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Article

Induced Electrostatic Fields in the Presence Of Conductors and Dielectrics: A Purely Laplace Equation-Based Treatment

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Abstract: This article studies electrostatic fields and potentials in the presence of conductors and point charges under the framework of solving Laplace's equation with specified boundary conditions. The results demonstrate that many problems posed and solved in elementary electrostatics through various heuristics such as the method of images, can be more rigorously treated under the solution framework of Laplace's equation.

1. Introduction

Since the times when electricity was studied by rubbing objects like amber and pith [1], empirical formulas [2] have been proposed for the electric force created by a charge at a given distance. By the 18th century, several prominent mathematicians and natural scientists were already aware of the inverse square dependence of the force, and in 1785, French physicist Charles–Augustin de Coulomb published his famous papers [3,4] stating the law now commonly referred to as *Coulomb's law*. In modern terminology and notation, the law states that the *electric field* $\mathbf{E}$ produced at a point $\mathbf{r}$ due to a stationary charge q placed at the origin, is given by

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}},$$

or due to a volume charge density $\rho(\mathbf{r}')$ distributed over a set $\mathbf{r}' \in \mathscr{D}$ by

$$\mathbf{E}(\mathbf{r}) = \int_{\mathscr{D}} \frac{\rho}{4\pi\epsilon_0 \left\|\mathbf{r}-\mathbf{r}'\right\|^3}\left(\mathbf{r}-\mathbf{r}'\right)\, d\mathscr{V}'. \tag{1}$$

This formulation provided considerable mathematical advances in electrostatics [5], and mathematician Carl Friedrich Gauss used Coulomb's law to formulate the so-called Gauss's law or Gauss's flux theorem [4,6], which states that the electric flux through a closed surface is proportional to the total charge enclosed by the surface, or, equivalently in differential form, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, where ρ is the electrostatic charge density and $\nabla \cdot \mathbf{A}$ for a vector field $\mathbf{A}(\mathbf{r}) \equiv A_x(\mathbf{r})\hat{\mathbf{x}} + A_y(\mathbf{r})\hat{\mathbf{y}} + A_z(\mathbf{r})\hat{\mathbf{z}}$ is defined as

$$\nabla \cdot \mathbf{A} := \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

With Maxwell's unification of electromagnetism [7], it was proved that Gauss's law is more general than Coulomb's law, and continues to hold beyond electrostatics, even for time-varying fields and charge distributions, and ultimately, from the point of view of modern quantum electrodynamics [8–10], is a limitation on the degrees of freedom of the photon, the fundamental particle carrying the electromagnetic field.

An electrostatic field satisfying Coulomb's law (1) is known to be *conservative*, i.e., the line integral of $\mathbf{E}$ from a point A to a point B depends only on the vector $\mathbf{r}_{A\to B}$ from A to B. Potential theory then tells us [11] that any such electrostatic field $\mathbf{E}$ can be expressed as $\mathbf{E} = -\nabla V$ for some differentiable scalar field V. Combining this with Gauss's law yields the *Poisson equation* [12]

$$\nabla^2 V = -\rho/\epsilon_0.$$

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If a region of space has no net charge density, Poisson's equation reduces to $\nabla^2 V = 0$, which is the so-called *Laplace's equation* [13] arising in diverse application areas such as fluid flow [14], gravitation, electrodynamics [15], and general relativity and cosmology [16]. Due to the broad application and richness of Laplace's equation, it has spawned a new mathematical field of *harmonic functions* [17]. In this work, we examine some solutions of Laplace's equation with various boundary conditions in the context of calculating electric fields and potentials produced by static charge distributions.

2. Laplace's Equation and Electric Fields

We will solve Laplace's equation $\nabla^2 V = 0$ with various boundary conditions to determine the electric potentials (and thereby, the electric fields) produced by various charge distributions and conductors and dielectrics [18]. Since the boundary conditions and charge distributions we will consider will typically have some sort of spherical symmetry, it will be easier to solve Laplace's equation in the *spherical polar* coordinates rather than Cartesian coordinates. To facilitate the solution, we first state the following lemma on the form of the Laplace operator in spherical polar coordinates.

Lemma 1. *The Laplacian operator ∇^2 can be represented in spherical polar coordinates (r, θ, ϕ) as*

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}. \tag{2}$$

For completeness, we provide the proof of Lemma 1 in the Appendix.

Remark 1. *Similar to Lemma 1, one can readily establish that the Laplacian operator ∇^2 can be written in the cylindrical coordinates (ρ, ϕ, z) as*

$$\nabla^2 \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

We now establish the solution of Laplace's equation in spherical coordinates with spherical boundary conditions.

Proposition 1. *Consider Laplace's equation $\nabla^2 V = 0$ with one or more boundary conditions of the form*

$$V(r_i, \theta, \phi) \equiv f_i(\theta), \tag{4}$$

$$\left.\frac{\partial V}{\partial r}\right|_{r=r_j} \equiv g_j(\theta), \tag{5}$$

for piecewise continuous functions $\{f_i : [0, \pi] \to \mathbb{R}\}_i$ and $\{g_j : [0, \pi] \to \mathbb{R}\}_j$. Then, the solution, if it exists, is of the form

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta),$$

where the constants $\{A_l\}_{l=0}^{\infty}$ and $\{B_l\}_{l=0}^{\infty}$ are determined by the functions $\{f_i\}_i$ and $\{g_j\}_j$, and the function $P_l(\cdot)$ is the l^{th} order Legendre polynomial given as [19]

$$P_l(t) = \frac{1}{2^l l!}\frac{d^l}{dt^l}\left((t^2 - 1)^l\right).$$

In order to prove Proposition 1, we need 2 additional lemmas, whose proofs are omitted.

