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Article

Exact Solutions of Nonlocal Continuous and Semi-Discrete Matrix Nonlinear Schrödinger Equations

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Abstract: In this paper, seven nonlocal matrix nonlinear Schrödinger (NLS) equations in continuous and semi-discrete cases are derived in the framework of bidifferential calculus. By making use of the direct method and reductions, different forms of exact solutions governed by Sylvester equation are obtained respectively in focusing and defocusing cases. Multisoliton solutions are listed in the Tables and rank one solutions are presented as illustrations.

Keywords: nonlinear Schrödinger equations; nonlocal; bidifferential calculus; exact solutions

1. Introduction

Integrable systems play important roles in describing physical phenomena. A few phenomena are found to be related to many events occurring in different places upon different times which may have some mutual interactions rather than a single event happening in a local area, nonlocal nonlinear equations are employed to depict these phenomenon mathematically and physically. Since Ablowitz and Musslimani first proposed the nonlocal nonlinear Schrödinger (nNLS) equation [1] describing wave propagation in nonlinear PT symmetric media, more integrable nonlocal systems were proposed and further investigated in different ways [2–8]. It is noted that a local integrable system usually has many types of nonlocal versions. There exist the reverse-time NLS equation, the reverse-space NLS equation and the reverse space-time NLS equation, which are all concerned in this paper.

Exact solution is always an important issue on integrable systems. Besides the inverse scattering method [1,9,11–13], Darboux transformations [10,14–19] and the bilinear method [20–22] are subsequently developed in the nonlocal level, which reveals some interesting behaviors such as the simultaneous existence of soliton and kink solutions [19]. Since many nonlocal systems including the nNLS equation are generated from reductions of Ablowitz-Kaup-Newell-Suger (AKNS) hierarchy, it is more convenient to impose the reduction conditions on the results of AKNS equation than directly solving those nonlinear equations. Thus double Wronskian forms of exact solutions to those nonlocal equations are derived by employing certain reduction technique when reducing double Wronskian solutions of the AKNS hierarchy [23], and this approach is applied to nonlocal continuous and semi-discrete nonlinear Schrödinger equations respectively in [24] and [26]. Also Cauchy matrix type solutions for the nonlocal nonlinear Schrödinger equations are obtained from the Cauchy matrix type solutions of AKNS equation through nonlocal reduction in [25].

This paper considers the noncommutative nonlocal nonlinear Schrödinger equation in the framework of bidifferential graded algebras. Bidifferential graded algebras (or bidifferential calculus) [27] are found to be valid to investigate several noncommutative integrable systems including the self-dual Yang-Mills equation, and many properties involving conservation laws [28], bi-Hamiltonian constructions [29], binary Darboux transformations [30] and the direct method for generating solutions [31,32] are established in this framework. Since [32] showed matrix nonlinear Schrödinger equation could be derived in the framework of bidifferential calculus, we are inspired to investigate the nonlocal matrix NLS equation through reductions, in respect of continuous and semi-discrete versions.

The paper is organized as follows. In Section 2, we give a brief introduction to bidfferential calculus and derive the nonlocal matrix nonlinear Schrödinger equation through reductions in this

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framework. In section 3, based on the reductions of the direct method we obtain different forms of exact solutions of nonlocal continuou NLS equations and semi-discrete NLS equations. In section 4, we illustrate the solutions for each nonlocal case. Finally conclusions are given in section 5.

2. Nonlocal Matrix NLS Equation in Bidifferential Calculus

A bidfferential calculus is a graded associative algebra $\Omega(\mathcal{A}) = \bigoplus_{s\in\mathbb{N}_0} \Omega^s$ equipped with two graded derivations $d, \bar{d}$: $\Omega^s \to \Omega^{s+1}$, with the properties

$$d^2 = 0, \ \ \bar{d}^2 = 0, \ \ d\bar{d} + \bar{d}d = 0. \tag{1}$$

In this framework, equations

$$\bar{d}d\phi = d\phi d\phi \tag{2}$$

and

$$d\left[(\bar{d}g)g^{-1}\right] = 0 \tag{3}$$

are found to have correspondences to many integrable equations by suitably choosing d, $\bar{d}$. Furthermore the Miura transformation between equation (2) and (3) are established by $(\bar{d}g)g^{-1} = d\phi$.

For any $\mathcal{A}$, the corresponding graded algebra is given by $\Omega = \mathcal{A} \otimes \wedge(\mathbb{C}^N)$ where $\wedge(\mathbb{C}^N)$ denotes the exterior algebra of $\mathbb{C}^N(N > 1)$ and further the max grade is N. For a given algebra $\mathcal{B}$, we introduce that $\mathrm{Mat}(N_1, N_2, \mathcal{B})$ denotes the set of $N_1 \times N_2$ matrices over $\mathcal{B}$, and $Mat_{N_0}(\mathcal{B})$ denotes the set of $\oplus_{N_1,N_2\geq N_0} Mat(N_1, N_2, \mathcal{B})$. Usually $\mathcal{B}$ is an extension of an algebra $\mathcal{B}_0$ consisting of defined functions, by adding related partial derivatives or shifts.

2.1. Nonlocal Continuous Matrix NLS Equation

Here let $\mathcal{B}_0$ be the space of smooth complex functions on $\mathbb{R}^2$ and extend it to an algebra $\mathcal{B}$ by adjoining the partial derivatives with respect to x, t. Then we have $\Omega = \mathcal{A} \otimes \wedge(\mathbb{C}^2)$ where $\mathcal{A} = Mat_2(\mathcal{B})$. For any $f \in \Omega$, we define the graded derivations

$$df = [\partial_x, f]\zeta_1 + \frac{1}{2}[J, f]\zeta_2, \ \ \bar{d}f = [-\mathrm{i}\partial_t, f]\zeta_1 + [\partial_x, f]\zeta_2, \tag{4}$$

where $\zeta_i(i = 1, 2)$ represents a $\mathcal{A}$ left-module basis in $\wedge(\mathbb{C}^N)$ and $J = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ and $\mathrm{i}^2 = -1$.

The equation (2) takes the form

$$-\mathrm{i}[J, \phi]_t - 2\phi_{xx} = [\phi_x, [J, \phi]]. \tag{5}$$

Making use of the substitution

$$\phi = J\begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix} \tag{6}$$

with $n \times n$ matrices $p, q, \bar{p}, \bar{q}$, the equation (5) is expanded as

$$\begin{pmatrix} -p_{xx} & -\mathrm{i}q_t - q_{xx} \\ \bar{q}_{xx} - \mathrm{i}\bar{q}_t & \bar{p}_{xx} \end{pmatrix} = \begin{pmatrix} q_x\bar{q} + q\bar{q}_x & p_xq + q\bar{p}_x \\ -\bar{q}p_x - \bar{p}_x\bar{q} & -\bar{q}_xq - \bar{q}q_x \end{pmatrix}$$

which reduces to

$$\mathrm{i}q_t + q_{xx} - 2q\bar{q}q = 0, \tag{7a}$$
$$-\mathrm{i}\bar{q}_t + \bar{q}_{xx} - 2\bar{q}q\bar{q} = 0, \tag{7b}$$

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with the relations

$$p_x = -q\overline{q}, \ \ \overline{p}_x = -\overline{q}q. \tag{8}$$

Interestingly if we let $\overline{q} = \delta q(\epsilon x, -t), \epsilon^2 = 1$, two equations in the system (7) are equivalent to the single nonlocal equation

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) - 2\delta q(x,t)q(\epsilon x, -t)q(x,t) = 0. \tag{9}$$

With the similar procedures, we obtain the following nonlocal equations

$$\overline{q} = \delta q^{\mathsf{T}}(\epsilon x, -t): \ \ \mathrm{i}q_t(x,t) + q_{xx}(x,t) - 2\delta q(x,t)q^{\mathsf{T}}(\epsilon x, -t)q(x,t) = 0, \tag{10}$$

$$\overline{q} = \delta q^{*}(\epsilon x, t): \ \ \mathrm{i}q_t(x,t) + q_{xx}(x,t) - 2\delta q(x,t)q^{*}(\epsilon x, t)q(x,t) = 0, \tag{11}$$

$$\overline{q} = \delta q^{\dagger}(\epsilon x, t): \ \ \mathrm{i}q_t(x,t) + q_{xx}(x,t) - 2\delta q(x,t)q^{\dagger}(\epsilon x, t)q(x,t) = 0. \tag{12}$$

Here the variables ϵ, δ satisfy $\epsilon^2 = \delta^2 = 1$ and *, $^{\mathsf{T}}$, † respectively denote complex conjugation, transpose and Hermitian conjugation.

