}},$$
$$\frac{1}{G_{k+1}} + \frac{3}{G_{k+2}} + \frac{2}{G_{k+3}} = \frac{G_{k+3}G_{k+5} - G_k G_{k+4}}{G_{k+1}G_{k+2}G_{k+3}},$$
$$\frac{3}{G_{k+1}} + \frac{4}{G_{k+2}} + \frac{1}{G_{k+3}} = \frac{G_{k+4}G_{k+5} - G_k G_{k+5}}{G_{k+1}G_{k+2}G_{k+3}}.$$

Then, for example,

$$\frac{1}{G_k G_{k+4} G_{k+5}} \cdot \frac{G_{k+4}G_{k+5} - G_k G_{k+1}}{G_{k+1}G_{k+2}G_{k+3}} = \frac{1}{G_k \cdots G_{k+3}} - \frac{1}{G_{k+2} \cdots G_{k+5}}$$

from which the first identity follows. □

Remark 1. *We note that in Corollary 4 the second and the sixth generate all of the cases. This is because*

$$span\{(6,3,-3),(3,-1,-4),(-1,2,3),(2,1,-1),(1,3,2),(3,4,1)\} = span\{(3,-1,-4),(3,4,1)\}.$$

Despite our best efforts, we were unable to find the exceptional identity with (rational) coefficients A, B and C, that is, the identity which left-hand side is

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+4} G_{k+5}} \left(\frac{A}{G_{k+1}} + \frac{B}{G_{k+2}} + \frac{C}{G_{k+3}} \right),$$

such that

$$span\{(3, -1, -4), (3, 4, 1), (A, B, C)\} = \mathbb{R}^3.$$

This would imply the closed form for all identities with arbitrary coefficients A, B and C.

4. Miscellaneous Identities

Theorem 7. *We have*

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+2} G_{k+3}} = \frac{1}{2G_1 G_2 G_3}. \tag{24}$$

Proof. Use

$$\frac{1}{G_k G_{k+2} G_{k+3}} = \frac{1}{2} \frac{G_{k+3} - G_k}{G_k G_{k+1} G_{k+2} G_{k+3}}$$

from which (T1) can be applied. □

Theorem 8. *For all $m \geq 0$ we have*

$$\sum_{k=1}^{\infty} \frac{1}{G_k \cdots G_{k+2m} G_{k+2m+2} \cdots G_{k+4m+2}} = \frac{1}{L_{2m+1} G_1 \cdots G_{4m+2}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+2}} = \frac{1}{G_1 G_2}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+1} G_{k+2} G_{k+4} G_{k+5} G_{k+6}} = \frac{1}{4G_1 \cdots G_6}.$$

Proof. We use (11). Take an odd m and make the replacement $m \to 2m+1$ in (11) to get

$$L_{2m+1} G_{k+2m+1} = G_{k+4m+2} - G_k.$$

Applying (T1) gives for a fixed $n > 1$:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{L_{2m+1} G_{k+2m+1}}{G_k \cdots G_{k+2m} G_{k+2m+1} G_{k+2m+2} \cdots G_{k+4m+2}} &= \sum_{k=1}^{n} \frac{G_{k+4m+2} - G_k}{G_k \cdots G_{k+4m+2}} \\
&= \sum_{k=1}^{n} \left(\frac{1}{G_k \cdots G_{k+4m+1}} - \frac{1}{G_{k+1} \cdots G_{k+4m+2}} \right) \\
&= \frac{1}{G_1 \cdots G_{4m+2}} - \frac{1}{G_{n+1} \cdots G_{n+4m+2}}.
\end{aligned}$$

□

The next theorem showcases a simple example of an infinite family of identities with similar structure.

Theorem 9. *We have*

$$\sum_{k=1}^{\infty} \frac{1}{G_{k+2}\cdots G_{k+m}}\left(\frac{F_m}{G_k}+\frac{F_{m-1}-1}{G_{k+1}}\right)=\frac{1}{G_1\cdots G_m}. \tag{25}$$

Proof. Use

$$\frac{F_m}{G_k}+\frac{F_{m-1}-1}{G_{k+1}}=\frac{G_{k+1}F_m-G_k+F_{m-1}G_k}{G_kG_{k+1}}=\frac{G_{k+m}-G_k}{G_kG_{k+1}}$$

and apply (T1). □

The the next relation is an alternating version of (25) in case $m=4$.

Theorem 10. *We have*

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{G_{k+2}G_{k+3}G_{k+4}}\left(\frac{3}{G_k}+\frac{1}{G_{k+1}}\right)=\frac{1}{G_1^2-G_0G_2}\left(\frac{1}{G_1G_2}+\frac{1}{G_1G_4}-\frac{1}{G_3G_4}-\frac{1}{2G_2G_3}\right)-\frac{1}{2(G_1^2-G_0G_2)}\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+1}}.$$

Proof. Start with

$$\frac{3}{G_k}+\frac{1}{G_{k+1}}=\frac{3G_{k+1}+G_k}{G_kG_{k+1}}=\frac{G_{k+4}-G_k}{G_kG_{k+1}}$$

and

$$\frac{(-1)^k}{G_{k+2}G_{k+3}G_{k+4}}\left(\frac{3}{G_k}+\frac{1}{G_{k+1}}\right)=\frac{(-1)^k}{G_k\cdots G_{k+3}}+\frac{(-1)^{k+1}}{G_{k+1}\cdots G_{k+4}}.$$