Lemma 2 ([20])**.** *The solution to the Laplace equation $\nabla^2 V = 0$ with boundary conditions $V(r_0, \theta, \phi) = f(\theta, \phi)$ is uniquely determined by the function f and the radius r_0 (> 0). The solution to the Laplace equation*

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with boundary conditions $\partial V/\partial r|_{r=r_0} = g(\theta,\phi)$ *is uniquely determined by the function* g *and the radius* r_0 (> 0)*, up to an additive constant.*

Lemma 3 ([21])**.** *Given any piecewise continuous function* $f(x)$ *with finitely many discontinuities in the interval* $[-1,1]$*, consider the sequence of sums*

$$f_n(x) = \sum_{l=0}^{n} a_l P_l(x),$$

where $P_l(x)$ *is the* Legendre polynomial *defined as*

$$P_l(t) = \frac{1}{2^l l!}\frac{d^l}{dt^l}\left((t^2-1)^l\right).$$

Then, we have

$$\lim_{n\to\infty}\int_{-1}^{1} |f_n(x) - f(x)|^2\, dx = 0,$$

provided we take

$$a_l = \frac{2l+1}{2}\int_{-1}^{1} f(x)P_l(x)\, dx.$$

Proof of Proposition 1. To solve Laplace's equation with the stated boundary conditions [22], we will use the so-called *separation of variables* method [23]. We note that the setting of Proposition 1 is similar to that of Lemma 2, except that the functions f_i and g_j are now functions of θ only; therefore, by Lemma 2, if we can find a solution $V(r,\theta,\phi) \equiv V(r,\theta)$ matching the boundary conditions, then that would be the unique solution, at most up to an additive constant. To this end, let us define a *trial solution* of Laplace's equation of the form $V(r,\theta) \equiv R(r)\Theta(\theta)$, where $R : \mathbb{R}^+ \to \mathbb{R}$ and $\Theta : [0,\pi] \to \mathbb{R}$ are twice differentiable functions. Using (2), the Laplace equation then reduces to

$$\frac{\Theta}{r^2}\frac{d}{dr}\left(r^2 R'(r)\right) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cdot\Theta'(\theta)\right) = 0,$$

or

$$\frac{1}{R}\frac{d}{dr}\left(r^2 R'(r)\right) + \frac{1}{\Theta\cdot\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cdot\Theta'(\theta)\right) = 0. \tag{6}$$

Now, Equation (6) has to hold for every (r,θ) in the domain, therefore, we must have

$$\frac{1}{R}\frac{d}{dr}\left(r^2 R'(r)\right) = -\frac{1}{\Theta\cdot\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cdot\Theta'(\theta)\right) = K,$$

where K is a constant independent of the coordinates (r,θ). Examining the θ equation first, we obtain, through a slight rearrangement,

$$\frac{d}{d\theta}\left(\sin\theta\cdot\Theta'(\theta)\right) + K\Theta\cdot\sin\theta = 0.$$

Writing $\xi(\cos\theta) := \Theta(\theta)$ enables us to write $\Theta'(\theta) = -\xi'(\cos\theta)\sin\theta$, and the equation therefore reduces to

$$-\frac{d}{d\theta}\left(\sin^2\theta\xi'(\cos\theta)\right) + K\xi(\cos\theta)\sin\theta = 0.$$

Now, dividing throughout by $\sin\theta$, writing $u := \cos\theta$, and noting that

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin^2\theta\xi'(\cos\theta)\right) = \frac{d}{du}\left(\sin^2\theta\xi'(u)\right) = \frac{d}{du}\left((1-u^2)\xi'(u)\right),$$

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the θ equation becomes, in terms of $u := \cos\theta$ as the independent variable,

$$\frac{d}{du}\left((1-u^2)\theta\xi'(u)\right) + K\xi(u) = 0. \tag{7}$$

Equation (7) is the Legendre equation [19] with $K = l(l+1)$. For non-integer $l \in \mathbb{R}$, the solutions to Legendre's equation are power series with radius of convergenece smaller than 1. For the current problem, however, the domain is $u \in [-1,1]$. Therefore, the only possible solutions to (7) that would make the trial solution valid, should be polynomials. Through the application of Sturm–Liouville theory (see, for example, [21]), we can conclude that the solutions are indeed polynomials when l is a non-negative integer. Therefore, for the trial solution to be valid, we must have $K = l(l+1)$ for a non-negative integer l, and the corresponding solution of the θ equation becomes

$$\Theta_l(\theta) = P_l(\cos\theta) \tag{8}$$

up to a multiplicative constant. To tackle the r equation, writing $\zeta(\log r) := R(r)$ enables us to write $\zeta'(\log r) = rR'(r)$, and a change of independent variable to $v := \log r$ enables us to write the v equation as

$$\frac{d}{dr}\left(r\zeta'(v)\right) - l(l+1)\zeta(v) = 0,$$

and replacing r with e^v leads to

$$e^{-v}\frac{d}{dv}\left(e^v\zeta'(v)\right) - l(l+1)\zeta(v) = 0,$$

which simplifies to

$$\zeta''(v) + \zeta'(v) - l(l+1)\zeta(v) = 0. \tag{9}$$

From elementary calculus, Equation (9) has the general solution of the form

$$\zeta(v) = A \cdot \exp(\alpha v) + A \cdot \exp(\beta v),$$

where α and β are solutions of the quadratic equation $x^2 + x - l(l+1) = 0$. We immediately obtain $\alpha = l$ and $\beta = -(l+1)$, which yields the solution

$$\zeta(v) = A \cdot \exp(lv) + B \cdot \exp\left(-(l+1)v\right),$$

and the solution to the r equation then becomes

$$R_l(r) = A_l r^l + \frac{B_l}{r^{l+1}}. \tag{10}$$

Finally, noting that since the Laplace equation is linear, any linear combination of a set of valid solutions will also be a solution, we come up with the largest set of soltuions that the trial solution enables us to get, by combining (10) and (8):

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta). \tag{11}$$

Now, by uniqueness (Lemma 2), if we can find constants $\{A_l, B_l\}$ matching the boundary conditions (4) and/or (5), then (11) with the determined constants will be the unique solution to the problem. Let us now focus on the boundary conditions (4). The result will similarly follow for the boundary

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conditions (5). Since f_i is a piecewise continuous function, by the completeness of Legendre polynomials (Lemma 3), there exist constants $\{A_l, B_l\}$ for which

$$f_i(\theta) = \sum_{l=0}^{\infty} \left(A_l r_i^l + \frac{B_l}{r_i^{l+1}} \right) P_l(\cos\theta).$$

More specifically, we have, in this case,

$$A_l r_i^l + \frac{B_l}{r_i^{l+1}} = \frac{2l+1}{2} \int_0^{\pi} f_i(\theta) P_l(\cos\theta) \sin\theta \, d\theta. \tag{12}$$

Note that at least 2 such boundary conditions are needed to uniquely determine the constants A_l and B_l. For such *consistent* boundary conditions, (11) with the constants A_l and B_l determined by the boundary conditions (12) is the unique solution to the Laplace equation. □

We will now directly use Proposition 1 to calculate electric potentials (and, thereby, electric fields) rigorously for various electrostatics problems commonly posed in many textbooks (see, for example, [15]).