2.2. Nonlocal Semi-Discrete Matrix NLS Equation

Let $\mathcal{B}_0$ be the space of functions on $\mathbb{Z} \times \mathbb{R}$ and extend it to an algebra $\mathcal{B}$ by adjoining the shift operator $\mathbb{S}$ with respect to the discrete variable x such that $\mathbb{S}f = \tilde{f} = f(x+1)$ with the notation $\tilde{}$. In the bi-differential calculus $\Omega = Mat_2(\mathcal{B}) \otimes \wedge(\mathbb{C}^2)$, we define the graded derivations

$$df = [\mathbb{S}, f]\zeta_1 + \frac{1}{2}[J, f]\zeta_2, \ \ \overline{d}f = [-\mathrm{i}\partial_t, f]\zeta_1 - [\mathbb{S}^{-1}, f]\zeta_2, \tag{13}$$

for any $f \in \Omega$. Then the equation (2) takes the form

$$-\mathrm{i}[J, \phi]_t + 4\phi - 2\underset{\sim}{\phi} - 2\tilde{\phi} = [[\mathbb{S}, \phi], [J, \phi]]. \tag{14}$$

After the transformation

$$\phi = J\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}\mathbb{S}^{-1}$$

with $n \times n$ matrices $p, q, \overline{p}, \overline{q}$, the equation (14) becomes

$$\mathrm{i}q_t + \tilde{q} - 2q + \underset{\sim}{q} - \tilde{q}\tilde{\overline{q}}q - q\overline{q}\underset{\sim}{q} = 0, \tag{15a}$$

$$-\mathrm{i}\overline{q}_t + \tilde{\overline{q}} - 2\overline{q} + \underset{\sim}{\overline{q}} - \tilde{\overline{q}}q\overline{q} - \overline{q}q\underset{\sim}{\overline{q}} = 0, \tag{15b}$$

with $\tilde{p} - p = -\tilde{q}\overline{q}$, $\tilde{\overline{p}} - \overline{p} = -\tilde{\overline{q}}q$, which admits the following nonlocal reductions:

$$\overline{q} = \delta q(x, -t): L(q) = \delta q(x+1, t)q(x, -t)q(x, t) + \delta q(x, t)q(x, -t)q(x-1, t), \tag{16}$$

$$\overline{q} = \delta q^{\dagger}(-x, t): L(q) = \delta q(x+1, t)q^{\dagger}(-x, t)q(x, t) + \delta q(x, t)q^{\dagger}(-x, t)q(x-1, t), \tag{17}$$

$$\overline{q} = \delta q^{\mathsf{T}}(-x, -t): L(q) = \delta q(x+1, t)q^{\mathsf{T}}(-x, -t)q(x, t) + \delta q(x, t)q^{\mathsf{T}}(-x, -t)q(x-1, t), \tag{18}$$

with $L(q) = \mathrm{i}q_t(x,t) + q(x+1,t) - 2q(x,t) + q(x-1,t)$.

3. Nonlocal Reductions of Solutions for Matrix NLS Equations

In the framework of bidifferential calculus, [32] have obtained the following theorem by employing the direct linearization method, i.e.

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Theorem 1. *Theorem Let $(\Omega, d, \overline{d})$ be a bidifferential graded algebra with $\Omega = \mathcal{A} \otimes \wedge(\mathbb{C}^2)$ and $\mathcal{A} = Mat_{N_0}(\mathcal{B})$. Let $\Theta \in Mat(m, m, \mathcal{B})$ satisfy*

$$\overline{d}\Theta = (d\Theta)P \tag{19}$$

and

$$[P, K\Theta] = VU\Theta \tag{20}$$

with d and $\overline{d}$ constant matrices $P, K \in Mat(m, m, \mathcal{B}), U \in Mat(n, m, \mathcal{B}), V \in Mat(m, n, \mathcal{B})$. Then

$$\phi = U\Theta(I_m - K\Theta)^{-1}V \in Mat(n, n, \mathcal{B}) \tag{21}$$

is a solution of (2), with invertible $I_m - K\Theta$. Here I_m denotes the $m \times m$ identity matrix.

3.1. Nonlocal Reductions for Continuous Cases

Substituting (4) and (6) to the Theorem 1, exact solutions for equation (7) are directly written by the following:

Proposition 1. *Proposition Let $U, \overline{U} \in Mat(n, m, \mathcal{B})$, $V, \overline{V} \in Mat(m, n, \mathcal{B})$ and $S, \overline{S}, K, \overline{K} \in Mat(m, m, \mathcal{B})$ be constant complex matrices, and satisfy the Sylvester equations*

$$SK + K\overline{S} = VU, \tag{22a}$$

$$\overline{SK} + \overline{K}S = \overline{VU}. \tag{22b}$$

Then

$$q = U(\overline{\Theta}^{-1} - \overline{K}\Theta K)^{-1}\overline{V}, \qquad \overline{q} = \overline{U}(\Theta^{-1} - K\overline{\Theta}\overline{K})^{-1}V \tag{23}$$

solves (7), where $\Theta, \overline{\Theta} \in Mat(m, m, \mathcal{B})$ are defined as

$$\Theta = e^{-xS - itS^2}, \quad \overline{\Theta} = e^{-x\overline{S} + it\overline{S}^2}. \tag{24}$$

Imposing the reduction conditions on the solution (23) with (22), we derive the following result, where we can ignore the sign of q due to the odd degree of every term in the equations.

Corollary 1. *Corollary For the case of $\overline{q} = \delta q(\epsilon x, -t)$, the nonlocal equation (9) admits the solution*

$$q = \delta U(e^{\epsilon x S - itS^2} - \epsilon\delta K e^{-xS - itS^2} K)^{-1}V, \tag{25}$$

with the condition

$$SK + \epsilon KS = VU. \tag{26}$$

Particularly when $\overline{q} = \delta q(-x, -t)$, the function

$$q = i\delta^{1/2}U(Ke^{x\overline{S} - it\overline{S}^2} \pm e^{-xS - itS^2}K)^{-1}V, \tag{27}$$

with

$$SK + K\overline{S} = VU, \tag{28}$$

solves the nonlocal equation (9) with $\epsilon = -1$.

Proof of Corollary 1. Comparing the terms of the solutions

$$\begin{aligned} \overline{q} =& \overline{U}(e^{xS + itS^2} - Ke^{-x\overline{S} + it\overline{S}^2}\overline{K})^{-1}V, \\ \delta q(\epsilon x, -t) =& \delta U(e^{\epsilon x\overline{S} + it\overline{S}^2} - \overline{K}e^{-\epsilon xS + itS^2}K)^{-1}\overline{V}, \end{aligned}$$

it is reasonable to give assumptions,

$$\overline{U} = U,\ \ \overline{V} = \delta V,\ \ \overline{S} = \epsilon S,\ \ \overline{K} = \varepsilon K,\ \ \varepsilon^2 = 1.$$

To deal with (22), we find that $\varepsilon = \epsilon\delta$ merges the two conditions into one equation (26), and meantime it admits the relation

$$\delta q(\epsilon x, -t) = U(e^{xS+\mathrm{i}tS^2} - \varepsilon K e^{-\epsilon xS+\mathrm{i}tS^2}K)^{-1}V = \overline{q}(x,t).$$

If we rewrite

$$\begin{aligned}\delta q(\epsilon x, -t) =& \delta UK^{-1}(\overline{K}^{-1}e^{\epsilon x\overline{S}+\mathrm{i}t\overline{S}^2}K^{-1} - \overline{K}e^{-\epsilon xS+\mathrm{i}tS^2}K)^{-1}\overline{K}^{-1}\overline{V}\\ =& -\delta UK^{-1}(\overline{K}e^{-\epsilon xS+\mathrm{i}tS^2}K - \overline{K}^{-1}e^{\epsilon x\overline{S}+\mathrm{i}t\overline{S}^2}K^{-1})^{-1}\overline{K}^{-1}\overline{V},\end{aligned}$$

there exists another reduction. To admit the reduction $\overline{q} = \delta q(\epsilon x, -t)$, we assume

$$\epsilon = -1,\ \ \overline{U} = \mathrm{i}\delta^{1/2}UK^{-1},\ \ \overline{K}K = \varepsilon I_m,\ \ \overline{V} = -\mathrm{i}\delta^{-1/2}\overline{K}V.$$

Then the two equations (22) are equivalent to $SK + K\overline{S} = VU$. Ultimately the solution (27) with (28) solves the nonlocal equation (9) with $\epsilon = -1$. □

Corollary 2. *Corollary For the case of $\overline{q} = \delta q^{\mathsf{T}}(\epsilon x, -t)$, the nonlocal equation (9) admits the solution*

$$q = V^{\mathsf{T}}(e^{\epsilon xS^{\mathsf{T}} - \mathrm{i}t(S^{\mathsf{T}})^2} - \epsilon\delta\overline{K}e^{-xS-\mathrm{i}tS^2}K)^{-1}\overline{V} \tag{29}$$

with

$$SK + \epsilon KS^{\mathsf{T}} = VV^{\mathsf{T}},\ \ S^{\mathsf{T}}\overline{K} + \epsilon\overline{K}S = \overline{V}\overline{V}^{\mathsf{T}},\ \ K^{\mathsf{T}} = \epsilon K,\ \ \overline{K}^{\mathsf{T}} = \epsilon\overline{K}. \tag{30}$$

Particularly when $\overline{q} = \delta q^{\mathsf{T}}(-x, -t)$, the function

$$q = \mathrm{i}\delta^{1/2}V^{\mathsf{T}}(e^{x\overline{S}-\mathrm{i}t\overline{S}^2}K^{-1} \pm \overline{K}e^{-xS-\mathrm{i}tS^2})^{-1}\overline{V} \tag{31}$$

with

$$S + K\overline{S}K^{-1} = VV^{\mathsf{T}},\ \ \overline{S} + \overline{K}S\overline{K}^{-1} = \overline{V}\overline{V}^{\mathsf{T}},\ \ S^{\mathsf{T}} = S,\ \ \overline{S}^{\mathsf{T}} = \overline{S},\ \ KK^{\mathsf{T}} = \overline{K}\overline{K}^{\mathsf{T}} = I_m, \tag{32}$$

solves the nonlocal equation (10) with $\epsilon = -1$.