We now apply the Cassini identity to have:

$$\begin{aligned}
&\frac{(-1)^k}{G_k\cdots G_{k+3}}+\frac{(-1)^{k+1}}{G_{k+1}\cdots G_{k+4}}\\
&=\frac{1}{G_1^2-G_0G_2}\left(\frac{G_{k+1}G_{k+3}-G_{k+2}^2}{G_k\cdots G_{k+3}}+\frac{G_{k+2}G_{k+4}-G_{k+3}^2}{G_{k+1}\cdots G_{k+4}}\right)\\
&=\frac{1}{G_1^2-G_0G_2}\left(\frac{1}{G_kG_{k+2}}+\frac{G_{k+1}-G_{k+3}}{G_kG_{k+1}G_{k+3}}+\frac{1}{G_{k+1}G_{k+3}}-\frac{G_{k+1}+G_{k+2}}{G_{k+1}G_{k+2}G_{k+4}}\right)\\
&=\frac{1}{G_1^2-G_0G_2}\left(\frac{1}{G_kG_{k+2}}+\frac{1}{G_kG_{k+3}}-\frac{1}{G_kG_{k+1}}+\frac{1}{G_kG_{k+3}}-\frac{1}{G_{k+2}G_{k+4}}-\frac{1}{G_{k+1}G_{k+4}}\right).
\end{aligned}$$

From that we get, using (T1),

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{G_{k+2}G_{k+3}G_{k+4}}\left(\frac{3}{G_k}+\frac{1}{G_{k+1}}\right)=\frac{1}{G_1^2-G_0G_2}\left(\frac{1}{G_1G_2}+\frac{1}{G_1G_4}-\frac{1}{G_3G_4}-\frac{1}{2G_2G_3}\right)-\frac{1}{2(G_1^2-G_0G_2)}\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+1}},$$

as

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+2}} = \frac{1}{G_1 G_2}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{G_k G_{k+3}} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{G_k G_{k+1}} - \frac{1}{2} \frac{1}{G_2 G_3}.$$

□

One of the original Brousseau sums is [7, Identity (9)]:

$$\sum_{k=1}^{\infty} \frac{F_{k+3}}{F_k F_{k+2} F_{k+4} F_{k+6}} = \frac{17}{480}. \tag{26}$$

This sum is a special case of the following sum with the structure similar to the previous sums from this section.

Theorem 11. *Let p, q, m be odd positive integers. Then*

$$\sum_{k=1}^{\infty} \frac{G_{km+pqm}}{G_{km} G_{km+2qm} \cdots G_{km+2pqm}} = \frac{1}{L_{pqm}} \sum_{j=1}^{2q} \frac{1}{G_{jm} G_{jm+2qm} \cdots G_{jm+2(p-1)qm}}.$$

Proof. Write (with the use of (11))

$$\frac{L_{pqm} G_{km+pqm}}{G_{km} G_{km+2qm} \cdots G_{km+2pqm}} = \frac{G_{km+2pqm} - G_{km}}{G_{km} G_{km+2qm} \cdots G_{km+2pqm}},$$

simplify and apply (T1). Note that the first term in the telescoping sum is

$$\frac{1}{G_m G_{m+2qm} \cdots G_{m+2(p-1)qm}} - \frac{1}{G_{m+2qm} G_{m+3qm} \cdots G_{m+2pqm}},$$

which justifies the sum obtained in the final form. □

If we now set $m = q = 1$ and $p = 3$ in Theorem 11, we restore (26).

Theorem 12. *Let $m > 0$ be odd. Then we have*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{G_{mk} G_{m(k+3)}} \left(\frac{G_{m(k+1)}}{G_{m(k+2)}} + \frac{G_{m(k+2)}}{G_{m(k+1)}} \right) = \frac{(-1)^{n+1}}{G_{m(n+1)} G_{m(n+3)}} + \frac{1}{G_m G_{3m}}$$

and in the limiting case we also have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{G_{mk} G_{m(k+3)}} \left(\frac{G_{m(k+1)}}{G_{m(k+2)}} + \frac{G_{m(k+2)}}{G_{m(k+1)}} \right) = \frac{1}{G_m G_{3m}}.$$

Proof. The identity follows from

$$\frac{1}{G_{mk}G_{m(k+3)}}\left(\frac{G_{m(k+1)}}{G_{m(k+2)}}+\frac{G_{m(k+2)}}{G_{m(k+1)}}\right)=\frac{G^2_{m(k+1)}+G^2_{m(k+2)}}{G_{mk}G_{m(k+1)}G_{m(k+2)}G_{m(k+3)}}$$

and further noting that, with the aid of Identity (13),

$$\begin{aligned}G^2_{m(k+1)}+G^2_{m(k+2)}&=(-1)^{m+1}G^2_{m(k+1)}+G_{m(k+2)}(L_mG_{m(k+1)}+G_{mk})\\&=G_{m(k+1)}(L_mG_{m(k+2)}+(-1)^{m+1}G_{m(k+1)})+G_{mk}G_{m(k+2)}\\&=G_{m(k+1)}G_{m(k+3)}+G_{mk}G_{m(k+2)}.\end{aligned}$$

We can now apply (T2) with $f_k=\frac{1}{G_{mk}G_{m(k+2)}}$. □

5. Series with Higher Powers

This part of our article is inspired by another problem proposal by Ohtsuka from 2024. In Problem H-938 [26] the readers are asked to prove the following relation:

$$\sum_{k=1}^{\infty}\frac{1}{F_k^2F_{k+1}^2}=\sum_{k=1}^{\infty}\frac{1}{F_k^2F_{k+2}^2}+\frac{13-5\sqrt{5}}{2}. \tag{27}$$

First, we derive a Gibonacci generalization of this statement. Second, we study similar series those closed forms involve squares of certain Gibonacci numbers.

Theorem 13. *We have*

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{G_k^2G_{k+1}^2}&=\sum_{k=1}^{\infty}\frac{1}{G_k^2G_{k+2}^2}+\frac{G_3^2-G_2^2-G_2}{G_1G_2^2G_3^2}\\&\quad+\frac{1}{G_1^2-G_0^2-G_0G_1}\left(\frac{G_3+G_5}{G_1G_2G_3}-\frac{5}{G_1^2\alpha+G_0G_1}\right).\end{aligned} \tag{28}$$

Proof. Let Q_G denote the difference of the sums in question, i.e.,

$$Q_G=\sum_{k=1}^{\infty}\left(\frac{1}{G_k^2G_{k+1}^2}-\frac{1}{G_k^2G_{k+2}^2}\right).$$