2.1. Conducting Sphere in Uniform Electric Field

Consider a conducting sphere of radius $a > 0$ (with center at the origin) placed in a uniform electric field $\mathbf{E}_0$. Without loss of generality, let $\mathbf{E}_0 = E_0\hat{\mathbf{z}} = E_0 \cos\theta\hat{\mathbf{r}} - E_0 \sin\theta\hat{\boldsymbol{\theta}}$. If a unique solution exists in the region $r > a$, it must only be determined by the vector $\mathbf{E}_0$, and therefore, $V(r, \theta, \phi)$ must be independent of ϕ. (This property is referred to as *azimuthal symmetry*.) We can then write a trial solution as

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta),$$

which yields

$$\frac{\partial V}{\partial r} = \sum_{l=0}^{\infty} \left(l A_l r^{l-1} - \frac{(l+1)B_l}{r^{l+2}} \right) P_l(\cos\theta).$$

Since the conductor is finite, the distortion caused by it to the electric field [24,25] is *local* and therefore, as $r \to \infty$, the electric field must approach $\mathbf{E}_0$. We thus have $\lim_{r\to\infty} \partial V/\partial r = -E_0 \cos\theta$, which is only possible if $A_l = 0$ for $l \geq 2$, and $A_1 = -E_0$. The solution then becomes

$$V(r, \theta, \phi) = A_0 + \frac{B_0}{r} + \left(\frac{B_1}{r^2} - E_0 r \right) \cos\theta + \sum_{l=2}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{13}$$

for some constants $A_0, B_0, B_1, \ldots$. Now, for a conductor, the electric field at the surface is purely along the normal to the surface, and therefore, we have

$$\frac{\partial V}{\partial \theta}|_{r=a} = 0,$$

which, combining with (13), yields

$$\left(\frac{B_1}{a^2} - E_0 a \right) \sin\theta + \sin\theta \sum_{l=2}^{\infty} \frac{B_l}{a^{l+1}} P_l'(\cos\theta) = 0$$

for $\theta \in [0, \pi]$. This is satisfied if $B_1 = E_0 a^3$ and $B_l = 0$ for $l \geq 2$, and the solution therefore becomes

$$V(r, \theta, \phi) = A_0 + \frac{B_0}{r} + \left(\frac{a^3}{r^2} - r \right) E_0 \cos\theta,$$

which yields the electric field

$$\begin{aligned}
\mathbf{E}(r,\theta,\phi) &= -\nabla V \\
&= -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} \\
&= \left(\frac{B_0}{r^2} + \left(\frac{2a^3}{r^3} + 1\right) E_0 \cos\theta\right)\hat{\mathbf{r}} + \left(\frac{a^3}{r^3} - 1\right) E_0 \sin\theta\hat{\boldsymbol{\theta}} \\
&= \mathbf{E}_0 + \frac{a^3}{r^3}\left(2(\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - (\mathbf{E}_0\cdot\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}}\right) + \frac{B_0}{r^2}\hat{\mathbf{r}} \\
&= \mathbf{E}_0 + \frac{a^3}{r^3}\left(2(\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - [\mathbf{E}_0 - (\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\right) + \frac{B_0}{r^2}\hat{\mathbf{r}} \\
&= \mathbf{E}_0 + \frac{a^3}{r^3}\left(3(\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{E}_0\right) + \frac{B_0}{r^2}\hat{\mathbf{r}} \\
&= \left(1 - \frac{a^3}{r^3}\right)\mathbf{E}_0 + \left(\frac{3a^3}{r^3}(\mathbf{E}_0\cdot\hat{\mathbf{r}}) + \frac{B_0}{r^2}\right)\hat{\mathbf{r}}. \qquad (14)
\end{aligned}$$

Note that (14) satisfies $\lim_{r\to\infty}\mathbf{E}(r,\theta,\phi) = \mathbf{E}_0$, as required. However, the constant B_0 is still undetermined. It is a measure of the *state* of the conductor, as can be seen from the following. From (14), we have

$$\begin{aligned}
E_r(a,\theta,\phi) &= 3(\mathbf{E}_0\cdot\hat{\mathbf{r}}) + \frac{B_0}{a^2} \\
&= 3E_0\cos\theta + \frac{B_0}{a^2}. \qquad (15)
\end{aligned}$$

Since the electric field inside a conductor is zero, we have that the surface charge density induced on the sphere is given by $\sigma(\theta,\phi) = \epsilon_0 E_r(a,\theta,\phi) = 3\epsilon_0 E_0\cos\theta + \frac{\epsilon_0 B_0}{a^2}$. The *total* charge on the conductor is then given by

$$\begin{aligned}
Q &= a^2\int_0^{\pi}\int_0^{2\pi}\sigma(\theta,\phi)\sin\theta d\theta d\phi \\
&= 4\pi\epsilon_0 B_0, \qquad (16)
\end{aligned}$$

which enables us to finally write the electric field in terms of the physical invariants of the system as

$$\mathbf{E} = \left(1 - \frac{a^3}{r^3}\right)\mathbf{E}_0 + \left(\frac{3a^3}{r^3}(\mathbf{E}_0\cdot\hat{\mathbf{r}}) + \frac{Q}{4\pi\epsilon_0 r^2}\right)\hat{\mathbf{r}}.$$

This essentially says that the total net charge on the conductor gets distributed in such a way as to produce the same field (outside the conductor) as a point charge placed at the center. In particular, if we place an uncharged conductor inside a uniform electrostatic field $\mathbf{E}_0$, then the final electrostatic field will be given by

$$\mathbf{E} = \left(1 - \frac{a^3}{r^3}\right)\mathbf{E}_0 + \frac{3a^3}{r^3}(\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}. \qquad (17)$$

We finally note that this expression for the electric field is "coordinate free" in the sense that it is only a function of the vector $\mathbf{E}_0$ and the radius vector $\mathbf{r}$ at each point, the latter of which essentially encodes the position of the sphere.