Proof of Corollary 2. Under the assumption

$$\overline{U} = \delta\overline{V}^{\mathsf{T}},\ \ U = V^{\mathsf{T}},\ \ \overline{S} = \epsilon S^{\mathsf{T}},\ \ \overline{K}^{\mathsf{T}} = \varepsilon\overline{K},\ \ K^{\mathsf{T}} = \varepsilon K,\ \ \varepsilon^2 = 1,$$

we have the identity

$$\delta q^{\mathsf{T}}(\epsilon x, -t) = \delta\overline{V}^{\mathsf{T}}\left(e^{\epsilon x\overline{S}^{\mathsf{T}} + \mathrm{i}t(\overline{S}^{\mathsf{T}})^2} - K^{\mathsf{T}}e^{-\epsilon xS^{\mathsf{T}} + \mathrm{i}t(S^{\mathsf{T}})^2}\overline{K}^{\mathsf{T}}\right)^{-1}U^{\mathsf{T}} = \overline{q}(x,t).$$

Based on the reduced condition $SK + \epsilon KS^{\mathsf{T}} = VV^{\mathsf{T}}$, we take its transposition written by

$$VV^{\mathsf{T}} = K^{\mathsf{T}}S^{\mathsf{T}} + \epsilon SK^{\mathsf{T}} = \varepsilon\epsilon(SK + \epsilon KS^{\mathsf{T}}) = \varepsilon\epsilon VV^{\mathsf{T}},$$

which reveals $\varepsilon = \epsilon$. Combining Proposition 2 with the above assumptions and renaming $\overline{K} \to \epsilon\delta\overline{K}$, we obtain the solution (29) with (30).

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Again, the assumption

$$\overline{U}=-\delta\overline{V}^{\mathsf{T}}(\overline{K}^{\mathsf{T}})^{-1},\ U=V^{\mathsf{T}}K,\ \overline{S}^{\mathsf{T}}=-\epsilon\overline{S},\ S^{\mathsf{T}}=-\epsilon S,\ KK^{\mathsf{T}}=\overline{KK}^{\mathsf{T}}=\varepsilon I_m,\ \varepsilon^2=1,$$

admits the reduction condition

$$\begin{aligned}\delta q^{\mathsf{T}}(\epsilon x,-t)=&-\delta\overline{V}^{\mathsf{T}}(\overline{K}^{\mathsf{T}})^{-1}\Big(e^{-\epsilon xS^{\mathsf{T}}+\mathrm{i}t(S^{\mathsf{T}})^2}-(K^{\mathsf{T}})^{-1}e^{\epsilon x\overline{S}^{\mathsf{T}}+\mathrm{i}t(\overline{S}^{\mathsf{T}})^2}(\overline{K}^{\mathsf{T}})^{-1}\Big)^{-1}(K^{\mathsf{T}})^{-1}U^{\mathsf{T}}\\=&\overline{U}(e^{xS+\mathrm{i}tS^2}-Ke^{-x\overline{S}+\mathrm{i}t\overline{S}^2}\overline{K})^{-1}V=\overline{q}(x,t).\end{aligned}$$

The condition (22) written by $S+K\overline{S}K^{-1}=VV^{\mathsf{T}}$, implies

$$VV^{\mathsf{T}}=S^{\mathsf{T}}+(K^{\mathsf{T}})^{-1}\overline{S}^{\mathsf{T}}K^{\mathsf{T}}=-\epsilon(S+K\overline{S}K^{-1})=-\epsilon VV^{\mathsf{T}}$$

which forces $\epsilon=-1$. Renaming $\overline{V}\to(-\varepsilon\delta)^{1/2}\overline{V},\ K\to\varepsilon^{1/2}K,\ \overline{K}\to\varepsilon^{1/2}\overline{K}$, we have a neat form of $q(x,t)$ (31) with (32). □

Furthermore, if we suppose $S^{\mathsf{T}}=\epsilon S,\overline{V}=cV,\overline{K}=c^2\epsilon K$ in the condition (30) , it reduces to

$$q=cV^{\mathsf{T}}(e^{xS-\mathrm{i}tS^2}-c^2\delta Ke^{-xS-\mathrm{i}tS^2}K)^{-1}V$$

with $SK+KS=VV^{\mathsf{T}},\ \ S^{\mathsf{T}}=\epsilon S$. While for the condition (32), if $\overline{S}=S,\ \overline{K}=K,\ \overline{V}=V$, it reduces to

$$q=\delta^{1/2}\mathrm{i}V^{\mathsf{T}}(e^{xS-\mathrm{i}tS^2}K^{-1}\pm Ke^{-xS-\mathrm{i}tS^2})^{-1}V$$

with

$$S+KSK^{-1}=VV^{\mathsf{T}},\ \ S^{\mathsf{T}}=S,\ \ KK^{\mathsf{T}}=I_m.$$

By the similar procedures, we derive the following solutions through certain assumptions:

- $\overline{q}=\delta q^*(\epsilon x,t)$: Assume

$$\overline{U}=U^*,\ \ \overline{V}=\delta V^*,\ \ \overline{S}=\epsilon S^*,\ \ \overline{K}=\epsilon\delta K^*,$$

the function

$$q=\delta U(e^{\epsilon xS^*-\mathrm{i}t(S^*)^2}-\epsilon\delta K^*e^{-xS-\mathrm{i}tS^2}K)^{-1}V^*, \tag{33}$$

with

$$SK+\epsilon KS^*=VU, \tag{34}$$

solves the nonlocal equation (11).

Particularly, assume

$$\overline{U}=-\delta U^*K^{*-1},\ \ \overline{V}=\overline{K}V^*,\ \ K\overline{K}^*=\varepsilon I_m,\ \ \epsilon=\delta,$$

the function

$$q=U(\varepsilon K^*e^{x\overline{S}-\mathrm{i}t\overline{S}^2}-e^{-xS-\mathrm{i}tS^2}K)^{-1}V^* \tag{35}$$

with

$$SK+K\overline{S}=VU,\ \ S^*=-\delta S,\ \ \overline{S}^*=-\delta\overline{S}, \tag{36}$$

solves the nonlocal equation (11) with $\epsilon=\delta$.

- $\overline{q}=\delta q^{\dagger}(\epsilon x,t)$: Assume

$$\overline{U}=\delta\overline{V}^{\dagger},\ \ U=V^{\dagger},\ \ \overline{S}=\epsilon S^{\dagger},$$

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the function

$$q = V^{\dagger}(e^{\epsilon x S^{\dagger} - it(S^{\dagger})^2} - \epsilon\delta \overline{K} e^{-xS - itS^2} K)^{-1}\overline{V} \tag{37}$$

with

$$SK + \epsilon K S^{\dagger} = VV^{\dagger}, \ \ S^{\dagger}\overline{K} + \epsilon \overline{K} S = \overline{V}\overline{V}^{\dagger}, \ \ K^{\dagger} = \epsilon K, \ \ \overline{K}^{\dagger} = \epsilon \overline{K}, \tag{38}$$

solves the nonlocal equation (12).

Particularly, assume

$$\overline{U} = \mathrm{i}\delta^{1/2}\overline{V}^{\dagger}\overline{K}^{\dagger\,-1}, \ \ U = -\mathrm{i}\delta^{-1/2}V^{\dagger}K,$$

the function

$$q = \mathrm{i}\delta^{1/2}V^{\dagger}(e^{x\overline{S} - \mathrm{i}t\overline{S}^2}K^{-1} - \overline{K}e^{-xS - \mathrm{i}tS^2})^{-1}\overline{V} \tag{39}$$

with

$$S + K\overline{S}K^{-1} = \mathrm{i}\delta^{1/2}VV^{\dagger}, \ \ \overline{S} + \overline{K}S\overline{K}^{-1} = \mathrm{i}\delta^{1/2}\overline{V}\overline{V}^{\dagger},$$
$$S^{\dagger} = -\delta S, \ \ \overline{S}^{\dagger} = -\delta \overline{S}, \ \ KK^{\dagger} = \overline{K}\overline{K}^{\dagger} = I_m,$$

both solve the nonlocal equation (12) with $\epsilon = \delta$.

Again, if we suppose $S^{\dagger} = \epsilon S$, $\overline{V} = cV$, $\overline{K} = c^2\epsilon K$ in the condition (38) , it reduces to

$$q = cV^{\dagger}(e^{xS - \mathrm{i}tS^2} - c^2\delta K e^{-xS - \mathrm{i}tS^2}K)^{-1}V$$

with $SK + KS = VV^{\mathsf{T}}, \ \ S^{\dagger} = \epsilon S$.