We have

$$\begin{aligned}Q_G&=\sum_{k=1}^{\infty}\frac{G_{k+2}^2-G_{k+1}^2}{G_k^2G_{k+1}^2G_{k+2}^2}\\&=\sum_{k=1}^{\infty}\frac{G_kG_{k+1}+G_kG_{k+2}}{G_k^2G_{k+1}^2G_{k+2}^2}\\&=\sum_{k=1}^{\infty}\left(\frac{1}{G_kG_{k+1}G_{k+2}^2}+\frac{1}{G_kG_{k+1}^2G_{k+2}}\right).\end{aligned}$$

Now, as

$$\frac{1}{G_kG_{k+1}G_{k+2}^2}=\frac{G_{k+3}}{G_kG_{k+1}G_{k+2}^2G_{k+3}}=\frac{1}{G_kG_{k+2}^2G_{k+3}}+\frac{1}{G_kG_{k+1}G_{k+2}G_{k+3}}$$

and

$$\frac{1}{G_kG_{k+1}^2G_{k+2}}=\frac{G_{k+1}-G_k}{G_{k-1}G_kG_{k+1}^2G_{k+2}}=\frac{1}{G_{k-1}G_kG_{k+1}G_{k+2}}-\frac{1}{G_{k-1}G_{k+1}^2G_{k+2}},$$

we can write

$$\begin{aligned}Q_G=\frac{1}{G_1^2G_2^2}-\frac{1}{G_1^2G_3^2}+\sum_{k=2}^{\infty}\left(\frac{1}{G_kG_{k+2}^2G_{k+3}}-\frac{1}{G_{k-1}G_{k+1}^2G_{k+2}}\right)\\+\sum_{k=2}^{\infty}\left(\frac{1}{G_kG_{k+1}G_{k+2}G_{k+3}}+\frac{1}{G_{k-1}G_kG_{k+1}G_{k+2}}\right).\end{aligned}$$

We apply the telescoping principle (T1) and end up with

$$Q_G=\frac{G_3^2-G_2^2}{G_1G_2^2G_3^2}-\frac{1}{G_1G_3^2G_4}-\frac{1}{G_1G_2G_3G_4}+2\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+1}G_{k+2}G_{k+3}}.$$

The statement follows from (23) and some basic simplifications. □

When $G_n=F_n$ then we get (27) from (28). When $G_n=L_n$ then (28) yields

$$\sum_{k=1}^{\infty}\frac{1}{L_k^2L_{k+1}^2}=\sum_{k=1}^{\infty}\frac{1}{L_k^2L_{k+2}^2}+\frac{25-9\sqrt{5}}{90}.$$

Theorem 14. *We have for all $m,q>0$:*

$$\sum_{k=1}^{\infty}\frac{1}{G_{km}G_{km+qm}G_{km+2qm}}\left(\frac{1}{G_{km}}+\frac{(-1)^{m+1}}{G_{km+2qm}}\right)=\frac{1}{L_{qm}}\sum_{j=1}^{q}\frac{1}{G_{jm}^2G_{jm+qm}^2}. \tag{29}$$

In particular,

$$\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+1}G_{k+2}}\left(\frac{1}{G_k}+\frac{1}{G_{k+2}}\right)=\frac{1}{G_1^2G_2^2} \tag{30}$$

and

$$\sum_{k=1}^{\infty}\frac{1}{G_{2k}G_{2(k+1)}G_{2(k+2)}}\left(\frac{1}{G_{2k}}-\frac{1}{G_{2(k+2)}}\right)=\frac{1}{3G_2^2G_4^2}.$$

Proof. Notice that by (11):

$$\begin{aligned}\frac{1}{G_{km}G_{km+qm}G_{km+2qm}}\left(\frac{1}{G_{km}}+\frac{(-1)^{m+1}}{G_{km+2qm}}\right) &= \frac{(G_{km+2qm}+(-1)^{m+1}G_{km})G_{km+qm}}{G_{km}^2G_{km+qm}^2G_{km+2qm}^2}\\ &= \frac{(G_{km+2qm}+(-1)^{m+1}G_{km})(G_{km+2qm}+(-1)^{m}G_{km})}{L_{qm}G_{km}^2G_{km+qm}^2G_{km+2qm}^2}\\ &= \frac{G_{km+2qm}^2-G_{km}^2}{L_{qm}G_{km}^2G_{km+qm}^2G_{km+2qm}^2}\\ &= \frac{1}{L_{qm}}\left(\frac{1}{G_{km}^2G_{km+qm}^2}-\frac{1}{G_{km+qm}^2G_{km+2qm}^2}\right).\end{aligned}$$

Use (T1). □

Theorem 15. *We have*

$$\sum_{k=1}^{\infty}\frac{1}{G_k^2G_{k+1}\cdots G_{k+3}G_{k+4}^2}\left(\frac{1}{G_{k+1}}+\frac{1}{G_{k+3}}\right)=\frac{1}{3(G_1\cdots G_4)^2}. \tag{31}$$

Proof. Use $G_{k+4}+G_k=3G_{k+2}$ and $G_{k+4}-G_k=G_{k+1}+G_{k+3}$ to obtain

$$\begin{aligned}\frac{1}{G_k^2G_{k+1}\cdots G_{k+3}G_{k+4}^2}\left(\frac{1}{G_{k+1}}+\frac{1}{G_{k+3}}\right) &= \frac{G_{k+1}G_{k+2}+G_{k+2}G_{k+3}}{(G_k\cdots G_{k+4})^2}\\ &= \frac{G_{k+4}^2-G_k^2}{3(G_k\cdots G_{k+4})^2}.\end{aligned}$$

□

A more general result, similar to (31) can also be obtained.