Figure 1 illustrates the electric field lines on a vertical plane (i.e., containing the z axis) obtained from the electric field in (17). As expected, far away from the sphere, the lines are vertical, while close by, they are "distorted" by the presence of the conductor. We also note that the field lines always enter and leave the sphere normally, and field abruptly drops to 0 as soon as we cross the boundary into

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the conductor. Finally, the field lines are seen to be the densest around the poles of the conductor $(r = a, \theta = 0, \pi)$, which can also be verified analytically from Equation (17). An analytical derivation yields a maximum field strength of $3\mathbf{E}_0$.

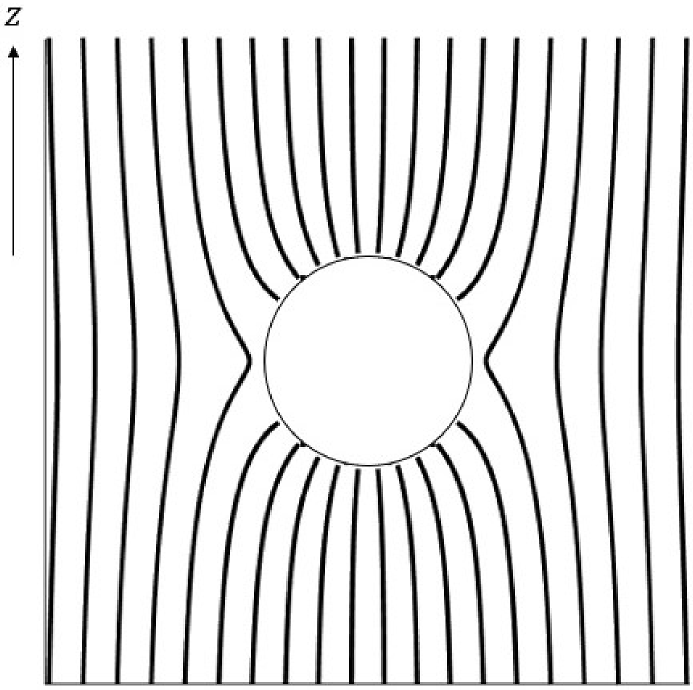

Figure 1. Electric field lines.

2.2. Conducting Sphere Near a Point Charge

Consider a conducting sphere of radius $a > 0$ (with center at the origin) placed near a point charge q located at $r_0\hat{\mathbf{z}}$ with $r_0 > a$ (see Figure 2). Since the conducting sphere is an equipotential, let it be at potential V_0. The overall potential $V(r, \theta, \phi)$ [26] for $r > a$ can be written as $V(r, \theta, \phi) = V_q(r, \theta, \phi) + \tilde{V}(r, \theta, \phi)$, where V_q is the potential due to the charge q and $\tilde{V}$ satisfies Laplace equation with appropriate boundary conditions.

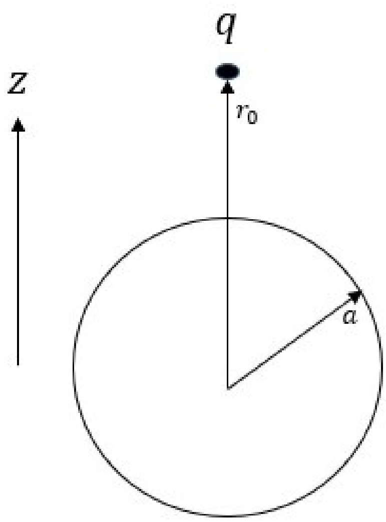

Figure 2. Conducting sphere near a point charge.

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To determine these boundary conditions, we note that the potential due to the charge q at a point (a, θ, ϕ) on the sphere (assuming the boundary conditions $\lim_{r\to\infty} V(r, \theta, \phi) = 0$) is given by Coulomb's law as

$$V_q(a, \theta, \phi) = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{\left(r_0^2 - 2r_0 a \cos\theta + a^2\right)^{1/2}}.$$

Therefore, $\tilde{V}$ satisfies the boundary condition

$$\begin{aligned} \tilde{V}(a, \theta, \phi) &= V_0 - \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{\left(r_0^2 - 2r_0 a \cos\theta + a^2\right)^{1/2}} \\ &= V_0 - \frac{q}{4\pi\epsilon_0 r_0} \cdot \frac{1}{\left(1 - 2\frac{a}{r_0}\cos\theta + \frac{a^2}{r_0^2}\right)^{1/2}}. \end{aligned} \tag{18}$$

Writing a trial solution $\tilde{V}$ as

$$\tilde{V}(r, \theta, \phi) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta),$$

we observe that since we have the boundary conditions $\lim_{r\to\infty} \tilde{V}(r, \theta, \phi) = 0$, we must have $A_l = 0$ for all $l \geq 0$. To obtain the coefficients B_l using (18), we need the following lemma.

Lemma 4 ([13]). *The series*

$$\sum_{l=0}^{\infty} P_l(x) t^l$$

converges absolutely for $x \in [-1, 1]$ and $t \in (0, 1)$, and

$$\sum_{l=0}^{\infty} P_l(x) t^l = \frac{1}{\left(1 - 2tx + t^2\right)^{1/2}}. \tag{19}$$

Remark 2. *The expansion (19) is the basis for the so-called "multi-pole" expansion (see, for example, [27]).*

Since we have $|\cos\theta| \leq 1$ and $a < r_0$, we can use Lemma 4 to rewrite (18) as

$$\begin{aligned} \tilde{V}(a, \theta, \phi) &= V_0 - \frac{q}{4\pi\epsilon_0 r_0} \sum_{l=0}^{\infty} \left(\frac{a}{r_0}\right)^l P_l(\cos\theta) \\ &= V_0 - \frac{qa/r_0}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left(\frac{a^2}{r_0}\right)^l \frac{1}{a^{l+1}} P_l(\cos\theta). \end{aligned} \tag{20}$$

Therefore, by the uniqueness theorem (Lemma 2), the coefficients $\{B_l\}_l$ are given by

$$B_0 = aV_0 - \frac{qa}{4\pi\epsilon_0 r_0}, \tag{21}$$

$$B_l = -\frac{qa}{4\pi\epsilon_0 r_0} \left(\frac{a^2}{r_0}\right)^l \quad \text{for } l \geq 1. \tag{22}$$