3.2. Nonlocal Reductions for Semi-Discrete Cases

Likewise from Theorem 1, the solutions for (15) are expressed by the following:

Proposition 2. *Proposition Let $U, \overline{U} \in Mat(n, m, \mathcal{B})$, $V, \overline{V} \in Mat(m, n, \mathcal{B})$ and $S, \overline{S}, K, \overline{K} \in Mat(m, m, \mathcal{B})$ be constant complex matrices, and satisfy the Sylvester equations*

$$S^{-1}K - K\overline{S} = VU, \tag{40a}$$
$$\overline{S}^{-1}\overline{K} - \overline{K}S = \overline{V}\overline{U}. \tag{40b}$$

Then the function (23) solves (15), where $\Theta, \overline{\Theta} \in Mat(m, m, \mathcal{B})$ are defined as

$$\Theta = S^x e^{-\mathrm{i}t\omega(S)}, \quad \overline{\Theta} = \overline{S}^x e^{\mathrm{i}t\omega(\overline{S})}, \quad \omega(S) = S + S^{-1} - 2I_m. \tag{41}$$

Likewise, we have the following exact solutions for nonlocal cases:

- $\overline{q} = \delta q(x, -t)$: Assume

$$\overline{U} = \delta U, \ \ \overline{K} = \delta K, \ \ \overline{S} = S, \ \ \overline{V} = V,$$

the function

$$q = U(S^{-x}e^{-\mathrm{i}t\omega(S)} - \delta K S^x e^{-\mathrm{i}t\omega(S)}K)^{-1}V, \tag{42}$$

with

$$S^{-1}K - KS = VU. \tag{43}$$

solves the nonlocal equation (16).

- $\overline{q} = \delta q^{\dagger}(-x, t)$: Both the functions

$$q = V^{\dagger}\Big((S^{\dagger})^x e^{-\mathrm{i}t\omega(S^{\dagger})} - \delta\overline{K}S^x e^{-\mathrm{i}t\omega(S)}K\Big)^{-1}\overline{V} \tag{44}$$

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with

$$S^{-1}K - K(S^{\dagger})^{-1} = VV^{\dagger}, \ \ S^{\dagger}\overline{K} - \overline{K}S = \overline{V}\overline{V}^{\dagger}, \tag{45}$$

and

$$q = V^{\dagger}\left(\overline{S}^{-x}e^{-\mathrm{i}t\omega(\overline{S})}K^{-1} - \overline{K}S^{x}e^{-\mathrm{i}t\omega(S)}\right)^{-1}\overline{V} \tag{46}$$

with

$$S^{-1} - K\overline{S}K^{-1} = VV^{\dagger}, \ \ \overline{S}^{-1} - \overline{K}S\overline{K}^{-1} = -\delta\overline{V}\overline{V}^{\dagger}, \ \ S^{\dagger} = S, \ \ \overline{S}^{\dagger} = \overline{S}, \ \ KK^{\dagger} = \overline{K}\overline{K}^{\dagger} = I_m, \tag{47}$$

solve the nonlocal equation (17). Additionally, the solutions can be reduced to

$$q = cV^{\dagger}\left(S^{-x}e^{-\mathrm{i}t\omega(S)} - \delta c^2 KS^{x}e^{-\mathrm{i}t\omega(S)}K\right)^{-1}V$$

with $S^{\dagger}K - KS = VV^{\dagger}, \ \ SS^{\dagger} = I_m$, and

$$q = \pm V^{\dagger}\left((-\delta S)^{-x}e^{-\mathrm{i}t\omega(-\delta S)}K^{-1} \pm KS^{x}e^{-\mathrm{i}t\omega(S)}\right)^{-1}V$$

with $S^{-1} + \delta KSK^{-1} = VV^{\dagger}, \ \ S^{\dagger} = S, \ \ KK^{\dagger} = I_m$.

- $\overline{q} = \delta q^{\mathsf{T}}(-x,-t)$: Both the functions

$$q = V^{\mathsf{T}}\left((S^{\mathsf{T}})^{x}e^{-\mathrm{i}t\omega(S^{\mathsf{T}})} - \delta\overline{K}S^{x}e^{-\mathrm{i}t\omega(S)}K\right)^{-1}\overline{V}, \tag{48}$$

with

$$S^{-1}K - K(S^{\mathsf{T}})^{-1} = VV^{\mathsf{T}}, \ \ S^{\mathsf{T}}\overline{K} - \overline{K}S = \overline{V}\overline{V}^{\mathsf{T}}. \tag{49}$$

and

$$q = \mathrm{i}\delta^{1/2}V^{\mathsf{T}}(\overline{S}^{-x}e^{-\mathrm{i}t\omega(\overline{S})}K^{-1} - \overline{K}S^{x}e^{-\mathrm{i}t\omega(S)})^{-1}\overline{V}, \tag{50}$$

with

$$S^{-1} - K\overline{S}K^{-1} = VV^{\mathsf{T}}, \ \ \overline{S}^{-1} - \overline{K}S\overline{K}^{-1} = \overline{V}\overline{V}^{\mathsf{T}}, \ \ S^{\mathsf{T}} = S, \ \ \overline{S}^{\mathsf{T}} = \overline{S}, \ \ KK^{\mathsf{T}} = \overline{K}\overline{K}^{\mathsf{T}} = I_m, \tag{51}$$

solve the nonlocal equation (18). Additionally, the solutions can be reduced to

$$q = cV^{\mathsf{T}}\left(S^{-x}e^{-\mathrm{i}t\omega(S)} - \delta c^2 KS^{x}e^{-\mathrm{i}t\omega(S)}K\right)^{-1}V$$

with $S^{\mathsf{T}}K - KS = VV^{\mathsf{T}}, \ \ SS^{\mathsf{T}} = I_m$, and

$$q = \pm\mathrm{i}\delta^{1/2}V^{\mathsf{T}}\left(S^{-x}e^{-\mathrm{i}t\omega(S)}K^{-1} \pm KS^{x}e^{-\mathrm{i}t\omega(S)}\right)^{-1}V$$

with $S^{-1} - KSK^{-1} = VV^{\mathsf{T}}, \ \ S^{\mathsf{T}} = S, \ \ KK^{\mathsf{T}} = I_m$.

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation as well as the experimental conclusions that can be drawn.

4. Illustrations of Solutions for the Nonlocal NLS Equations

Most of the auxiliary conditions which the constant matrices admit are in the form of Sylvester equation. By a similarity transformation, we can assume that $S, \overline{S}$ has the Jordan normal form without loss of generality and split them into Jordan blocks. The similar systematic analysis for solving

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Sylvester equation is given in [32] concretely and multi-soliton solutions can be constructed. This section only gives illustrations of solutions. To begin with, we assume

$$K = (k_{ij}), \quad U = \begin{pmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_m \end{pmatrix}, \quad V = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_m \end{pmatrix}^{\mathsf{T}}$$

with n-component column vectors $\boldsymbol{u}_i, \boldsymbol{v}_i, i = 1, ..., m$,

4.1. Solving the Sylvester Equations

4.1.1. Jordan Block Case

Let us consider a matrix S consisting of an $m \times m$ Jordan block

$$S = sI + \mathcal{N}, \tag{52}$$

where I is the $m \times m$ identity matrix and the components of $\mathcal{N}$ are defined by $\mathcal{N}_{ij} = \delta_{i,j-1}$.

For example, we consider the Sylvester equation (26) in the continuous case of $\overline{q} = \delta q(\epsilon x, -t)$. It takes the form

$$(s + \epsilon s)k_{ij} + k_{i+1,j} + k_{i,j-1} = \boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{u}_j \qquad i, j = 1, ..., m, \tag{53}$$

where $k_{r+1,j} = k_{i,0} = 0$. To secure the Sylvester equation has a unique solution, we consider the case $\epsilon = 1$. Then the components of K can be recursively determined by

$$k_{m1} = \frac{\boldsymbol{v}_m^{\mathsf{T}} \boldsymbol{u}_1}{2s}, \qquad k_{mj} = \frac{\boldsymbol{v}_m^{\mathsf{T}} \boldsymbol{u}_j - k_{m,j-1}}{2s} \quad j = 2, ..., m,$$
$$k_{i1} = \frac{\boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{u}_1 - k_{i+1,1}}{2s} \quad i = m-1, ..., 1.$$

If we take $m = 2$ and $\epsilon = 1$, it reads

$$K = \begin{pmatrix} \frac{\boldsymbol{v}_1^{\mathsf{T}} \boldsymbol{u}_1}{2s} - \frac{\boldsymbol{v}_2^{\mathsf{T}} \boldsymbol{u}_1}{4s^2} & \frac{\boldsymbol{v}_1^{\mathsf{T}} \boldsymbol{u}_2}{2s} - \frac{\boldsymbol{v}_1^{\mathsf{T}} \boldsymbol{u}_1 + \boldsymbol{v}_2^{\mathsf{T}} \boldsymbol{u}_2}{4s^2} + 2\frac{\boldsymbol{v}_2^{\mathsf{T}} \boldsymbol{u}_1}{8s^3} \\ \frac{\boldsymbol{v}_2^{\mathsf{T}} \boldsymbol{u}_1}{2s} & \frac{\boldsymbol{v}_2^{\mathsf{T}} \boldsymbol{u}_2}{2s} - \frac{\boldsymbol{v}_2^{\mathsf{T}} \boldsymbol{u}_1}{4s^2} \end{pmatrix}. \tag{54}$$

Particularly if $\boldsymbol{u}_1 = 0$ or $\boldsymbol{v}_m = 0$, we observe that the first column or the last row of K vanishes respectively. In this case, Jordan block solutions can be constructed through superpositions.