Theorem 16. *For odd $m>0$ we have*

$$\sum_{k=1}^{\infty}\frac{1}{G_k^2\cdots G_{k+m-2}^2G_{k+m-1}G_{k+m}G_{k+m+1}G_{k+m+2}^2\cdots G_{k+2m}^2}\left(\frac{1}{G_{k+m-1}}+\frac{1}{G_{k+m+1}}\right) = \frac{1}{F_{2m}(G_1\cdots G_{2m})^2}, \tag{32}$$

where the case $m=1$ is understood as

$$\sum_{k=1}^{+\infty}\frac{1}{G_kG_{k+1}G_{k+2}}\left(\frac{1}{G_k}+\frac{1}{G_{k+2}}\right)=\frac{1}{G_1^1G_2^2}. \tag{33}$$

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Proof. Rewrite

$$\begin{aligned}
&\frac{1}{G_k^2 \cdots G_{k+m-2}^2 G_{k+m-1} G_{k+m} G_{k+m+1} G_{k+m+2}^2 \cdots G_{k+2m}^2}\left(\frac{1}{G_{k+m-1}}+\frac{1}{G_{k+m+1}}\right)\\
&\qquad = \frac{G_{k+m}(G_{k+m-1}+G_{k+m+1})}{(G_k\cdots G_{k+2m})^2}\\
&\qquad = \frac{1}{F_m}\frac{1}{L_m}\frac{(G_{k+2m}-G_k)(G_{k+2m}+G_k)}{(G_k\cdots G_{k+2m})^2}\\
&\qquad = \frac{G_{k+2m}^2-G_k^2}{F_{2m}(G_k\cdots G_{k+2m})^2}.
\end{aligned}$$

Note that we have used $G_{k+2m} - G_k = L_m G_{k+m}$ (cf. Identity (11)) as well as $G_{k+2m} + G_k = F_m(G_{k+m-1} + G_{k+m+1})$ for odd m (cf. Identity (12)). □

When $G_n = F_n$ or $G_n = L_n$, then we get from (33) the sums

$$\sum_{k=1}^{+\infty}\frac{1}{F_kF_{k+1}F_{k+2}}\left(\frac{1}{F_k}+\frac{1}{F_{k+2}}\right)=1,\qquad \sum_{k=1}^{+\infty}\frac{1}{L_kL_{k+1}L_{k+2}}\left(\frac{1}{L_k}+\frac{1}{L_{k+2}}\right)=\frac{1}{4}$$

and we get from (32) the sum

$$\sum_{k=1}^{+\infty}\frac{1}{F_k^2F_{k+1}^2F_{k+2}F_{k+3}F_{k+4}F_{k+5}^2F_{k+6}^2}\left(\frac{1}{F_{k+2}}+\frac{1}{F_{k+4}}\right)=\frac{1}{460\,800}.$$

Theorem 17. *We have*

$$\sum_{k=1}^{\infty}\frac{1}{G_k\cdots G_{k+3}}\left(\frac{G_{k+3}}{G_{k+2}}+\frac{G_k}{G_{k+1}}\right)=\frac{1}{G_1G_2^2G_3}.$$

Proof. We calculate

$$\begin{aligned}
\frac{1}{G_k\cdots G_{k+3}}\left(\frac{G_{k+3}}{G_{k+2}}+\frac{G_k}{G_{k+1}}\right) &= \frac{G_{k+1}G_{k+3}+G_kG_{k+2}}{G_kG_{k+1}^2G_{k+2}^2G_{k+3}}\\
&= \frac{1}{G_kG_{k+1}G_{k+2}^2}+\frac{1}{G_{k+1}^2G_{k+2}G_{k+3}}.
\end{aligned}$$

Set

$$A(k)=\frac{1}{G_kG_{k+1}G_{k+2}^2},\qquad B(k)=\frac{1}{G_k^2G_{k+1}G_{k+2}}.$$

Then,

$$\begin{aligned}
\frac{1}{G_k\cdots G_{k+3}}\left(\frac{G_{k+3}}{G_{k+2}}+\frac{G_k}{G_{k+1}}\right) &= A(k)+B(k+1)\\
&= \big(A(k)+B(k)\big)+\big(B(k+1)-B(k)\big).
\end{aligned}$$

Thus, using (30) (where the series involving $A(k)+B(k)$ is computed) we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{G_k\cdots G_{k+3}}\left(\frac{G_{k+3}}{G_{k+2}}+\frac{G_k}{G_{k+1}}\right)&=\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+1}G_{k+2}}\left(\frac{1}{G_k}+\frac{1}{G_{k+2}}\right)+\sum_{k=1}^{\infty}(B(k+1)-B(k))\\&=\frac{1}{G_1^2G_2^2}-B(1)=\frac{1}{G_1^2G_2^2}-\frac{1}{G_1^2G_2G_3}\\&=\frac{1}{G_1G_2^2G_3}.\end{aligned}$$

The proof is completed. □

This is a generalization of the previous result.

Theorem 18. *We have for all odd $m>0$*

$$\sum_{k=1}^{\infty}\frac{1}{G_{km}G_{km+qm}G_{km+2qm}G_{km+3qm}}\left(\frac{G_{km+3qm}}{G_{km+2qm}}+\frac{(-1)^{m+1}G_{km}}{G_{km+qm}}\right)=\frac{1}{L_{qm}}\sum_{k=1}^{q}\frac{1}{G_{jm}G_{jm+qm}^2G_{jm+2qm}} \tag{34}$$

and for all even $m>0$

$$\sum_{k=1}^{\infty}\frac{1}{G_{km}G_{km+qm}G_{km+2qm}G_{km+3qm}}\left(\frac{G_{km+3qm}}{G_{km+2qm}}+\frac{(-1)^{m+1}G_{km}}{G_{km+qm}}\right)=\frac{1}{L_{qm}}\sum_{j=1}^{q}\frac{G_{jm+2qm}+L_{qm}G_{jm+qm}}{G_{jm}^2G_{jm+qm}^2G_{jm+2qm}}. \tag{35}$$

Proof. Calculate

$$\begin{aligned}&\frac{1}{G_{km}G_{km+qm}G_{km+2qm}G_{km+3qm}}\left(\frac{G_{km+3qm}}{G_{km+2qm}}+\frac{(-1)^{m+1}G_{km}}{G_{km+qm}}\right)\\&=\frac{G_{km+qm}G_{km+3qm}+(-1)^{m+1}G_{km+2qm}G_{km}}{G_{km}G_{km+qm}^2G_{km+2qm}^2G_{km+3qm}}\\&=\frac{1}{G_{km}G_{km+qm}G_{km+2qm}^2}+\frac{(-1)^{m+1}}{G_{km+qm}^2G_{km+2qm}G_{km+3qm}}.\end{aligned}$$

