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Article

Interaction Solutions for the Fractional KdVSKR Equations in (1+1)-Dimension and (2+1)-Dimension

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Abstract: In this paper, we extend the KdVSKR equations in (1+1)-dimension and (2+1)-dimension to fractional KdVSKR equations with modified Riemann-Liouville derivative. The (2+1)-dimensional KdVSKR equation, which is a recent extension of (1+1)-dimensional KdVSKR equation, can model the resonances of solitons in shallow water. By means of the Hirota bilinear method, finite symmetry group method and consistent Riccati expansion method, many new interaction solutions have been derived. Soliton-cnoidal interaction solution of the (1+1)-dimensional fractional KdVSKR equation has been derived for the first time. For the (2+1)-dimensional fractional KdVSKR equation, two-wave interaction solutions and three-wave interaction solutions including dark-soliton-sine interaction solution, bright-soliton-elliptic interaction solution, and lump-hyperbolic-sine interaction solution. Impact of fractional order γ on the shapes of these solutions has been illustrated by figures. The three-wave interaction solution of fractional system has not been reported in the existing references. The research idea in this paper can be applied to other fractional differential equations.

Keywords: Korteweg-de Vries Sawada-Kotera-Ramani (KdVSKR) equation; extended Korteweg-deVries (KdV) equation; Sawada-Kotera (SK) equation; finite symmetry groups; Lie symmetry; exact solutions; consistent Riccati expansion (CRE); Hirota bilinear method; fractional

1. Introduction

In recent decades, fractional nonlinear systems have developed into effective mathematical tools to describe real-world problems with the rapid development of fractional calculus. Fractional differential equations (FDEs) can describe evolutionary phenomena that depend on both the time instant and time history, and they have been widely used in many fields such as economics and finance [1,2], epidemiology of disasters [3,4], physics [5–8], engineering [9] and so on. To better model practical problems, different fractional derivatives have been defined and applied, such as Riemann-Liouville [10,11], Caputo [12], modified Riemann-Liouville [13], Atangana-Baleanu derivative [14] and so on [15]. For further information on the fractional derivatives, we refer the readers to [10–12] and the cited references.

Since FDEs play an important role in expressing the practical problems mathematically, extraction of exact solutions for these FDEs is imperative. Exact solutions of the governing FDEs can be the benchmark solutions to verify the outcomes and codes of numerical solutions, and even to develop various numerical methods such as their differencing schemes and grid generation skills [16]. In addition, the effort to find these solutions is significant for the more profound understanding of many physical phenomena, thus they may give more insight into the physical aspects of the problems. For example, the wave distributions observed in fluid dynamics, plasma and elastic media are often described by soliton solutions [17,18]. In the past several decades, many effective methods for directly obtaining exact solutions of nonlinear FDEs have been presented, such as Lie symmetry method [5,19–23], fractional sub-equation method [24–26], G'/G -expansion method [27], exp-

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function method [28] and many more [29–32]. Among those, the fractional complex transform builds a bridge between fractional differential equations, partial differential equations (PDEs) or ordinary differential equations (ODEs) [6,29,30]. Through the fractional complex transform, many fractional differential equations can be transformed into PDEs. Therefore, methods which are used to find exact solutions of PDEs, such as Hirota bilinear method (HBM) [33,34], finite symmetry group method (FSGM) [35,36], consistent Riccati expansion method (CREM) [37,38] and so on can be applied to extract exact solutions of FDEs.

Hirota bilinear method (HBM) was first proposed by Hirota in 1971 [33,34]. For almost all nonlinear PDEs, HBM is very effective in exploring explicit soliton excitation solutions, including lump, breather and soliton molecules. These solutions provide a better explanation of the physical problems. Several fractional PDEs have been exactly solved by combining the fractional complex transform and Hirota bilinear method [39–42]. The finite symmetry group method (FSGM) was proposed by Lou [35] in 2005, which is a direct method to find finite symmetry groups of PDEs. By the FSGM, we can get Lie point symmetry group as special case. Particularly, we can get the relationship between known solutions and new solutions. Recently, a consistent Riccati expansion method (CREM) [37] has been proposed to find soliton or soliton-cnoidal solutions. The CREM has been applied to many PDEs, for example, Korteweg-deVries, Kadomtsev-Petviashvili, sine-Gordon, Sawada-Kotera, Kaup-Kupershmidt, Broer-Kaup, dispersive water wave, and Burgers systems [37,38].