For the scalar defocusing case, we write

$$U = \begin{pmatrix} u_1 & u_2 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{\mathsf{T}}$$

Here the variables s_1,u_i,v_i $i = 1, 2$ are arbitrary constants. And the Cauchy matrix type Jordan block solution is

$$q(x,t) = \frac{P_1 \mathrm{e}^{s(\mathrm{i}st - x)} + P_2 \mathrm{e}^{s(\mathrm{i}st + x)}}{Q_0 + Q_1 \mathrm{e}^{2xs} + Q_2 \mathrm{e}^{-2xs}}$$
$$P_1 = -u_1^2 v_2^2 \Big[u_1 v_2 (2\mathrm{i}st + x) - (u_1 v_1 + u_2 v_2) + 2 v_2 u_1 s^{-1} \Big]$$
$$P_2 = 16 s^4 [u_1 v_2 (2\mathrm{i}st - x) + u_1 v_1 + u_2 v_2]$$
$$Q_0 = -16 s^4 u_1^2 t^2 v_2^2 - 4((u_1 v_2 x - u_1 v_1 - v_2 u_2)s + u_1 v_2)^2 - 2 u_1^2 v_2^2$$
$$Q_1 = 16 s^4, \qquad Q_2 = \frac{1}{16 s_1^4} u_1^4 v_2^4.$$

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4.1.2. Diagonal Case

In the following, we consider $S = \mathrm{diag}(s_1, ..., s_m)$ for solving the Sylvester equation (34) in the case of $\delta q^*(\epsilon x, t)$. It is reasonable to assume the spectrum condition $\sigma(S) \cap \sigma(-\epsilon S) = 0$, i.e. $s_i + \epsilon s_j^* \neq 0$ for all $i, j = 1, ..., m$. With no restriction of generality, we can then assume that the eigenvalues s_i are pairwise different.

Writing

$$\Theta = \mathrm{diag}(\xi_1, ..., \xi_m), \qquad K = (k_{ij}), \quad i, j = 1, ..., m, \tag{55}$$

the Sylvester equation (34) in the continuous case takes the form

$$(s_i + \epsilon s_j^*) k_{ij} = \boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{u}_j, \quad i, j = 1, ..., m,$$

which implies

$$k_{ij} = \frac{\boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{u}_j}{s_i + \epsilon s_j^*}. \tag{56}$$

Then the function

$$q = \delta U (e^{\epsilon x S^* - \mathrm{i}t(S^*)^2} - \epsilon\delta K^* e^{-xS - \mathrm{i}tS^2} K)^{-1} V^*, \tag{57}$$

with (56) and $\forall V, U, s_i \neq s_j$ solves the nonlocal equation (11).

In the semi-discrete case, solving the Sylvester equation (43) which takes the form

$$(s_i^{-1} - s_j) k_{ij} = \boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{u}_j, \quad i, j = 1, ..., m,$$

we have the solution

$$q = U (S^{-x} e^{-\mathrm{i}t\omega(S)} - \delta K S^x e^{-\mathrm{i}t\omega(S)} K)^{-1} V, \tag{58}$$

with

$$k_{ij} = \frac{s_i \boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{u}_j}{1 - s_i s_j}, \qquad \forall V, \ U, \ s_i s_j \neq 1$$

solves the nonlocal equation (16).

In this case, multisolition solutions are constructed. All the cases we derived in the previous section are similarly solved one by one, listed in Table 1 and 2 respectively for the continuous and semi-discrete cases.

When the denominator is zero in element condition of the tables, the corresponding Sylvester equation has not an unique solution. For instance, when we consider the equation (9) with $m = 2$ and $\epsilon = \delta = -1$, we choose

$$S = \mathrm{diag}(a\mathrm{i}, b), \quad \overline{S} = \mathrm{diag}(-a\mathrm{i}, b), \quad U = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} \mathrm{i} & \mathrm{i} \\ 0 & 1 \end{pmatrix}$$

with the real numbers a, b, then we can calculate

$$K = \begin{pmatrix} 0 & \frac{2\mathrm{i}}{\mathrm{i}a+b} \\ \mathrm{i}a - b & \frac{1}{2b} \end{pmatrix}$$

where the k_{11} can be arbitrary. It turns out the solution

$$q = \frac{1}{c} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

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where

$$
\begin{aligned}
c &= 4b(e^{bx-\mathrm{i}b^2t}+e^{\mathrm{i}a(at-x)})(e^{-bx-\mathrm{i}b^2t}+e^{\mathrm{i}a(at-x)}),\\
q_{11} &= -(\mathrm{i}a+b)\Big((\mathrm{i}a-b)e^{bx-\mathrm{i}b^2t}+(\mathrm{i}a-3b)e^{-bx-\mathrm{i}b^2t}-2be^{\mathrm{i}a(at-x)}\Big),\\
q_{12} &= (a^2-3b^2+4\mathrm{i}ab)e^{bx-\mathrm{i}b^2t}+(a^2+3b^2+2\mathrm{i}ab)e^{-bx-\mathrm{i}b^2t}+(6\mathrm{i}ab-b^2)e^{\mathrm{i}a(at-x)},\\
q_{21} &= (\mathrm{i}a+b)\Big((\mathrm{i}a-b)e^{bx-\mathrm{i}b^2t}+(\mathrm{i}a+b)e^{-bx-\mathrm{i}b^2t}+2be^{\mathrm{i}a(at-x)}\Big),\\
q_{22} &= (3b^2-a^2-4\mathrm{i}ab)e^{bx-\mathrm{i}b^2t}+(b^2-a^2+2\mathrm{i}ab)e^{-bx-\mathrm{i}b^2t}+(6b^2-2\mathrm{i}ab)e^{\mathrm{i}a(at-x)}.
\end{aligned}
$$

Besides, if we let

$$
S=\begin{pmatrix}\mathrm{i}a & 1\\ 0 & \mathrm{i}a\end{pmatrix},\quad \overline{S}=\begin{pmatrix}-\mathrm{i}a & 1\\ 0 & -\mathrm{i}a\end{pmatrix},
$$

we choose certain U, V and derive the corresponding solutions for equation (9) written by

$$
q=\frac{e^{-\mathrm{i}a(at-x)}}{(2at-x)^2+1}\begin{pmatrix}1 & x-2at\\ x-2at & -1\end{pmatrix},
$$

and

$$
q=\frac{e^{-\mathrm{i}a(at-x)}}{2at-x}\mathrm{diag}(1,1).
$$

The absolute values of them are both in rational form, the former is regular and the latter corresponds to the scalar case.

Table 1. Solutions for nonlocal continuous NLS equations.