Let

$$A(k)=\frac{1}{G_{km}G_{km+qm}G_{km+2qm}^2},\qquad B(k)=\frac{(-1)^{m+1}}{G_{km}^2G_{km+qm}G_{km+2qm}}.$$

Then each term of the desired sum equals

$$A(k)+B(k+q)=(A(k)+B(k))+(B(k+q)-B(k)).$$

We now use (29) to compute $\sum_{k=1}^{\infty}(A(k)+B(k))$ and we notice that

$$\sum_{k=1}^{\infty}(B(k+q)-B(k)) = -\sum_{j=1}^{q}B(j),$$

thus the desired sum equal, with the aid of (11),

$$\frac{1}{L_{qm}}\sum_{k=1}^{q}\frac{1}{G_{jm}^2 G_{jm+qm}^2} - \sum_{j=1}^{q}\frac{(-1)^{m+1}}{G_{jm}^2 G_{jm+qm}G_{jm+2qm}} = \frac{1}{L_{qm}}\sum_{j=1}^{q}\frac{G_{jm+2qm}+(-1)^m L_{qm}G_{jm+qm}}{G_{jm}^2 G_{jm+qm}^2 G_{jm+2qm}}.$$

If m is odd, then the numerator simplifies to G_{jm} via (11), which concludes (34). Otherwise, for even m we have (35). □

6. A New Look at Brousseau Sums

Particular cases of the sums discussed in this section appeared in [4, Corollary 1], but the final form is different than ours.

Theorem 19. *If m and q are positive integers, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{mk}}{G_{mk}G_{mk+mq}} = \frac{1}{\varepsilon_G F_{mq}}\left(q\alpha - \sum_{k=1}^{q}\frac{G_{mk+1}}{G_{mk}}\right). \tag{36}$$

Proof. It is known that [2]

$$\sum_{k=1}^{n}\frac{(-1)^{mk}}{G_{mk}G_{mk+mq}} = \frac{F_{mn}}{F_{mq}}\sum_{k=1}^{q}\frac{(-1)^{mk}}{G_{mk}G_{mk+mn}}, \quad m,q\in\mathbb{Z}^+. \tag{37}$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{(-1)^{mk}}{G_{mk}G_{mk+mq}} = \frac{1}{F_{mq}}\sum_{k=1}^{q}\frac{(-1)^{mk}}{G_{mk}}\lim_{n\to\infty}\frac{F_{mn}}{G_{mk+mn}},$$

from which (36) follows on account of (16). □

Theorem 20. *If m and q are positive integers, then*

$$\sum_{k=1}^{\infty}\frac{1}{G_{mk}G_{mk+2mq}} = \frac{1}{\varepsilon_G F_{2mq}}\sum_{k=1}^{2q}(-1)^{mk-1}\frac{G_{mk+1}}{G_{mk}}$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^{k(m-1)}}{G_{mk}G_{mk+2mq}} = \frac{1}{\varepsilon_G F_{2mq}}\sum_{k=1}^{2q}(-1)^{k-1}\frac{G_{mk+1}}{G_{mk}}.$$

Proof. Similar to the proof of Theorem 19. We use [2]:

$$\sum_{k=1}^{2n}\frac{(\pm 1)^{k(m-1)}}{G_{mk}G_{mk+2mq}} = \frac{F_{2mn}}{F_{2mq}}\sum_{k=1}^{2q}\frac{(\pm 1)^{k(m-1)}}{G_{mk}G_{mk+2mn}}. \tag{38}$$

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□

Theorem 21. *If m and n are non-negative integers, then*

$$\sum_{k=1}^{2n}\frac{1}{G_kG_{k+2m}}=\frac{F_{2n}}{F_{2m}}\sum_{k=1}^{2m}\frac{1}{G_kG_{k+2n}} \tag{39}$$

and

$$\sum_{k=1}^{2n}\frac{1}{G_kG_{k+2m+1}}=\frac{1}{F_{2m+1}}\sum_{k=1}^{2n}\frac{1}{G_kG_{k+1}}-\frac{F_{2n}}{F_{2m+1}}\sum_{k=1}^{2m}\frac{1}{G_kG_{k+2n}}-\frac{F_{2m}}{F_{2m+1}G_{2n+1}G_{2n+2m+1}}+\frac{F_{2m}}{F_{2m+1}G_1G_{2m+1}}. \tag{40}$$

Proof. Set $m=1$ in (38) and write m for q to obtain (39).

Write $2m+1$ for r in (10):

$$\frac{1}{G_k}-\frac{F_{2m}}{G_{k+2m+1}}=\frac{F_{2m+1}G_{k+1}}{G_kG_{k+2m+1}},$$

and arrange as

$$\frac{F_{2m+1}}{G_kG_{k+2m+1}}=\frac{1}{G_kG_{k+1}}-\frac{F_{2m}}{G_{k+1}G_{k+2m+1}}.$$

Now sum, to obtain, after a shift of summation index,

$$\sum_{k=1}^{n}\frac{1}{G_kG_{k+2m+1}}=\frac{1}{F_{2m+1}}\sum_{k=1}^{n}\frac{1}{G_kG_{k+1}}-\frac{F_{2m}}{F_{2m+1}}\sum_{k=1}^{n}\frac{1}{G_kG_{k+2m}}-\frac{F_{2m}}{F_{2m+1}G_{n+1}G_{n+2m+1}}+\frac{F_{2m}}{F_{2m+1}G_1G_{2m+1}}.$$

Write $2n$ for n and use (39) to re-write the second term on the right hand side; this gives (40). □

As a corollary, we obtain the infinite series version of the above theorem.