The general solution $\tilde{V}$ is thus given by

$$\tilde{V}(r, \theta, \phi) = \frac{aV_0}{r} - \frac{qa}{4\pi\epsilon_0 r_0} \sum_{l=0}^{\infty} \left(\frac{a^2}{r_0}\right)^l \frac{1}{r^{l+1}} P_l(\cos\theta) \tag{23}$$

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for $r > a$. Equation (23) can be written in a more illuminating form as

$$\begin{aligned}
\tilde{V}(r,\theta,\phi) &= \frac{aV_0}{r} - \frac{qa/r_0}{4\pi\epsilon_0 r}\sum_{l=0}^{\infty}\left(\frac{a^2}{rr_0}\right)^l P_l(\cos\theta) \\
&\overset{(19)}{=} \frac{aV_0}{r} - \frac{qa/r_0}{4\pi\epsilon_0 r}\cdot\frac{1}{\left(1-\frac{2a^2}{rr_0}\cos\theta+\left(\frac{a^2}{rr_0}\right)^2\right)^{1/2}} \\
&= \frac{aV_0}{r} - \frac{qa/r_0}{4\pi\epsilon_0}\cdot\frac{1}{\left(r^2-2r\cdot\frac{a^2}{r_0}\cos\theta+\left(\frac{a^2}{r_0}\right)^2\right)^{1/2}}. \qquad (24)
\end{aligned}$$

The overall potential is then given by

$$\begin{aligned}
V(r,\theta,\phi) &= V_q(r,\theta,\phi)+\tilde{V}(r,\theta,\phi) \\
&= \frac{q}{4\pi\epsilon_0}\cdot\frac{1}{\left(r^2-2rr_0\cos\theta+r_0^2\right)^{1/2}} - \frac{qa/r_0}{4\pi\epsilon_0}\cdot\frac{1}{\left(r^2-2r\cdot\frac{a^2}{r_0}\cos\theta+\left(\frac{a^2}{r_0}\right)^2\right)^{1/2}} + \frac{aV_0}{r}. \qquad (25)
\end{aligned}$$

The potential (25) is the same as that produced by the charge q at $z = r_0$, together with "virtual" charges $-qa/r_0$ at $z = a^2/r_0$ and $4\pi\epsilon_0 aV_0$ at the center of the conducting sphere [28]. This can also be obtained through the so-called *method of images* as used in many treatments of this topic [15]. Equation (25) also enables us to write the surface charge density on the sphere as

$$\begin{aligned}
\sigma(\theta,\phi) &= -\epsilon_0\frac{\partial V}{\partial r}|_{r=a} \\
&= \left[\frac{q}{4\pi}\cdot\frac{r-r_0\cos\theta}{\left(r^2-2rr_0\cos\theta+r_0^2\right)^{3/2}} - \frac{qa/r_0}{4\pi}\cdot\frac{r-\frac{a^2}{r_0}\cos\theta}{\left(r^2-2r\cdot\frac{a^2}{r_0}\cos\theta+\left(\frac{a^2}{r_0}\right)^2\right)^{3/2}} + \frac{a\epsilon_0 V_0}{r^2}\right]_{r=a} \\
&= \left(\frac{q}{4\pi}\cdot\frac{a-r_0\cos\theta}{\left(a^2-2ar_0\cos\theta+r_0^2\right)^{3/2}}\right) - \left(\frac{qa/r_0}{4\pi}\cdot\frac{a-\frac{a^2}{r_0}\cos\theta}{\left(a^2-2a\cdot\frac{a^2}{r_0}\cos\theta+\left(\frac{a^2}{r_0}\right)^2\right)^{3/2}}\right) + \frac{\epsilon_0 V_0}{a} \\
&= \left(\frac{q}{4\pi}\cdot\frac{a-r_0\cos\theta}{\left(a^2-2ar_0\cos\theta+r_0^2\right)^{3/2}}\right) - \left(\frac{qr_0^2}{4\pi a^2}\cdot\frac{a-\frac{a^2}{r_0}\cos\theta}{\left(a^2-2ar_0\cos\theta+r_0^2\right)^{3/2}}\right) + \frac{\epsilon_0 V_0}{a} \\
&= \frac{\epsilon_0 V_0}{a} - \left(\frac{q}{4\pi a^2}\cdot\frac{a^2(r_0\cos\theta-a)+ar_0(r_0-a\cos\theta)}{\left(a^2-2ar_0\cos\theta+r_0^2\right)^{3/2}}\right) \\
&= \frac{\epsilon_0 V_0}{a} - \left(\frac{q}{4\pi a}\cdot\frac{r_0^2-a^2}{\left(a^2-2ar_0\cos\theta+r_0^2\right)^{3/2}}\right). \qquad (26)
\end{aligned}$$

The charge density (26) can be integrated to obtain the total charge on the sphere as

$$\begin{aligned}
Q_{\text{tot}} &= a^2\int_0^{\pi}\int_0^{2\pi}\sigma(\theta,\phi)\sin\theta\, d\phi\, d\theta \\
&= 4\pi\epsilon_0 aV_0 - \frac{aq}{2}\int_0^{\pi}\frac{(r_0^2-a^2)}{\left(a^2-2ar_0\cos\theta+r_0^2\right)^{3/2}}\sin\theta\, d\theta. \qquad (27)
\end{aligned}$$

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Writing $u := \left(a^2 - 2ar_0\cos\theta + r_0^2\right)^{1/2}$, we have

$$u\frac{du}{d\theta} = ar_0\sin\theta. \tag{28}$$

We can then continue (27) as

$$\begin{aligned}Q_{\text{tot}} &= 4\pi\epsilon_0 aV_0 - \frac{q(r_0^2 - a^2)}{2r_0}\int_{r_0-a}^{r_0+a}\frac{1}{u^2}\,du \\ &= 4\pi\epsilon_0 aV_0 - \frac{qa}{r_0}.\end{aligned} \tag{29}$$

Equation (29) is often interpreted (see, for example, [15,27]) as the sum of the *image charges* $-\frac{qa}{r_0}$ and $4\pi\epsilon_0 aV_0$.