Equations	Solutions	Elements
(9):$\overline{q} = \delta q(\epsilon x, -t)$	$q = \delta U(e^{\epsilon x S - \mathrm{i}tS^2} - \epsilon\delta K e^{-xS-\mathrm{i}tS^2}K)^{-1}V$	$k_{ij} = \frac{\boldsymbol{v}_i^{\mathsf{T}}\boldsymbol{u}_j}{s_i+\epsilon s_j}, \ \forall U, V, s_i + \epsilon s_j \neq 0$
(9):$\overline{q} = \delta q(-x, -t)$	$q = \mathrm{i}\delta^{1/2}U(Ke^{x\overline{S}-\mathrm{i}t\overline{S}^2} \pm e^{-xS-\mathrm{i}tS^2}K)^{-1}V$	$k_{ij} = \frac{\boldsymbol{v}_i^{\mathsf{T}}\boldsymbol{u}_j}{s_i+\overline{s}_j}, \ \forall U, V, s_i + \overline{s}_j \neq 0$
(10):$\overline{q} = \delta q^{\mathsf{T}}(\epsilon x, -t)$	$q = V^{\mathsf{T}}(e^{\epsilon x S^{\mathsf{T}} - \mathrm{i}t(S^{\mathsf{T}})^2} - \epsilon\delta \overline{K} e^{-xS-\mathrm{i}tS^2}K)^{-1}\overline{V}$	$k_{ij} = \frac{\boldsymbol{v}_i^{\mathsf{T}}\boldsymbol{v}_j}{s_i+\epsilon s_j}, \overline{k}_{ij} = \frac{\overline{\boldsymbol{v}}_i^{\mathsf{T}}\overline{\boldsymbol{v}}_j}{s_i+\epsilon s_j}, \forall V, \overline{V}, s_i + \epsilon s_j \neq 0$
(10):$\overline{q} = \delta q^{\mathsf{T}}(-x, -t)$	$q = \mathrm{i}\delta^{1/2}V^{\mathsf{T}}(e^{x\overline{S}-\mathrm{i}t\overline{S}^2}K^{-1} \pm \overline{K}e^{-xS-\mathrm{i}tS^2})^{-1}\overline{V}$	[1] $S + K\overline{S}K^{-1} = VV^{\mathsf{T}}, \ \overline{S} + \overline{K}S\overline{K}^{-1} = \overline{V}\overline{V}^{\mathsf{T}}$,
		$\forall s_i, \overline{s}_j$ and orthogonal matrix $K, \overline{K}$
(11):$\overline{q} = \delta q^*(\epsilon x, t)$	$q = \delta U(e^{\epsilon x S^* - \mathrm{i}t(S^*)^2} - \epsilon\delta K^* e^{-xS-\mathrm{i}tS^2}K)^{-1}V^*$	$k_{ij} = \frac{\boldsymbol{v}_i^{\mathsf{T}}\boldsymbol{u}_j}{s_i+\epsilon s_j^*}, \ \forall V, U, s_i + \epsilon s_j^* \neq 0$
(11):$\overline{q} = \delta q^*(\delta x, t)$	$q = U(K^* e^{x\overline{S}-\mathrm{i}t\overline{S}^2} \pm e^{-xS-\mathrm{i}tS^2}K)^{-1}V^*$	$k_{ij} = \frac{\boldsymbol{v}_i^{\mathsf{T}}\boldsymbol{u}_j}{s_i+\overline{s}_j^*}$, [2] $s_i^* + \delta s_i = \overline{s}_i^* + \delta \overline{s}_i = 0$,
		$s_i + \overline{s}_j^* \neq 0, \forall V, U$
(12):$\overline{q} = \delta q^{\dagger}(\epsilon x, t)$	$q = V^{\dagger}(e^{\epsilon x S^{\dagger} - \mathrm{i}t(S^{\dagger})^2} - \epsilon\delta \overline{K} e^{-xS-\mathrm{i}tS^2}K)^{-1}\overline{V}$	$k_{ij} = \frac{\boldsymbol{v}_i^{\mathsf{T}}\boldsymbol{v}_j^*}{s_i+\epsilon s_j^*}, \ \overline{k}_{ij} = \frac{\overline{\boldsymbol{v}}_i^{\mathsf{T}}\overline{\boldsymbol{v}}_j^*}{s_i+\epsilon s_j^*}, \ \forall V, \overline{V}, s_i + \epsilon s_j^* \neq 0$
(12):$\overline{q} = \delta q^{\dagger}(\delta x, t)$	$q = \mathrm{i}\delta^{1/2}V^{\dagger}(e^{\mathrm{i}\delta^{1/2}x\overline{S}+\mathrm{i}t\delta\overline{S}^2}K^{-1} - \overline{K}e^{-\mathrm{i}\delta^{1/2}xS+\mathrm{i}t\delta S^2})^{-1}\overline{V}$	$S + K\overline{S}K^{-1} = VV^{\dagger}, \ \overline{S} + \overline{K}S\overline{K}^{-1} = \overline{V}\overline{V}^{\dagger}$,
		$\forall$ real $s_i, \overline{s}_j$ and unitary matrix $K, \overline{K}$

[1] If we give arbitrary V or $\overline{V}$, the conditions are not always solvable for $K, \overline{K}$. Thus for any given $K, \overline{K}$ we aim to solve conditions for V and $\overline{V}$, which are always solvable due to the fact that the left sides are symmetric matrices.

[2] For the focusing case, $s_i, \overline{s}_j$ are real numbers, while they are pure imaginary numbers for the local case.

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Table 2. Solutions for nonlocal semi-discrete NLS equations.

Equations	Solutions	Elements
(16):$\overline{q} = \delta q(x,-t)$	$q = U(S^{-x}e^{-\mathrm{i}t\omega(S)} - \delta K S^{x} e^{-\mathrm{i}t\omega(S)} K)^{-1} V$	$k_{ij} = \frac{\boldsymbol{v}_i^{\top}\boldsymbol{u}_j}{s_i^{-1}-s_j}$, $\forall V, U, s_i s_j \neq 1$
(17):$\overline{q} = \delta q^{\dagger}(-x,t)$	$q = V^{\dagger}\left(S^{\dagger x}e^{-\mathrm{i}t\omega(S^{\dagger})} - \delta \overline{K} S^{x} e^{-\mathrm{i}t\omega(S)} K\right)^{-1}\overline{V}$	$k_{ij} = \frac{\boldsymbol{v}_i^{\top}\boldsymbol{v}_j}{s_i^{-1}-s_j^{*-1}}, \overline{k}_{ij} = \frac{\overline{\boldsymbol{v}}_i^{\top}\overline{\boldsymbol{v}}_j}{s_i^{*}-s_j}$, $\forall V, \overline{V}, s_i s_j^{*} \neq 1$
(17):$\overline{q} = \delta q^{\dagger}(-x,t)$	$q = \mathrm{i}\delta^{1/2} V^{\dagger}\left(\overline{S}^{-x}e^{-\mathrm{i}t\omega(\overline{S})}K^{-1} - \overline{K} S^{x} e^{-\mathrm{i}t\omega(S)}\right)^{-1}\overline{V}$	$S^{-1} - K\overline{S}K^{-1} = VV^{\dagger}$, $\overline{S}^{-1} - \overline{K}S\overline{K}^{-1} = \overline{V}\overline{V}^{\dagger}$,
		$\forall$ real $s_i, \overline{s}_j \neq 0$ and unitary matrix $K, \overline{K}$
(18):$\overline{q} = \delta q^{\top}(-x,-t)$	$q = V^{\top}\left(S^{\top x}e^{-\mathrm{i}t\omega(S^{\top})} - \delta \overline{K} S^{x} e^{-\mathrm{i}t\omega(S)} K\right)^{-1}\overline{V}$	$k_{ij} = \frac{\boldsymbol{v}_i^{\top}\boldsymbol{v}_j}{s_i^{-1}-s_j^{-1}}$, $\overline{k}_{ij} = \frac{\overline{\boldsymbol{v}}_i^{\top}\overline{\boldsymbol{v}}_j}{s_i-s_j}$, $\forall V, U, s_i \neq s_j$
(18):$\overline{q} = \delta q^{\top}(-x,-t)$	$q = \mathrm{i}\delta^{1/2} V^{\top}(\overline{S}^{-x}e^{-\mathrm{i}t\omega(\overline{S})}K^{-1} - \overline{K} S^{x} e^{-\mathrm{i}t\omega(S)})^{-1}\overline{V}$	$S^{-1} - K\overline{S}K^{-1} = VV^{\top}$, $\overline{S}^{-1} - \overline{K}S\overline{K}^{-1} = \overline{V}\overline{V}^{\top}$,
		$\forall s_i, \overline{s}_j$ and orthogonal matrix $K, \overline{K}$

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4.2. Rank One Solutions for Nonlocal NLS Equations

In this subsection, we consider rank one solutions one by one, i.e. $m = 1$. In a regular manner, we write

$$S = s,\ \ \overline{S} = \overline{s},\ \ K = k,\ \ \overline{K} = \overline{k},\ \ U = \boldsymbol{u},\ \ V = \boldsymbol{v}^{\top},\ \ \overline{V} = \overline{\boldsymbol{v}}^{\top} \tag{59}$$

where $\boldsymbol{u}, \boldsymbol{v}, \overline{\boldsymbol{v}}$ are columns and the others are complex numbers. And the parameters $\alpha, \beta, \kappa, \theta, \overline{\theta}$ below are real numbers.

4.2.1. Continuous Cases

In general, the function formed by

$$q = e^{xs+\mathrm{i}ts^2}\boldsymbol{u}\boldsymbol{v}^{\top},$$

solves all continuous cases (9)-(12) respectively with the condition

$$\boldsymbol{v}^{\top}\boldsymbol{u} = 0, \quad (\boldsymbol{v}^{\top}\boldsymbol{v})(\boldsymbol{u}^{\top}\boldsymbol{u}) = 0, \quad \boldsymbol{v}^{\dagger}\boldsymbol{u} = 0, \quad (\boldsymbol{v}^{\dagger}\boldsymbol{v})(\boldsymbol{u}^{\dagger}\boldsymbol{u}) = 0$$

where the corresponding Sylvester equation has not a unique solution. In fact, the equation is divided into the linear and nonlinear terms, each of which is respectively zero. Besides, the other type of solutions are presented in the following.

- Equation (9) (i.e. $\overline{q} = \delta q(\epsilon x, -t)$):
  For the case of $\epsilon = 1$ (i.e. $\overline{q} = \delta q(x, -t)$), we have the solution from the first line in Table 1,

$$q = \frac{\delta e^{\mathrm{i}ts^2}\boldsymbol{u}\boldsymbol{v}^{\top}}{e^{xs} - \delta k^2 e^{-xs}},\ \ k = \frac{\boldsymbol{v}^{\top}\boldsymbol{u}}{2s},\ \ s \neq 0.$$

Due to the arbitrariness of $\boldsymbol{u}$, we replace $\boldsymbol{u}$ by $\frac{2\delta^{1/2}s\mathrm{i}}{\boldsymbol{v}^{\top}\boldsymbol{u}}\boldsymbol{u}$ to secure $k^2 = -\delta$, and consequently $q = \mathrm{i}\delta^{1/2}se^{\mathrm{i}ts^2}\frac{\boldsymbol{u}\boldsymbol{v}^{\top}}{\boldsymbol{v}^{\top}\boldsymbol{u}}\operatorname{sech}(sx)$.