To the best of our knowledge, soliton-cnoidal wave solutions for FDEs have not been reported up to now. Furthermore, FSGM has not been applied to derive solutions for FDEs. In this paper, we will apply the HBM, FSGM and CREM to extract new solutions for the two FDEs. The research objects are the (1+1)-dimensional fractional Korteweg-de Vries Sawada-Kotera-Ramani (FKdVSKR) equation

$$u_t^\gamma + \alpha(u_{xxx} + 6uu_x) - \beta\left(u_{xxxxx} + 45u^2u_x + 15uu_{xxx} + 15u_xu_{xx}\right) = 0, \quad (1)$$

and (2+1)-dimensional FKdVSKR equation

$$\begin{aligned} &u_t^\gamma + \alpha(u_{xxx} + 6uu_x) - \beta\Big(u_{xxxxx} + 45u^2u_x + 15uu_y + 15uu_{xxx} + 15u_xu_{xx} + 5u_{xxy} \\ &-5\int u_{yy}dx + 15u_x\int u_y dx\Big) = 0, \end{aligned} \quad (2)$$

in the sense of the modified Riemann-Liouville derivative, where $D_t^\gamma(.)$ is Jumarie's modified Riemann-Liouville derivative [13], $0 < \gamma \le 1$, γ is a constant, α and β are both constants.

FKdVSKR equations (1) and (2) include a lot of KdV-type equations as their special cases. When $\gamma=1, \alpha = 1, \beta = 0$, (1) becomes the well-known KdV equation

$$u_t + u_{xxx} + 6uu_x = 0.$$

When $\gamma=1, \alpha = 0, \beta = 1$, (1) becomes the well-known Sawada-Kotera (SK) equation

$$u_t + u_{xxxxx} + 45u^2u_x + 15uu_{xxx} + 15u_xu_{xx} = 0.$$

When $\gamma=1$, (1) is the (1+1)-dimensional KdVSKR equation [43–49]

$$u_t + \alpha(u_{xxx} + 6uu_x) - \beta\left(u_{xxxxx} + 45u^2u_x + 15uu_{xxx} + 15u_xu_{xx}\right) = 0, \quad (3)$$

which is also named KdV-SK equation [46] or extended KdV equation [48,49] and is a combination of the KdV equation and the SK equation.

Soliton molecules and asymmetric soliton for (3) are obtained in [43] by means of the velocity resonance condition. Also, finite symmetry groups of (3) are obtained. Lie symmetry, optimal system, symmetry reductions, power series solutions and N-soliton solution of (3) are derived in [44]. Exact solitary wave solutions and quasi-periodic travelling wave solutions of (3) are studied in [45] by a new method. A Kaup-Kupershmidt soliton wave solution has been obtained in [46]. The authors in [46] regard (3) as the higher order KdV equation, or KdV-SK equation. Soliton-cnoidal wave interaction solutions of (3) have been obtained by the consistent Riccati expansion method (CREM) [47]. For the (1+1)-dimensional KdVSKR equation with variable coefficients, rational function solutions, multi-wave rational function solutions and two-soliton rational solutions have been

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derived by the unified method and its generalized form [48]. We should point out that all the above work is on (3), and it is the first time to research the FKdVSKR equation (1).

When $\gamma=1, \alpha=0, \beta=-1,$ (2) becomes the integrable Sawada-Kotera (SK) equation [50]

$$u_t+u_{xxxxx}+15u_xu_{xx}+15uu_{xxx}+45u^2u_x+5u_{xxy}+15uu_y+15u_x\int u_y dx-5\int u_{yy}dx=0.$$