2.3. Uniform Dielectric Sphere in Uniform Electric Field

Consider a sphere of radius $a > 0$, made of a *linear deielectric material* of electric susceptibility χ (with center at the origin) placed in a uniform electric field $\mathbf{E}_0$. Without loss of generality, let $\mathbf{E}_0 = E_0\hat{\mathbf{z}} = E_0\cos\theta\hat{\mathbf{r}} - E_0\sin\theta\hat{\boldsymbol{\theta}}$. We now wish to find the field both inside and outside the sphere. As we shall see later (cf. Remark 4), this situation can be thought of as a generalization to the problem studied in Section 2.1. As argued in Section 2.1, this problem exhibits azimuthal symmetry since the boundary conditions are independent of ϕ. We can then write trial solutions for the potentials inside and outside the sphere as

$$V^{\text{in}}(r,\theta,\phi) = \sum_{l=0}^{\infty}\left(A_l^{\text{in}}r^l + \frac{B_l^{\text{in}}}{r^{l+1}}\right)P_l(\cos\theta), \tag{30}$$

$$V^{\text{out}}(r,\theta,\phi) = \sum_{l=0}^{\infty}\left(A_l^{\text{out}}r^l + \frac{B_l^{\text{out}}}{r^{l+1}}\right)P_l(\cos\theta), \tag{31}$$

We will now discuss the boundary conditions to this problems, which are what will turn out to cause the crucial differences in the solutions for this case. First, as argued in Section 2.1, the electric field must approach $\mathbf{E}_0$ as $r \to \infty$, and we therefore conclude that $A_l^{\text{out}} = 0$ for $l \geq 2$, and $A_1^{\text{out}} = -E_0$. Further, since the field inside the sphere is finite, V^{in} must have finite derivatives as $r \to 0$, and (30) then leads us to conclude that $B_l^{\text{in}} = 0$ for all l. Equations (30) and (31) then enable us to write down a global trial solution as

$$V(r,\theta,\phi) = \begin{cases}\sum_{l=0}^{\infty}A_l^{\text{in}}r^l P_l(\cos\theta), & r < a, \\ A_0^{\text{out}} + \frac{B_0^{\text{out}}}{r} + \left(\frac{B_1^{\text{out}}}{r^2} - E_0 r\right)\cos\theta + \sum_{l=2}^{\infty}\frac{B_l^{\text{out}}}{r^{l+1}}P_l(\cos\theta), & r > a.\end{cases} \tag{32}$$

By the continuity of V at the surface of the dielectric, the two expressions in (32) must match at $r = a$. Rearrangement leads to

$$\left(A_0^{\text{out}} + \frac{B_0^{\text{out}}}{a} - A_0^{\text{in}}\right) + \left(\frac{B_1^{\text{out}}}{a^2} - E_0 a - A_1^{\text{in}}a\right)\cos\theta + \sum_{l=2}^{\infty}\left(\frac{B_l^{\text{out}}}{a^{l+1}} - A_l^{\text{in}}a^l\right)P_l(\cos\theta) = 0. \tag{33}$$

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Using the completeness of Legendre polynomials (Lemma 3) and noting that $P_1(\cos\theta) = \cos\theta$, Equation (33) enables us to write

$$A_0^{\text{out}} + \frac{B_0^{\text{out}}}{a} = A_0^{\text{in}} =: V_0, \tag{34}$$

$$A_1^{\text{in}} = \frac{B_1^{\text{out}}}{a^3} - E_0, \tag{35}$$

$$A_l^{\text{in}} = \frac{B_l^{\text{out}}}{a^{2l+1}}, \; l \geq 2. \tag{36}$$

Finally, we come to the behavior of the electric field at the surface of the dielectric, which will yield additional boundary conditions and will be determined by the properties of the dielectric. The surface density of bound charges on the dielectric can be written as

$$\sigma(\theta, \phi) = \epsilon_0 \left[\frac{\partial V^{\text{in}}}{\partial r} - \frac{\partial V^{\text{out}}}{\partial r} \right]_{r=a}. \tag{37}$$

This bound charge density, however, is *also* given by the radial component of the polarization vector $\mathbf{P}$, i.e., $\sigma(\theta, \phi) = [\mathbf{P} \cdot \hat{\mathbf{r}}]_{r=a}$. Finally, since the material is a linear dielectric, we have $\mathbf{P}(\mathbf{r}) = \chi\epsilon_0 \mathbf{E}(\mathbf{r})$ for $\|\mathbf{r}\| < a$. We therefore have

$$\sigma(\theta, \phi) = -\chi\epsilon_0 \cdot \frac{\partial V^{\text{in}}}{\partial r}|_{r=a}. \tag{38}$$

Equations (37) and (38) enable us to write

$$\left[(1+\chi)\frac{\partial V^{\text{in}}}{\partial r} - \frac{\partial V^{\text{out}}}{\partial r} \right]_{r=a} = 0. \tag{39}$$

Using Lemma 3 again, the boundary condition (39) enables us to write

$$B_0^{\text{out}} = 0, \tag{40}$$

$$A_1^{\text{in}} = -\frac{1}{1+\chi}\left(\frac{2B_1^{\text{out}}}{a^3} + E_0 \right), \tag{41}$$

$$A_l^{\text{in}} = -\frac{(l+1)B_l^{\text{out}}}{l(1+\xi)a^{2l+1}}, \; l \geq 2. \tag{42}$$

For $l \geq 2$, (36) and (42) can simultaneously hold only if $A_l^{\text{in}} = B_l^{\text{out}} = 0$. Moreover, (35) and (41) enable us to solve for B_1^{out} as

$$\frac{B_1^{\text{out}}}{a^3} - E_0 = -\frac{1}{1+\chi}\left(\frac{2B_1^{\text{out}}}{a^3} + E_0 \right),$$

i.e.,

$$B_1^{\text{out}} = a^3 \frac{\left(1 - \frac{1}{1+\chi}\right) E_0}{1 + \frac{2}{1+\chi}} = \frac{\chi}{\chi+3} a^3 E_0.$$

This immediately yields

$$A_1^{\text{in}} = -\frac{3}{\chi+3} E_0,$$

and we can finally write down the full solution (32) as

$$V(r, \theta, \phi) = \begin{cases} V_0 - \frac{3}{\chi+3} E_0 r \cos\theta, & r < a, \\ V_0 + \left(\left(\frac{\chi}{\chi+3} \right) \frac{a^3}{r^2} - r \right) E_0 \cos\theta, & r > a. \end{cases} \tag{43}$$