For the case of $\epsilon = -1$ (i.e. $\overline{q} = \delta q(-x, -t)$), we have

$$q = \frac{\mathrm{i}\delta^{1/2}\boldsymbol{u}\boldsymbol{v}^{\top}}{k(e^{x\overline{s}-\mathrm{i}t\overline{s}^2} \pm e^{-xs-\mathrm{i}ts^2})},\ \ k = \frac{\boldsymbol{v}^{\top}\boldsymbol{u}}{s+\overline{s}},\ \ s+\overline{s} \neq 0,$$

which reduces to the one-soliton solution

$$q = \alpha\mathrm{i}\delta^{1/2}e^{(\beta x+(\alpha^2-\beta^2)t)\mathrm{i}}\operatorname{sech}(\alpha x - 2\alpha\beta t)\frac{\boldsymbol{u}\boldsymbol{v}^{\top}}{\boldsymbol{v}^{\top}\boldsymbol{u}},$$

if we let $s = \alpha + \beta\mathrm{i}$, $\overline{s} = s^*$.

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- Equation (10) (i.e. $\overline{q} = \delta q^{\mathsf{T}}(\epsilon x, -t)$):
  If $\epsilon = 1$, we obtain the solution

$$q = \frac{e^{its^2} \boldsymbol{v}\overline{\boldsymbol{v}}^{\mathsf{T}}}{e^{xs} - \delta k\overline{k} e^{-xs}},$$

with $k = \frac{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v}}{2s}$, $\overline{k} = \frac{\overline{\boldsymbol{v}}^{\mathsf{T}}\overline{\boldsymbol{v}}}{2s}$, which reduces to

$$q = \frac{\mathrm{i}\delta^{1/2} s e^{its^2} \boldsymbol{v}\overline{\boldsymbol{v}}^{\mathsf{T}}}{\sqrt{(\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v})(\overline{\boldsymbol{v}}^{\mathsf{T}}\overline{\boldsymbol{v}})}} \operatorname{sech}(sx)$$

after $\boldsymbol{v} \to \frac{2\delta^{1/2} s\mathrm{i}}{\sqrt{(\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v})(\overline{\boldsymbol{v}}^{\mathsf{T}}\overline{\boldsymbol{v}})}}\boldsymbol{v}$. While if $\epsilon = -1$, we find the solution

$$q = \frac{\mathrm{i}\delta^{1/2} \boldsymbol{v}\overline{\boldsymbol{v}}^{\mathsf{T}}}{e^{x\overline{s} - \mathrm{i}t\overline{s}^2} \pm e^{-xs - \mathrm{i}ts^2}},$$

where $s + \overline{s} = \boldsymbol{v}^{\mathsf{T}}\boldsymbol{v} = \overline{\boldsymbol{v}}^{\mathsf{T}}\overline{\boldsymbol{v}}$. Let $s = \alpha + \beta\mathrm{i}$, $\overline{s} = s^*, \boldsymbol{v} \to \sqrt{\frac{2\alpha}{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v}}}\boldsymbol{v}, \overline{\boldsymbol{v}} \to \sqrt{\frac{2\alpha}{\overline{\boldsymbol{v}}^{\mathsf{T}}\overline{\boldsymbol{v}}}}\overline{\boldsymbol{v}}$, it reads the one-soliton solution

$$q = \mathrm{i}\delta^{1/2}\alpha e^{(\beta x + (\alpha^2 - \beta^2)t)\mathrm{i}} \operatorname{sech}(\alpha x - 2\alpha\beta t) \frac{\boldsymbol{v}\overline{\boldsymbol{v}}^{\mathsf{T}}}{\sqrt{(\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v})(\overline{\boldsymbol{v}}^{\mathsf{T}}\overline{\boldsymbol{v}})}}.$$

- Equation (11) (i.e. $\overline{q} = \delta q^*(\epsilon x, t)$):
  The function

$$q = \frac{\boldsymbol{u}\boldsymbol{v}^{\dagger}}{e^{xs^* - \mathrm{i}t(s^*)^2} - \delta|k|^2 e^{-xs - \mathrm{i}ts^2}},$$

with $k = \frac{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}}{s+s^*}$, $\mathrm{Re}(s) \neq 0$ solves the local NLS equation (i.e. $\epsilon = 1$). Particularly let $s = \alpha + \beta\mathrm{i}$, $|\frac{s+s^*}{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}}| = e^{\kappa}$, the one-soliton solution of the focusing NLS equation is written by

$$q = \alpha e^{(\beta x + (\alpha^2 - \beta^2)t)\mathrm{i}} \operatorname{sech}(\alpha x - 2\alpha\beta t + \kappa) \frac{\boldsymbol{u}\boldsymbol{v}^{\dagger}}{|\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}|}.$$

While the defocusing NLS equation has solutions

$$q = \alpha e^{(\beta x + (\alpha^2 - \beta^2)t)\mathrm{i}} \operatorname{csch}(\alpha x - 2\alpha\beta t + \kappa) \frac{\boldsymbol{u}\boldsymbol{v}^{\dagger}}{|\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}|},$$

with $\frac{2|\alpha|}{|\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}|} = e^{\kappa}$, and

$$q = \frac{(\alpha + \beta)\boldsymbol{u}\boldsymbol{v}^{\dagger}\mathrm{i}}{(e^{\beta(x+\beta t)\mathrm{i}} \pm e^{\alpha(-x+\alpha t)\mathrm{i}})\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}}$$

which reduces to $q = \frac{\alpha e^{-\alpha^2 t\mathrm{i}}\mathrm{i}\boldsymbol{u}\boldsymbol{v}^{\dagger}}{\cos(\alpha x)\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}}$ or $q = \frac{\alpha e^{-\alpha^2 t\mathrm{i}}\boldsymbol{u}\boldsymbol{v}^{\dagger}}{\sin(\alpha x)\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}}$ when $\alpha = \beta$.

  The function

$$q = \frac{\boldsymbol{u}\boldsymbol{v}^{\dagger}}{e^{-xs^* - \mathrm{i}t(s^*)^2} + \delta|k|^2 e^{-xs - \mathrm{i}ts^2}},$$

with $k = \frac{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}}{s-s^*}$ satisfies the nonlocal equation (11) with $\epsilon = -1$. Suppose $s = \alpha + \beta\mathrm{i}, \boldsymbol{u} \to |\frac{2\beta}{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}}|\boldsymbol{u}$, we have

$$q = \frac{2\beta e^{\alpha x + (\alpha^2 - \beta^2)t\mathrm{i}}}{e^{\beta x\mathrm{i} - 2\alpha\beta t} + \delta e^{-\beta x\mathrm{i} + 2\alpha\beta t}} \frac{\boldsymbol{u}\boldsymbol{v}^{\dagger}}{|\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}|},$$

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which implies the solutions $q = \frac{\beta e^{-\beta^2 t\mathrm{i}}}{\cos(\beta x)} \frac{\boldsymbol{u}\boldsymbol{v}^\dagger}{|\boldsymbol{v}^\top \boldsymbol{u}|}$ and $q = \frac{\beta e^{-\beta^2 t\mathrm{i}}}{\sin(\beta x)\mathrm{i}} \frac{\boldsymbol{u}\boldsymbol{v}^\dagger}{|\boldsymbol{v}^\top \boldsymbol{u}|}$ respectively for the case of $\delta = 1$ and $\delta = -1$. Besides, the focusing case has another solution formed by

$$q = \frac{(\alpha+\beta)\boldsymbol{u}\boldsymbol{v}^\dagger}{(e^{\beta(x-\beta t\mathrm{i})} \pm e^{-\alpha(x+\alpha t\mathrm{i})})\boldsymbol{v}^\top \boldsymbol{u}}$$

which reduces to $q = \alpha e^{\alpha^2 t\mathrm{i}} \operatorname{sech}(\alpha x) \frac{\boldsymbol{u}\boldsymbol{v}^\dagger}{\boldsymbol{v}^\top \boldsymbol{u}}$ when $\alpha = \beta$.

- Equation (12) (i.e. $\overline{q} = \delta q^\dagger(\epsilon x, t)$):
  For the local case, the focusing NLS equation has the solution

$$q = \alpha e^{(\beta x + (\alpha^2-\beta^2)t)\mathrm{i}} \operatorname{sech}(\alpha x - 2\alpha\beta t + \kappa) \frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^\top}{\|\boldsymbol{v}\| \|\overline{\boldsymbol{v}}\|}, \tag{60}$$

which emerges in [32]. While the defocusing NLS equation has the solutions

$$q = \alpha e^{(\beta x + (\alpha^2-\beta^2)t)\mathrm{i}} \operatorname{csch}(\alpha x - 2\alpha\beta t + \kappa) \frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^\top}{\|\boldsymbol{v}\| \|\overline{\boldsymbol{v}}\|},$$

and

$$q = \frac{\mathrm{i}(\alpha+\beta)}{e^{\mathrm{i}(-\alpha x + t\alpha^2 + \theta)} - e^{\mathrm{i}(\beta x + t\beta^2 + \overline{\theta})}} \frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^\top}{\|\boldsymbol{v}\| \|\overline{\boldsymbol{v}}\|}.$$