When $\gamma=1,$ (2) becomes the (2+1)-dimensional KdVSKR equation

$$\begin{aligned}&u_t+\alpha(u_{xxx}+6uu_x)-\beta\Big(u_{xxxxx}+45u^2u_x+15uu_y+15uu_{xxx}+15u_xu_{xx}+5u_{xxy}\\&-5\int u_{yy}dx+15u_x\int u_y dx\Big)=0,\end{aligned} \tag{4}$$

which has been proposed recently by generalizing (3) to y direction and can be used to describe the resonances of solitons in shallow water [51]. In [51–55], the authors study soliton excitations or interaction solutions of (4) based on the Hirota bilinear method (HBM). Multi-order lumps, interaction between lump and solitons of (4) are derived by the long wave limit method [51]. Soliton molecule and multi-breather solutions of (4) are extracted by the velocity resonance mechanism and the complex conjugate relations in the parameters [52]. In [53], the fission solution and fusion solution of (4) are studied. The dynamic behaviors of lump molecules and y-type molecules are also illustrated. In [54], a novel restrictive condition has been given to show the nonlinear superposition between a lump soliton and other nonlinear localized excitations. Interaction solutions of lump solution with hyperbolic cosine function and lump solution with exponential function of (4) are obtained by choosing appropriate function in the bilinear form [55]. In [56], Painlevé analysis, Lie point symmetry and symmetry reductions for (4) have been studied. To the best of our knowledge, there is no further work studying (4). Therefore, CREM and FSGM have not been applied to the KdVSKR equation in (2+1)-dimension.

The framework of the rest is in the following. In Section 2, we construct new exact solutions of the (1+1)-dimensional FKdVSKR equation (1) by the fractional complex transform and the known solutions. In Section 3, we construct new exact solutions of the (2+1)-dimensional FKdVSKR equation (2) by the fractional complex transform and CREM. In Section 4, explicit solutions of the (2+1)-dimensional FKdVSKR equation (2) will be studied by the fractional complex transform and FSGM. We will build the relationship of new solutions with the known ones. New interaction solutions will be derived. Section 5 is devoted to discussion of the results and methods in this paper. In Section 6, some conclusions and future directions of the paper are presented.

2. Exact Solutions of the (1+1)-Dimensional FKdVSKR Equation

To reduce the (1+1)-dimensional FKdVSKR equation to a PDE, we introduce the fractional complex transform

$$u=u(x,\mathrm{T}), T=\frac{t^\gamma}{\Gamma(1+\gamma)}. \tag{5}$$

Substituting (5) into (1), we have the following (1+1)-dimensional KdVSKR equation [43–49,55]

$$u_T+\alpha(u_{xxx}+6uu_x)-\beta\left(u_{xxxxx}+45u^2u_x+15uu_{xxx}+15u_xu_{xx}\right)=0. \tag{6}$$

Works on (6) have been analyzed in the Introduction. Because of simplicity and directness, the Hirota bilinear method (HBM) is often used to construct localized nonlinear wave solutions such as soliton, lump and interaction solutions of a given PDE. A critical step in applying HBM is converting a PDE to its bilinear form. By means of the common logarithmic transformation

$$u=2(\ln\psi)_{xx},$$

(6) can be converted into the following bilinear form

$$(D_TD_x-\beta D_x^6+\alpha D_x^4)\psi\cdot\psi=0,$$

where D_x and D_T are Hirota's operators with respect to x, y and T respectively.

2.1. Soliton Solutions

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The KdVSKR equation (6) admits N-soliton solution as follows:

$$u = 2(\ln\psi)_{xx}, \psi = \sum_{\rho=0,1} \exp\left(\sum_{j=1}^{N} \rho_j \eta_j + \sum_{1\le j<i\le N}^{N} \rho_i \rho_j A_{ij}\right), \quad (7a)$$

with

$$\eta_i = \mu_i x - (\alpha\mu_i^3 - \beta\mu_i^5)T + \delta_i, \quad (7b)$$

and

$$e^{A_{ij}} = -\frac{(\mu_i-\mu_j)(v_i-v_j)+\alpha(\mu_i-\mu_j)^4-\beta(\mu_i-\mu_j)^6}{(\mu_i+\mu_j)(v_i+v_j)+\alpha(\mu_i+\mu_j)^4-\beta(\mu_i+\mu_j)^6}, \quad (7c)$$

$$v_i = -\alpha\mu_i^3 + \beta\mu_i^5, \ (1\le j<i\le N)$$

where $\{\mu_i, \delta_i\}(i=1,\ldots,N)$ are arbitrary constants, $\sum_{1\le j<i\le N}^{N}$ is the summation of all possible pairs taken from N elements with the condition $1\le j<i\le N$, $\sum_{\rho=0,1}$ indicates a summation over all possible combinations of $\rho_i, \rho_j = 0,1\,(i,j=1,\ldots,N)$.