The potential (43) yields the electric field

$$\begin{aligned}
\mathbf{E}(r,\theta,\phi) &= -\nabla V \\
&= -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} \\
&= \begin{cases} \left(\frac{3}{\chi+3}\right) E_0 \cos\theta\hat{\mathbf{r}} - \left(\frac{3}{\chi+3}\right) E_0 \sin\theta\hat{\boldsymbol{\theta}}, & r < a \\ \left(1 + \frac{2\chi a^3}{(\chi+3)r^3}\right) E_0 \cos\theta\hat{\mathbf{r}} + \left(\left(\frac{\chi}{\chi+3}\right)\frac{a^3}{r^3} - 1\right) E_0 \sin\theta\hat{\boldsymbol{\theta}}, & r \geq a. \end{cases}
\end{aligned}$$

The electric field inside can then be written as

$$\mathbf{E}^{\text{in}} = \left(\frac{3}{\chi+3}\right) E_0 \left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) = \left(\frac{3}{\chi+3}\right)\mathbf{E}_0, \tag{44}$$

which turns out to be uniform. The field outside can be written as

$$\begin{aligned}
\mathbf{E}^{\text{out}} &= \mathbf{E}_0 + \frac{a^3}{r^3}\left(2\left(\frac{\chi}{\chi+3}\right)(\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \left(\frac{\chi}{\chi+3}\right)(\mathbf{E}_0\cdot\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}}\right) \\
&= \mathbf{E}_0 + \frac{a^3}{r^3}\left(\frac{\chi}{\chi+3}\right)\left(2(\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - [\mathbf{E}_0 - (\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\right) \\
&= \mathbf{E}_0 + \frac{a^3}{r^3}\left(\frac{\chi}{\chi+3}\right)\left(3(\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{E}_0\right) && (45) \\
&= \left(1 - \frac{\chi a^3}{(\chi+3)r^3}\right)\mathbf{E}_0 + \left(\frac{3\chi a^3}{(\chi+3)r^3}(\mathbf{E}_0\cdot\hat{\mathbf{r}})\right)\hat{\mathbf{r}}. && (46)
\end{aligned}$$

Figure 3 illustrates the electric field lines on a vertical plane (i.e., containing the z axis) obtained from the electric field in (46) and (44). As expected, far away from the sphere, the lines are vertical, while close by, they are "distorted" by the presence of the dielectric. We also note that there is a discontinuity of the field at the boundary of the dielectric, but unlike Figure 1 in Section 2.1, the field inside the dielectric is nonzero. Finally, the field lines are no longer orthogonal to the surface of the dielectric; the field can have a tangential component and electrostatic conditions can still be maintained.

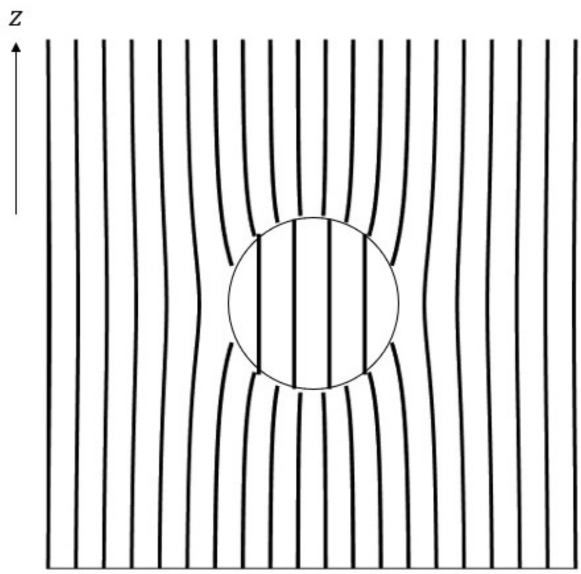

Figure 3. Electric field lines.

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Remark 3. *We note that the field when an uncharged conductor is placed in a uniform electric field (cf.* (17) *in Section* 2.1*) can also be written as*

$$\mathbf{E} = \mathbf{E}_0 + \frac{a^3}{r^3}\left(3(\mathbf{E}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{E}_0\right). \tag{47}$$

Both (47) *and* (45) *can be compared to the field of a dipole* $\mathbf{p}$ *placed at the origin (see, for example, [15]):*

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}\left(3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right).$$

This suggests that a sphere made of a conductor or a linear dielectric material placed in an uniform electrostatic field acquires a dipole moment equal to

$$\begin{aligned}\mathbf{p}^{\text{cond.}} &= 4\pi\epsilon_0 a^3 \mathbf{E}_0, \\ \mathbf{p}^{\text{diel.}} &= 4\pi\epsilon_0 a^3 \left(\frac{\chi}{\chi+3}\right)\mathbf{E}_0.\end{aligned} \tag{48}$$

The polarization $\mathbf{P}$ *is defined as dipole moment per unit volume, therefore this suggests that the spheres acquire polarizations* $\mathbf{P}^{\text{cond.}} = 3\epsilon_0\mathbf{E}_0$ *and* $\mathbf{P}^{\text{diel.}} = 3(\chi/(\chi+3))\epsilon_0\mathbf{E}_0$*. The electric susceptibility* χ*, therefore, can be interpreted as a parameter quantifying the amount of "freedom of movement" of charges inside a metrial – in a conductor they have perfect freedom of movement (i.e., ability to polarize), enabling complete cancellation of the electric field, but in a dielectric sphere, the polarization is* $\chi/(\chi+3)$ *times relative to a conductor.*

Remark 4. *We note that all results in Section* 2.1 *follow from those in this section by taking* $\chi \to \infty$*. This shows that in electrostatics, in many situations a conductor can be thought of as a dielectric within infinite susceptibility. Taking this limit for* (44) *for the field inside the dielectric immediately recovers the result that the field is zero inside the conductor.*

Remark 5. *Note that in this section, we tacitly assumed that the dielectric is uncharged when we equated* (37) *and* (38)*. The dielectric, in fact, could have a "free" surface charge density* $\sigma_{\text{f}}(\theta,\phi)$ *which would then have to be taken into account at that step. However, because charges cannot move about freely in a dielectric, unlike in a conductor, the complete charge density* $\sigma_{\text{f}}(\theta,\phi)$ *would have to be specified for this problem to have a unique solution.*

3. Discussions

The general setup described in this work can be used to find potentials and fields for a wide range of charge distributions commonly used to demonstrate the "method of images" in electrostatics texts. A popular and elementary problem setup not explored in this work is the *infinite conducting plane* near a point charge q, which can be similarly solved, but in the *cylindrical coordinates* using (3). For a detailed account of the history and development of Laplace's equation and solution techniques in the context of electrostatics, we refer the reader to [1].