Corollary 5. *If m is a non-negative integer, then*

$$\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+2m}}=\frac{1}{\varepsilon_GF_{2m}}\sum_{k=1}^{2m}(-1)^{k-1}\frac{G_{k+1}}{G_k}$$

and

$$\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+2m+1}}=\frac{1}{F_{2m+1}}\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+1}}-\frac{1}{\varepsilon_GF_{2m+1}}\sum_{k=1}^{2m}(-1)^{k-1}\frac{G_{k+1}}{G_k}+\frac{F_{2m}}{F_{2m+1}G_1G_{2m+1}}.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+2}}=\frac{1}{G_1G_2}$$

and

$$\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+3}}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{G_kG_{k+1}}-\frac{1}{2}\frac{1}{G_2G_3}.$$

Theorem 22. *If p is a positive integer and m and r are non-negative integers such that m is greater than r, then,*

$$\sum_{k=1}^{n} \frac{(-1)^{pk}}{G_{pk+pr}G_{pk+pm}} = (-1)^{pr}\frac{F_{pn}}{F_{pm-pr}}\sum_{k=r+1}^{m} \frac{(-1)^{pk}}{G_{pk}G_{pk+pn}}.$$

In particular,

$$\sum_{k=1}^{n} \frac{(-1)^{k}}{G_{k+r}G_{k+m}} = (-1)^{r}\frac{F_{n}}{F_{m-r}}\sum_{k=r+1}^{m} \frac{(-1)^{k}}{G_{k}G_{k+n}}.$$

Proof. Making the replacements $k \to pk$, $m \to pm$ and $r \to pr$ in (14), we have

$$\frac{F_{pm-pr}}{G_{pk+pr}G_{pk+pm}} = (-1)^{pr}\frac{F_{pm}}{G_{pk}G_{pk+pm}} - (-1)^{pr}\frac{F_{pr}}{G_{pk}G_{pk+pr}}.$$

Now multiply each term by $(-1)^{pk}$ and sum from $k=1$ to $k=n$, making use of (37). $\square$

Corollary 6. *If p is a positive integer and m and r are non-negative integers such that m is greater than r, then,*

$$\sum_{k=1}^{\infty} \frac{(-1)^{pk}}{G_{pk+pr}G_{pk+pm}} = \frac{(-1)^{pr}}{\varepsilon_G F_{pm-pr}}\left((m-r)\alpha - \sum_{k=r+1}^{m} \frac{G_{pk+1}}{G_{pk}}\right).$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k}}{G_{k+r}G_{k+m}} = \frac{(-1)^{r}}{\varepsilon_G F_{m-r}}\left((m-r)\alpha - \sum_{k=r+1}^{m} \frac{G_{k+1}}{G_{k}}\right).$$

Remark 2. *It should be noted that our results cannot be compared tothe sums obtained by Farhi in [9]. Our results concern arbitrary Gibonacci sequences, whereas his results are related to Lucas sequences of the first and second kinds.*

7. Sums with Products of Gibonacci Numbers in the Denominator

The identity (T3) can be generalized to the following, which can be easily proven, for example, by direct computation.

Lemma 2. *If $g(k) = (g_k)_{k\in\mathbb{Z}^+}$ is a sequence of complex numbers, n and r are integers and c and d are any numbers, then*

$$\begin{aligned}&\sum_{k=1}^{n} d^{\,n-k} c^{\,k-1} g_{k+1}g_{k+2}\cdots g_{k+r-1}(c\,g_{k+r} - d\,g_k) \\ &\quad = c^{\,n} g_{n+1}g_{n+2}\cdots g_{n+r} - d^{\,n} g_1 g_2\cdots g_r.\end{aligned} \tag{41}$$

The Lemma turns out to be extremely useful in finding a variety of identities of our interest. The remainder of this section showcases many of its applications

Theorem 23. *If r is a positive integer, then*

$$\sum_{k=1}^{n} \frac{(-1)^{k(r+1)}}{G_k G_{k+1}\cdots G_{k+r-1}G_{k+r+1}\cdots G_{k+2r}} = \frac{(-1)^{rn+n+r}}{L_r G_{n+1}G_{n+2}\cdots G_{n+2r}} - \frac{(-1)^{r}}{L_r G_1 G_2\cdots G_{2r}}. \tag{42}$$

In particular,

$$\sum_{k=1}^{n}\frac{(-1)^k}{G_kG_{k+1}G_{k+3}G_{k+4}}=\frac{(-1)^n}{3\,G_{n+1}G_{n+2}G_{n+3}G_{n+4}}-\frac{1}{3\,G_1G_2G_3G_4}$$

and

$$\sum_{k=1}^{n}\frac{1}{G_kG_{k+1}G_{k+2}G_{k+4}G_{k+5}G_{k+6}}=\frac{1}{4\,G_1G_2G_3G_4G_5G_6}-\frac{1}{4\,G_{n+1}G_{n+2}G_{n+3}G_{n+4}G_{k+5}G_{k+6}}.$$

Proof. With (9) in mind, identity (42) follows upon use of $g(k)=1/G_k$, $c=(-1)^r$ and $d=-1$ in (41). □

Corollary 7. *If r is a positive integer, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k(r+1)}}{G_kG_{k+1}\cdots G_{k+r-1}G_{k+r+1}\cdots G_{k+2r}}=\frac{(-1)^{r-1}}{L_rG_1G_2\cdots G_{2r}}.$$

Theorem 24. *If m and r are positive integers such that $r>m$, then*

$$\sum_{k=1}^{n}\frac{(-1)^{k(m+1)}(F_{r-m}/F_m)^k}{G_kG_{k+1}\cdots G_{k+m-1}G_{k+m+1}\cdots G_{k+r-1}G_{k+r}}=\frac{(-1)^{nm+n+m}(F_{r-m}/F_m)^nF_{r-m}}{F_rG_{n+1}G_{n+2}\cdots G_{n+r}}-\frac{(-1)^mF_{r-m}}{F_rG_1G_2\cdots G_r}. \tag{43}$$

In particular, at $m=2$, $r=3$, we find

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{G_kG_{k+1}G_{k+3}}=\frac{(-1)^{n-1}}{2G_{n+1}G_{n+2}G_{n+3}}+\frac{1}{2G_1G_2G_3}$$

and at $m=2$, $r=4$ we get

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{G_kG_{k+1}G_{k+3}G_{k+4}}=\frac{(-1)^{n-1}}{3G_{n+1}G_{n+2}G_{n+3}G_{n+4}}+\frac{1}{3G_1G_2G_3G_4}.$$

Proof. Choosing $c=F_{m-r}$ and $d=(-1)^rF_m$ in (41) together with (14) gives (43). □

Remark 3. *Identity (24) is reproduced as the limiting case of choosing $m=1$, $r=3$ in (43).*

Corollary 8. *If m and r are positive integers such that $r>m$, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k(m+1)}(F_{r-m}/F_m)^k}{G_kG_{k+1}\cdots G_{k+m-1}G_{k+m+1}\cdots G_{k+r-1}G_{k+r}}=\frac{(-1)^{m+1}F_{r-m}}{F_rG_1G_2\cdots G_r}.$$

Setting $r=m+1$ in (43) leads to the next result.