2.1.1. One-Soliton Solution

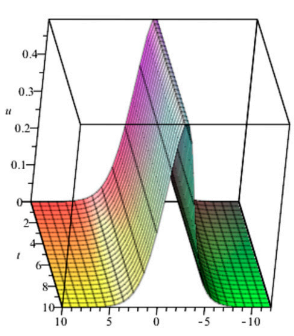
When $N=1$, $\psi = 1+e^{\eta_1}$, a one-soliton solution for the KdVSKR equation (6) is

$$u = \frac{2\mu^2 e^{\mu_1 x - \alpha\mu_1^3 T + \beta\mu_1^5 T + \delta_1}}{\left(1+e^{\mu_1 x - \alpha\mu_1^3 T + \beta\mu_1^5 T + \delta_1}\right)^2},$$

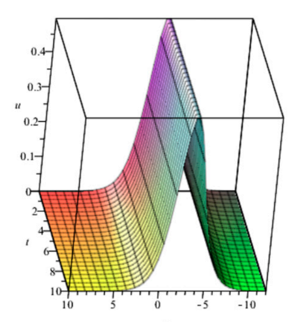
and the fractional one-soliton solution for the FKdVSKR equation (1) is

$$u = \frac{2\mu^2 e^{\mu_1 x + (-\alpha\mu_1^3 + \beta\mu_1^5)\frac{t^\gamma}{\Gamma(1+\gamma)} + \delta_1}}{\left(1+e^{\mu_1 x + (-\alpha\mu_1^3 + \beta\mu_1^5)\frac{t^\gamma}{\Gamma(1+\gamma)} + \delta_1}\right)^2}. \quad (8)$$

Graphs (a), (b) and (c) in Figure 1 illustrate the one-soliton solution (8) for different γ when taking $\alpha=1$, $\beta=2, \mu_1=1, \delta_1=0.1$. Graphs (d) in Figure 1 shows the relative locations for different γ at $\mathrm{t}=5$. As we can see from the graphs, locations of the one -soliton change as the fractional order parameter γ changes. The larger is γ, the more backward the soliton is located; the smaller is γ, the more forward the solution is located. Interestingly, the fractional soliton sulution for the KMM system is just the opposite [17]. For the KMM system, the location of the soliton is backwarder when the fractional order become smaller.



(a) One-soliton with $\gamma = 0.05$.



(b) One-soliton with $\gamma = 0.5$.

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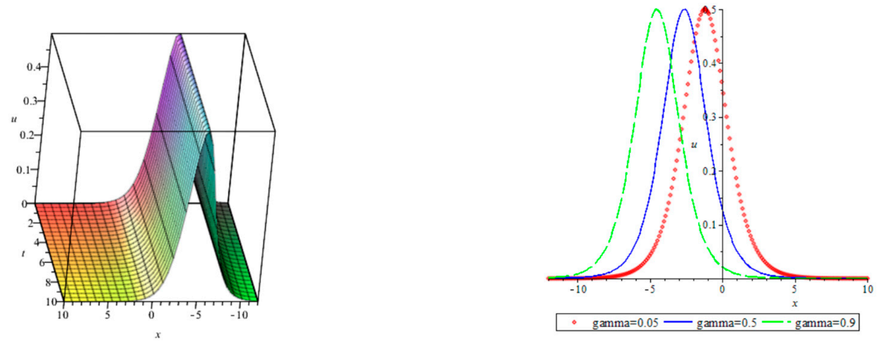

(c) One-soliton with $\gamma = 0.9$. (d) One-soliton with different γ.

Figure 1. One-soliton solution (10) with different γ..

2.1.2. Two-Soliton Solution

When $N = 2, \psi = 1 + e^{\eta 1} + e^{\eta 2} + e^{\eta 1+\eta 2+A_{12}}$, the fractional two-soliton solution for the FKdVSKR equation (1) is

$$u = \frac{2\left(\mu_1^{\ 2} e^{\eta_1} + \mu_2^{\ 2} e^{\eta 2} + (\mu_1+\mu_2)^2 A_{12} e^{\eta_1+\eta_2}\right)}{1+e^{\eta 1}+e^{\eta 2}+A_{12}e^{\eta 1+\eta 2}} - \frac{2\left(\mu_1 e^{\eta_1} + \mu_2 e^{\eta 2} + (\mu_1+\mu_2) A_{12} e^{\eta_1+\eta_2}\right)^2}{\left(1+e^{\eta 1}+e^{\eta 2}+A_{12}e^{\eta 1+\eta 2}\right)^2},$$

where $\eta_i = \mu_i x - (\alpha\mu_i^{\ 3} - \beta\mu_i^{\ 5})\dfrac{t^{\gamma}}{\Gamma(1+\gamma)} + \delta_i, (i = 1, 2)$, v_1, v_2 and $e^{A_{12}}$ are determined by (7).