For the nonlocal case, we obtain the solution

$$q = \frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^\top}{e^{-xs^* - \mathrm{i}t(s^*)^2} + \delta k \overline{k} e^{-xs - \mathrm{i}ts^2}},$$

where

$$k = \frac{\boldsymbol{v}^\top \boldsymbol{v}^*}{s - s^*}, \quad \overline{k} = \frac{\overline{\boldsymbol{v}}^\top \overline{\boldsymbol{v}}^*}{s^* - s}, \quad \mathrm{Im}(s) \neq 0.$$

More concretely for the focusing equation $\overline{q} = -q^\dagger(-x, t)$, making use of $s = \alpha + \beta\mathrm{i}$, we rewrite the solution in the form

$$q = \frac{e^{\alpha(x-2\beta t) + (\beta x + \alpha^2 t - \beta^2 t)\mathrm{i}} \boldsymbol{v}^* \overline{\boldsymbol{v}}^\top}{e^{-4\alpha\beta t + 2\beta x\mathrm{i}} - e^{2\kappa}} \tag{61}$$

where $e^\kappa = \frac{\|\boldsymbol{v}\| \|\overline{\boldsymbol{v}}\|}{2|\beta|}$, $\beta \neq 0$. The choice of $\alpha = 0$, $|\kappa| \neq 0$ leads to a nonsingular solution $q = \frac{e^{\beta(x-\beta t)\mathrm{i}} \boldsymbol{v}^* \overline{\boldsymbol{v}}^\top}{e^{2\beta x\mathrm{i}} - e^{2\kappa}}$ which norm $\|q\|$ is independent of t. Besides, the focusing equation allows another solution

$$q = \frac{(\alpha+\beta)e^{-\alpha x + \mathrm{i}t\alpha^2}}{e^{-x(\alpha+\beta) + \mathrm{i}(\alpha^2 t - \beta^2 t + \overline{\theta})} - e^{\mathrm{i}\theta}} \frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^\top}{\|\boldsymbol{v}\| \|\overline{\boldsymbol{v}}\|}. \tag{62}$$

The property of the form (62) is different from that of (61). (62) has single singular position $x = 0$ at the periodic moments $t = \frac{\theta - \overline{\theta} + 2k\pi}{s^2 - \overline{s}^2}$ $(k \in \mathbb{Z})$ while (61) has a series of singular positions $x = \frac{k\pi}{\beta}$ $(k \in \mathbb{Z})$ at the single moment $t = -\frac{\kappa}{2\alpha\beta}$.

4.2.2. Semi-Discrete Cases

Similarly with the continuous case, the function formed by

$$q = s^{\pm x} e^{\mathrm{i}t\omega(s)} \boldsymbol{u}\boldsymbol{v}^\top,$$

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solves all semi-discrete cases (16)-(18) respectively with the condition

$$\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u} = 0, \quad (\boldsymbol{v}^{\dagger}\boldsymbol{v})(\boldsymbol{u}^{\dagger}\boldsymbol{u}) = 0, \quad (\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v})(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{u}) = 0,$$

where the linear and nonlinear terms are both zeros.

For the case of $\overline{q} = \delta q(x, -t)$, the solution for (16) is written by

$$q = \frac{e^{\mathrm{i}t\omega(s)}\boldsymbol{u}\boldsymbol{v}^{\mathsf{T}}}{s^{-x} - \delta k^2 s^x}$$

where $k = \frac{\boldsymbol{v}^{\mathsf{T}}\boldsymbol{u}}{s^{-1}-s}$, $s^2 \neq 1$.

For the case of $\overline{q} = \delta q^{\dagger}(-x, t)$, two types of solutions for (17) are obtained, i.e.

$$q = \frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^{\mathsf{T}}}{(s^*)^x e^{-\mathrm{i}t\omega(s^*)} - \delta k\overline{k} s^x e^{-\mathrm{i}t\omega(s)}}$$

with $k\overline{k} = \frac{|s|^2\|\boldsymbol{v}\|^2\|\overline{\boldsymbol{v}}\|^2}{|s^*-s|^2}$ and

$$q = \frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^{\mathsf{T}}}{\beta^{-x} e^{-\mathrm{i}t\omega(\beta)+\mathrm{i}\theta} - \alpha^x e^{-\mathrm{i}t\omega(\alpha)+\mathrm{i}\theta}}$$

with $\|\boldsymbol{v}\|^2\|\overline{\boldsymbol{v}}\|^2 = \delta(\alpha^{-1} - \beta)(\alpha - \beta^{-1})$. Let $\frac{|s^*-s|}{|s|\|\boldsymbol{v}\|\|\overline{\boldsymbol{v}}\|} = e^{\kappa}, s = e^{\alpha+\beta\mathrm{i}}$ in the former solution, we have

$$q = \frac{2\sin(\beta)e^{-\alpha x - 2\mathrm{i}t(1-\cos(\beta)\cosh(\alpha))}}{e^{\mathrm{i}\beta x + 2\sin(\beta)\sinh(\alpha)t - \kappa} + \delta e^{-(\mathrm{i}\beta x + 2\sin(\beta)\sinh(\alpha)t - \kappa)}} \frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^{\mathsf{T}}}{\|\boldsymbol{v}\|\|\overline{\boldsymbol{v}}\|}.$$

For the case of $\overline{q} = \delta q^{\mathsf{T}}(-x, -t)$, the equation (18) has the solution

$$q = \frac{\mathrm{i}\delta^{1/2}\boldsymbol{v}\overline{\boldsymbol{v}}^{\mathsf{T}}}{\overline{s}^{-x} e^{-\mathrm{i}t\omega(\overline{s})} \pm s^x e^{-\mathrm{i}t\omega(s)}},$$

where $\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v} = s^{-1} - \overline{s}$, $\overline{\boldsymbol{v}}^{\mathsf{T}}\overline{\boldsymbol{v}} = \overline{s}^{-1} - s$, $s\overline{s} \neq 1$. Let $\frac{|s^*-s|}{|s|\|\boldsymbol{v}\|\|\overline{\boldsymbol{v}}\|} = e^{\kappa}, s = e^{\alpha+\beta\mathrm{i}}$, we have

$$q = \mathrm{i}\delta^{1/2}\sinh(\alpha)e^{-\mathrm{i}\beta x - 2\mathrm{i}t(1-\cos(\beta)\cosh(\alpha))}\mathrm{sech}(\alpha x + 2\sin(\beta)\sinh(\alpha)t)\frac{\boldsymbol{v}^* \overline{\boldsymbol{v}}^{\mathsf{T}}}{\|\boldsymbol{v}\|\|\overline{\boldsymbol{v}}\|},$$

which interestingly satisfies $-q^{\mathsf{T}}(-x, -t) = -q^{\dagger}(x, t)$ in focusing case and corresponds to the one-soliton solution of semi-discrete local NLS equation derived in [32].

5. Conclusions

This paper has shown a reduction approach to construct solutions of nonlocal continuous and semi-discrete matrix NLS equations. Bidifferential calculus is an applicable tool for studying noncommutative integrable systems. Based on this frame, this approach is a further work of [32]. Imposing suitable nonlocal reductions respectively on the (7) and (15), nonlocal continuous NLS equations and semi-discrete NLS equations are derived respectively in the forms of (9)-(12) and (16)-(18). To admit the reduction conditions, solutions for (7) and (15) are written in different ways and then we obtain different forms and properties of solutions for nonlocal cases.

This paper gives detail forms of multi-solutions listed in Table 1 and 2 for each case, and presents rank one solutions as illustrations. One could refer to [32] to get more details operations and discussions. Our result is pretty abundant that almost every nonlocal equation corresponds more than one types of solutions. There exist relations between the direct method of bidifferential calculus and Cauchy matrix approach, and the scalar case of our result involves the Cauchy matrix type solutions presented in [25]. From Table 1 and 2, we give solutions with more free parameters in the continuous cases of $\overline{q} = \delta q(-x,-t)$, $\delta q^{\mathsf{T}}(-x,-t)$, $\delta q^{*}(\delta x,t)$, $\delta q^{\dagger}(\delta x,t)$ and the semi-discrete cases of $\overline{q} = \delta q^{\dagger}(-x,t)$, $\delta q^{\mathsf{T}}(-x,-t)$. Among them, the parameters only in the continuous case of $\overline{q} = -q(-x,-t)$, $\delta q^{\mathsf{T}}(-x,-t)$ and the semi-discrete cases of $\overline{q} = \delta q^{\mathsf{T}}(-x,-t)$ are arbitrary complex numbers, while the others are real or pure imaginary numbers. It is noted that if $q = f(x,t)$ is the solution of the defocusing case without conjugation in $\overline{q}$, then $q = \mathrm{i}f(x,t)$ satisfies the focusing equation with the same situation. Therefore the result of solutions hardly makes difference between the focusing and defocusing case in both nonlocal continuous and semi-discrete NLS equations without conjugation in $\overline{q}$. And the solutions behaves much different if $\overline{q}$ involves conjugation. For example in the case of $\overline{q} = \delta q^{\dagger}(\epsilon x,t)$ (12), the regular one-soliton solution (60) emerges only in the focusing equation but vanishes in the defocusing equation. Particularly in the continuous nonlocal cases, the transpose in $\overline{q}$ does not effect the construction of solutions, that is, the properties of nonlocal equation (9) and (11) are respectively similar with (10) and (12). Besides this work, we believe this kind of reductions is also valid on the investigations of Darboux transformations and other characters which would be considered in the future.

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