Corollary 9. *If m is a positive integer, then*

$$\sum_{k=1}^{n}\frac{(-1)^{k(m+1)}}{F_m^kG_kG_{k+1}\cdots G_{k+m-1}G_{k+m+1}}=\frac{(-1)^{nm+n+m}}{F_m^nF_{m+1}G_{n+1}G_{n+2}\cdots G_{n+m+1}}-\frac{(-1)^m}{F_{m+1}G_1G_2\cdots G_{m+1}},$$

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with the limiting value

$$\sum_{k=1}^{\infty} \frac{(-1)^{k(m+1)}}{F_m^k G_k G_{k+1} \cdots G_{k+m-1} G_{k+m+1}} = \frac{(-1)^{m+1}}{F_{m+1} G_1 G_2 \cdots G_{m+1}}.$$

Theorem 25. *If $r > 1$ is a positive integer, then*

$$\sum_{k=1}^{n} \frac{F_{r-1}^{k-1}}{G_k G_{k+2} \cdots G_{k+r}} = \frac{1}{F_r G_1 G_2 \cdots G_r} - \frac{F_{r-1}^{n}}{F_r G_{n+1} G_{n+2} \cdots G_{n+r}},$$

with the limiting value

$$\sum_{k=1}^{\infty} \frac{F_{r-1}^{k-1}}{G_k G_{k+2} \cdots G_{k+r}} = \frac{1}{F_r G_1 G_2 \cdots G_r}.$$

Proof. Use $c = F_{r-1}$ and $d = 1$ in (41) in conjunction with (10). □

Theorem 26. *If n is a non-negative integer, $r > 1$ is a positive integer and $G_0 \neq 0$, then*

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{F_{k+1} \cdots F_{k+r-1}}{G_k G_{k+1} \cdots G_{k+r}} = \frac{1}{G_0 F_r} \left(\frac{F_1 F_2 \cdots F_r}{G_1 G_2 \cdots G_r} - \frac{F_{n+1} F_{n+2} \cdots F_{n+r}}{G_{n+1} G_{n+2} \cdots G_{n+r}} \right).$$

In particular,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{F_{k+r} F_{k+r+1}} = \frac{1}{F_r} \left(\frac{1}{F_{r+1}} - \frac{F_{n+1}}{F_{n+r+1}} \right) \tag{44}$$

and

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{F_{k+1}}{G_k G_{k+1} G_{k+2}} = \frac{1}{G_0} \left(\frac{1}{G_1 G_2} - \frac{F_{n+1} F_{n+2}}{G_{n+1} G_{n+2}} \right).$$

Proof. Set $c = 1 = d$ and $g(k) = F_k / G_k$ in (41) and use the following identity [29, Identity (21)]:

$$\frac{F_{k+r}}{G_{k+r}} - \frac{F_k}{G_k} = (-1)^k \frac{G_0 F_r}{G_k G_{k+r}}.$$

□

Corollary 10. *If $r > 1$ is a positive integer and $G_0 \neq 0$, then*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{F_{k+1} \cdots F_{k+r-1}}{G_k G_{k+1} \cdots G_{k+r}} = \frac{1}{G_0 F_r} \left(\frac{F_1 F_2 \cdots F_r}{G_1 G_2 \cdots G_r} - \frac{1}{(G_1 - G_0 \beta)^r} \right).$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{F_{k+r} F_{k+r+1}} = \frac{1}{F_r} \left(\frac{1}{F_{r+1}} - \frac{1}{\alpha^r} \right) \tag{45}$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{F_{k+1}}{G_k G_{k+1} G_{k+2}} = \frac{1}{G_0} \left(\frac{1}{G_1 G_2} - \frac{1}{(G_1 - G_0 \beta)^2} \right).$$

Remark 4. *We note that* (44) *and* (45) *are not new. These results are special cases of another generalization, namely,*

$$\sum_{k=1}^{n} \frac{(-1)^{m(k+1)}}{F_{km+r}F_{km+m+r}} = \frac{F_{mn}}{F_m F_{m+r} F_{mn+m+r}}, \quad m \geq 1, r \geq 0,$$

which can be found in [13] or [19]. When $m = 1$ then

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{F_{k+r}F_{k+1+r}} = \frac{F_n}{F_{r+1}F_{n+1+r}}.$$

The equivalence of (44) *and the last identity is readily seen by noting that*

$$F_{n+1+r} - F_{r+1}F_{n+1} = F_r F_n.$$

Theorem 27. *If n is a non-negative integer and $r > 1$ is a positive integer, then*

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{G_{k+1}G_{k+2}\cdots G_{k+r-1}}{G_{k+r}G_{k+r+1}\cdots G_{k+2r}} = \frac{1}{\varepsilon_G F_r^2}\left(\frac{G_{n+1}G_{n+2}\cdots G_{n+r}}{G_{n+r+1}G_{n+r+2}\cdots G_{n+2r}} - \frac{G_1 G_2\cdots G_r}{G_{r+1}G_{r+2}\cdots G_{2r}}\right).$$

In particular,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}G_{k+1}}{G_{k+2}G_{k+3}G_{k+4}} = \frac{1}{\varepsilon_G}\left(\frac{G_{n+1}G_{n+2}}{G_{n+3}G_{n+4}} - \frac{G_1G_2}{G_3G_4}\right).$$