When taking $\alpha = 1$, $\beta = 2, \mu_1 = 1, \delta_1 = 0.1, \mu_2 = 1.05, \delta_2 = 0.1$, diagrams of the two-soliton solution can be plotted as follows.

Figure 2 illustrates the two-soliton solution for $\gamma = 0.05$. As we can see from Figure 3, changes of the locations of the two-soliton is the same as the one-soliton. The location of the two-soliton is more forward when γ become smaller.

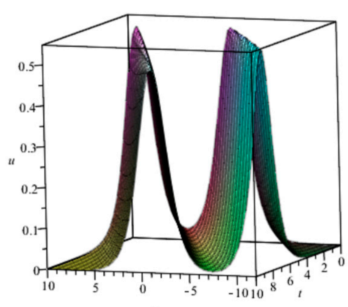

Figure 2. Two-soliton with $\gamma = 0.05$.

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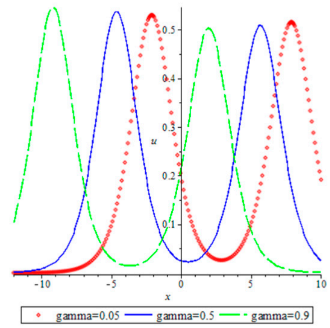

Figure 3. Two-soliton at $t=8.$.

2.1.3. Three-Soliton Solution

When $N=3$, if we take

$$\psi = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3},$$

where $\eta_i = \mu_i x - (\alpha\mu_i^3 - \beta\mu_i^5)\frac{t^\gamma}{\Gamma(1+\gamma)} + \delta_i, (i=1,2,3)$, v_1, v_2, v_3 and $e^{A_{12}}, e^{A_{13}}, e^{A_{23}}$ are determined by (7), the fractional three-soliton solution for the FKdVSKR equation (1) can be obtained by (7) and (5). When taking $\alpha=1, \beta=2, \mu_1=1, \mu_2=-1.24, \mu_3=1.23,$ and $v_1=1.25, v_2=1.3, v_3=1.4, \delta_1=0, \delta_2=0, \delta_3=0,$ diagrams of the above three- soliton solution can be obtained as follows. Figure 4 illustrates the three-soliton solution for $\gamma=0.9$. As we can see from Figure 5, changing tendency of the locations of the three-soliton is the same as the one-soliton and two-soliton. The location of the three -soliton is more forward when γ becomes smaller. However, in the neighborhood of $x=0$, trend of this change is not obvious.

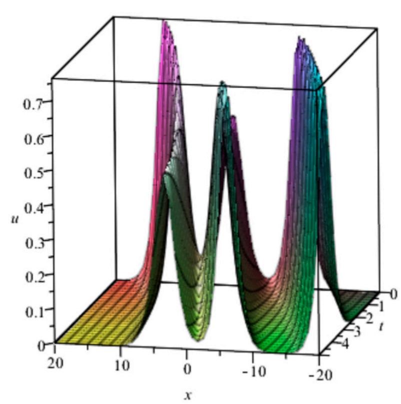

Figure 4. Three-soliton with $\gamma=0.9$.

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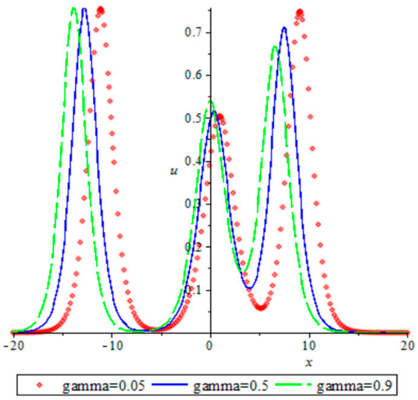

Figure 5. Three-soliton at $t = 2.$.