Appendix A Proofs

Proof of Lemma 1. We first establish some preliminary results that will make the proof considerably easier. The relations between Cartesian and spherical polar coordinates can be written [29] as

$$(r,\theta,\phi) \mapsto (x,y,z) : \begin{cases} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{cases} \tag{A1}$$

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and

$$(x,y,z) \mapsto (r,\theta,\phi) : \begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \operatorname{acos}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \phi &= \operatorname{atan2}(y,x), \end{cases} \tag{A2}$$

where $\operatorname{atan2}(y,x)$ is the *four-quadrant inverse tangent* defined as the real number λ satisfying $\cos\lambda = x/\sqrt{x^2+y^2}$ and $\sin\lambda = y/\sqrt{x^2+y^2}$. Treating the spherical unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ as functions of (r,θ,ϕ), we can then write

$$\begin{aligned} \hat{\mathbf{r}} &= \frac{\mathbf{r}}{r} \\ &= \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + x\hat{\mathbf{z}}}{r} \\ &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}, \end{aligned} \tag{A3}$$

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= \frac{\frac{\partial\hat{\mathbf{r}}}{\partial\theta}}{\left\|\frac{\partial\hat{\mathbf{r}}}{\partial\theta}\right\|} \\ &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}, \end{aligned} \tag{A4}$$

and

$$\begin{aligned} \hat{\boldsymbol{\phi}} &= \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \\ &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}. \end{aligned} \tag{A5}$$

Equations (A3)–(A5) can be easily inverted to yield

$$\begin{aligned} \hat{\mathbf{x}} &= \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}, \\ \hat{\mathbf{y}} &= \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}}, \\ \hat{\mathbf{z}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}, \end{aligned}$$

or, more compactly,

$$\begin{aligned} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} &= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} \\ &=: M(r,\theta,\phi) \cdot \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}. \end{aligned} \tag{A6}$$

One can verify that the matrix M is orthogonal, i.e., $M^{-1} = M^T$. Finally, the infinitesimal line element can be written in spherical spherical polar coordinates as

$$\begin{aligned} d\mathbf{r} &= d(r\hat{\mathbf{r}}) \\ &= dr\hat{\mathbf{r}} + r\frac{\partial\hat{\mathbf{r}}}{\partial\theta}d\theta + r\frac{\partial\hat{\mathbf{r}}}{\partial\phi}d\phi \\ &\overset{(a)}{=} dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}, \end{aligned} \tag{A7}$$

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where (a) follows from Equations (A3)–(A5). This also leads us to conclude immediately, that the infinitesimal volume element (since $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are orthonormal) is given by

$$d\mathscr{V} = r^2 \sin\theta \cdot dr \cdot d\theta \cdot d\phi.$$

Now, with these preliminaries established, let f be a differentiable scalar field. We have

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi. \tag{A8}$$

Now, writing $\nabla f \equiv g^{(r)}(r,\theta,\phi)\hat{\mathbf{r}} + g^{(\theta)}(r,\theta,\phi)\hat{\boldsymbol{\theta}} + g^{(\phi)}(r,\theta,\phi)\hat{\boldsymbol{\phi}}$ and using the gradient theorem [11] $df = \nabla f \cdot d\mathbf{r}$, (A7), and (A8), we have

$$g^{(r)}(r,\theta,\phi)dr + \left(rg^{(\theta)}(r,\theta,\phi)\right) d\theta + \left(r\sin\theta \cdot g^{(\phi)}(r,\theta,\phi)\right) d\phi \equiv \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi.$$

This implies that

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}. \tag{A9}$$

Equation (A9) also implies that

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{1}{r}\frac{\partial f}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix},$$

or in other words,

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = M \cdot \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r}\frac{\partial f}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \end{bmatrix}. \tag{A10}$$

We can write (A10) as the three scalar equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\sin\theta\cos\phi + \frac{1}{r}\frac{\partial f}{\partial \theta}\cos\theta\cos\phi - \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\sin\phi, \tag{A11}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\sin\theta\sin\phi + \frac{1}{r}\frac{\partial f}{\partial \theta}\cos\theta\sin\phi + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\cos\phi, \tag{A12}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial f}{\partial \theta}\sin\theta. \tag{A13}$$

Now, let $\mathbf{A}$ be a differentiable vector field. We have

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} \\ &\overset{(a)}{=} \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \cdot M \cdot \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} \\ &\equiv \begin{bmatrix} A_r & A_\theta & A_\phi \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}, \end{aligned}$$

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which leads to

$$\begin{bmatrix} A_x & A_y & A_z \end{bmatrix} = \begin{bmatrix} A_r & A_\theta & A_\phi \end{bmatrix} \cdot M^{-1},$$

or, since M is orthogonal,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = M \cdot \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}. \tag{A14}$$

We can write (A14) as the three scalar equations

$$A_x = A_r \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi, \tag{A15}$$
$$A_y = A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi, \tag{A16}$$
$$A_z = A_r \cos\theta - A_\theta \sin\theta. \tag{A17}$$

Using Equations (A11)–(A13) and (A15)–(A17), collecting terms, and simplifying, we obtain

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\left(\frac{\partial A_\phi}{\partial\phi}\right). \end{aligned} \tag{A18}$$

Combining (A9) and (A18), we can then write, for a twice differentiable scalar field f,

$$\begin{aligned} \nabla^2 f &= \nabla \cdot (\nabla f) \\ &\overset{(A9)}{=} \nabla \cdot \left(\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}\right) \\ &\overset{(a)}{=} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{1}{r}\frac{\partial f}{\partial\theta}\right) + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}, \end{aligned} \tag{A19}$$

where (a) follows from (A18), by using $A_r \leftrightarrow \partial f/\partial r$, $A_\theta \leftrightarrow (1/r)\partial f/\partial\theta$, $A_\phi \leftrightarrow (1/r\sin\theta)\partial f/\partial\phi$. Equation (A19) establishes the result. □

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