Proof. Use $g(k) = G_k/G_{k+r}$ and $c = 1 = d$ in (41), noting that (variation on Tagiuri's identity)

$$\frac{G_{k+r}}{G_{k+2r}} - \frac{G_k}{G_{k+r}} = (-1)^{k-1}\frac{F_r^2\varepsilon_G}{G_{k+r}G_{k+2r}}.$$

□

Corollary 11. *If $r > 1$ is a positive integer, then*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{G_{k+1}G_{k+2}\cdots G_{k+r-1}}{G_{k+r}G_{k+r+1}\cdots G_{k+2r}} = \frac{1}{\varepsilon_G F_r^2}\left((-1)^r\left(\beta F_{r^2} + F_{r^2-1}\right) - \frac{G_1G_2\cdots G_r}{G_{r+1}G_{r+2}\cdots G_{2r}}\right).$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}G_{k+1}}{G_{k+2}G_{k+3}G_{k+4}} = \frac{1}{\varepsilon_G}\left(3\beta + 2 - \frac{G_1G_2}{G_3G_4}\right).$$

Theorem 28. *If n is a non-negative integer and $r \geq 2$ is an even integer, then*

$$\sum_{k=1}^{n} \frac{G_{2k+r}}{F_kG_kF_{k+1}G_{k+1}\cdots F_{k+r}G_{k+r}} = \frac{1}{F_r}\left(\frac{1}{F_1G_1\cdots F_rG_r} - \frac{1}{F_{n+1}G_{n+1}\cdots F_{n+r}G_{n+r}}\right).$$

In particular,

$$\sum_{k=1}^{n} \frac{G_{2k+2}}{F_kG_kF_{k+1}G_{k+1}F_{k+2}G_{k+2}} = \frac{1}{G_1G_2} - \frac{1}{F_{n+1}G_{n+1}F_{n+2}G_{n+2}}.$$

Proof. Use $g(k) = 1/(F_k G_k)$ and $c = 1 = d$ in (41), noting that

$$\frac{1}{F_n G_n} - \frac{1}{F_{n+r} G_{n+r}} = \frac{F_r G_{2n+r}}{F_n G_n F_{n+r} G_{n+r}},$$

which itself is a consequence of

$$F_r G_{2n+r} = F_{n+r} G_{n+r} - F_n G_n, \qquad r \text{ even.}$$

□

Corollary 12. *If $r \geq 2$ is an even integer, then*

$$\sum_{k=1}^{\infty} \frac{G_{2k+r}}{F_k G_k F_{k+1} G_{k+1} \cdots F_{k+r} G_{k+r}} = \frac{1}{F_r} \frac{1}{F_1 G_1 \cdots F_r G_r}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{G_{2k+2}}{F_k G_k F_{k+1} G_{k+1} F_{k+2} G_{k+2}} = \frac{1}{G_1 G_2}.$$

8. Conclusions

It should be clear that the possibilities do not end here and we (or the reader) could derive many more identities based on the telescoping principle. We focused mainly on (T1), (T3) and (41) and these simple rules enabled us to discover a multitude of identities with Gibonacci numbers in the denominator and, in some cases, in the numerator, including the alternating versions of some identities.

In the Introduction we announced that we would not give that much attention to (T2). However, using the ideas introduced in various sections of this article, it possible to utilize that telescoping identity. For example, we can present the following variation of Corollary 4.

Corollary 13. *We have the following identities:*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{G_k G_{k+4} G_{k+5}} \left(\frac{3}{G_{k+1}} - \frac{3}{G_{k+2}} \right) = \frac{1}{G_2 G_3 G_4 G_5},$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{G_k G_{k+4} G_{k+5}} \left(\frac{1}{G_{k+1}} + \frac{2}{G_{k+2}} - \frac{1}{G_{k+3}} \right) = \frac{1}{G_1 G_3 G_4 G_5},$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{G_k G_{k+4} G_{k+5}} \left(\frac{4}{G_{k+1}} - \frac{1}{G_{k+2}} - \frac{1}{G_{k+3}} \right) = \frac{1}{G_1 G_2 G_4 G_5},$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{G_k G_{k+4} G_{k+5}} \left(\frac{5}{G_{k+1}} + \frac{1}{G_{k+2}} - \frac{2}{G_{k+3}} \right) = \frac{1}{G_1 G_2 G_3 G_5},$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{G_k G_{k+4} G_{k+5}} \left(\frac{9}{G_{k+1}} - \frac{3}{G_{k+3}} \right) = \frac{1}{G_1 G_2 G_3 G_4}.$$

Proof. Use the following set of identities in order:

$$
\begin{aligned}
3G_{k+2}G_{k+3} - 3G_{k+1}G_{k+3} &= G_{k+5}G_k + G_{k+1}G_k,\\
G_{k+2}G_{k+3} + 2G_{k+1}G_{k+3} - G_{k+1}G_{k+2} &= G_{k+5}G_{k+1} + G_{k+2}G_k,\\
4G_{k+2}G_{k+3} - G_{k+1}G_{k+3} - G_{k+1}G_{k+2} &= G_{k+5}G_{k+2} + G_{k+3}G_k,\\
5G_{k+2}G_{k+3} + G_{k+1}G_{k+3} - 2G_{k+1}G_{k+2} &= G_{k+5}G_{k+3} + G_{k+4}G_k,\\
9G_{k+2}G_{k+3} - 3G_{k+1}G_{k+2} &= G_{k+5}G_{k+4} + G_{k+5}G_k.
\end{aligned}
$$

Follow the proof of Corollary 4 and apply (T2). □

We observe that the identities appearing in the proof of Corollary 13 exhibit an elegant structure. In each respective case, the right-hand side is $G_{k+5}G_{k+i} + G_{k+i+1}G_k$ for $i = 1, \ldots, 4$. Therefore, it is interesting to find similar patterns that could be used for Gibonacci identities, including patterns involving products of more than two terms.

We encourage readers to explore the potential applications of (T2) and other telescoping principles to summation problems.

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