2.1.4. Generalized Kaup-Kupershmidt Solitary Waves

Based on [46], (6) has a Generalized Kaup-Kupershmidt (GKK) solitary wave as follows

$$u = \frac{2}{5}\frac{\delta_4 \cosh\left(\frac{x - \frac{4}{25}T}{\sqrt{5}}\right) + 1}{\left(\cosh\left(\frac{x - \frac{4}{25}T}{\sqrt{5}}\right) + \delta_4\right)^2},$$

where δ_4 is a constant. By means of (5), one can get a GKK solitary wave solution for the (1+1)-dimensional FKdVSKR equation (1)

$$u = \frac{2}{5}\frac{\delta_4 \cosh\left(\frac{x - \frac{4}{25}\frac{t^\gamma}{\Gamma(1+\gamma)}}{\sqrt{5}}\right) + 1}{\left(\cosh\left(\frac{x - \frac{4}{25}\frac{t^\gamma}{\Gamma(1+\gamma)}}{\sqrt{5}}\right) + \delta_4\right)^2}. \quad (9)$$

We should remark that it is different from the one-soliton solution (8).

2.2. Soliton-Cnoidal Interaction Wave Solution

The consistent Riccati expansion (CRE) method is an effective and straightforward method to identify CRE solvable systems and find interaction solutions between soliton and other types of nonlinear waves [37,38]. In [48], it has been proved that the KdVSKR (6) equation is CRE solvable, and a soliton-cnoidal wave solution has been found.

Taking advantage of the known solutions in [48], an interaction solution between soliton and Jacobi elliptic function for (1) can be obtained as follows

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$$u=\frac{1}{15\beta}-\frac{k_0^2m^2}{4\left(2n\mathrm{Sn}-\sqrt{2(n^2+1)}\right)}\left[2n\mathrm{Sn}(2n^2\mathrm{Sn}^2+3n^2+3)-\sqrt{2(n^2+1)}(6n^2\mathrm{Sn}^2+n^2+1)\right]$$
$$\times\tanh\left\{\frac{k_0m\sqrt{2(n^2+1)}}{-80\beta}\left[-20\beta x+4\frac{t^\gamma}{\Gamma(1+\gamma)}-5\beta^2k_0^4m^4(n^4+62n^2+1)\frac{t^\gamma}{\Gamma(1+\gamma)}\right]+\frac{1}{2}\ln(\mathrm{Dn}-n\mathrm{Cn})+\bar{C}\right\}^2 \quad (10)$$
$$+k_0^2m^2n\mathrm{Cn}\,\mathrm{Dn}\tanh\left\{\frac{k_0m\sqrt{2(n^2+1)}}{-80\beta}\left[-20\beta x+4\frac{t^\gamma}{\Gamma(1+\gamma)}-5\beta^2k_0^4m^4(n^4+62n^2+1)\frac{t^\gamma}{\Gamma(1+\gamma)}\right]\right.$$
$$\left.+\frac{1}{2}\ln(\mathrm{Dn}-n\mathrm{Cn})+\bar{C}\right\}-\frac{k_0^2m^2}{12(2n\mathrm{Sn}-\sqrt{2(n^2+1)})}\left[2n\mathrm{Sn}(6n^2\mathrm{Sn}^2-7n^2-7)+\sqrt{2(n^2+1)}(6n^2\mathrm{Sn}^2+n^2+1)\right],$$

with

$$\eta=\frac{k_2}{5\beta}(-5\beta x+\frac{t^\gamma}{\Gamma(1+\gamma)})-\frac{1}{4}\beta k_2^5m^4(19n^4+26n^2+19)\frac{t^\gamma}{\Gamma(1+\gamma)},$$

where $\mathrm{Sn}=\mathrm{Sn}(m\eta,n),\mathrm{Cn}=\mathrm{Cn}(m\eta,n),\mathrm{Dn}=\mathrm{Dn}(m\eta,n)$, $n\ (0<n<1)$ denotes the modulus of the Jacobi elliptic function, $k_0,\bar{C}$ and m are constants.

2.3. Lump-Periodic Interaction Wave Solution

In the expression of N-soliton solution (7), if we set

$$\psi=k_1\cosh(\eta_1)+k_2\cos(\eta_2),\ \eta_1=\bar{\mu}_1x+\bar{w}_1T+\bar{\delta}_1,\ \eta_2=\bar{\mu}_2x+\bar{w}_2T+\bar{\delta}_2,$$

where $\bar{\mu}_1,\bar{\mu}_2,\bar{w}_1,\bar{w}_2,\bar{\delta}_1$ and $\bar{\delta}_2$ are constants, we can get a lump-periodic interaction solutions for the KdVSKR equation (6). By means of (5), a lump-periodic interaction solution for the FKdVSKR equation (1) can be obtained as follows.

$$u=\frac{2(k_1\bar{\mu}_2^{\,2}\cosh(\eta_1)-k_2\bar{\mu}_2^{\,2}\cos(\eta_2))}{k_1\cosh(\eta_1)+k_2\cos(\eta_2)}-\frac{2(k_1\bar{\mu}_2\cosh(\eta_1)-k_2\bar{\mu}_2\cos(\eta_2))^2}{(k_1\cosh(\eta_1)+k_2\cos(\eta_2))^2},$$
$$\eta_1=i\bar{\mu}_2x+4\bar{\mu}_2^{\,3}(4\beta\bar{\mu}_2^{\,2}+\alpha)\frac{t^\gamma}{\Gamma(1+\gamma)}+\bar{\delta}_1,$$
$$\eta_2=\bar{\mu}_2x+(16\beta\bar{\mu}_2^{\,5}+4\alpha\bar{\mu}_2^{\,3})\frac{t^\gamma}{\Gamma(1+\gamma)}+\bar{\delta}_2. \quad (11)$$

When $\gamma=1$, the solution (11) become a solution for the KdVSKR equation (6), and it is exactly the same as that in [55].

3. New Solutions of the (2+1)-Dimensional FKdVSKR Equation by CREM

To reduce (2) to a PDE, we introduce the fractional complex transform

$$u=u(x,y,\mathrm{T}),T=\frac{t^\gamma}{\Gamma(1+\gamma)}. \quad (12)$$

Substituting (12) into (2), we get the following (2+1) dimensional KdVSKR equation:

$$u_T+\alpha(u_{xxx}+6uu_x)-\beta\Big(u_{xxxxx}+45u^2u_x+15uu_y+15uu_{xxx}+15u_xu_{xx}+5u_{xxy}$$
$$-5\int u_{yy}dx+15u_x\int u_ydx\Big)=0. \quad (13)$$

Removing integral operation, we can transform (13) to the following KdVSKR system:

$$\begin{cases}u_T+\alpha(u_{xxx}+6uu_x)-\beta\left(u_{xxxxx}+45u^2u_x+15uu_y+15uu_{xxx}+15u_xu_{xx}+5u_{xxy}-5v_y+15u_xv\right)=0,\\ u_y=v_x.\end{cases} \quad (14)$$

In the following, we first give an explanation of the CREM, then derive solutions of (14) by the CREM.

3.1. Explanation of CREM

The consistent Riccati expansion method (CREM) [37,38] is proposed to derive exact solutions based on the known solutions of the Riccati equation. Consider a nonlinear PDE system as follows

$$\begin{cases} p_1(x,y,T,u,v,u_x,v_x,u_y,v_y,u_T,v_T,u_{xx},v_{xx},\ldots)=0, \\ p_2(x,y,T,u,v,u_x,v_x,u_y,v_y,u_T,v_T,u_{xx},v_{xx},\ldots)=0. \end{cases} \quad (15)$$

Suppose that the solutions of (15) are as follows

$$\begin{cases} u=\sum_{j=0}^{J_1} u_j R^j(W), \\ v=\sum_{j=0}^{J_2} v_j R^j(W), \end{cases} \quad (16)$$

where u_j, v_j and W are undetermined functions concerning x, y and T. The positive integers J_1 and J_2 are dertermined by balancing the derivative term of the highest-order with the nonlinear term of the highest-order in (15). The function $R(W)$ is a solution of the Riccati equation

$$R_W = A + BR + MR^2, \quad (17)$$

with A, B, M *being constants.* Exact solutions of (17) have been reported in many references, such as [57].

Substituting (16) with (17) into (15), we can get a system of PDEs composed by the coefficients of different $R^j(W)$. If u_j and v_j can be determined by W and its derivatives, then the expansion (16) is a CRE and the nonlinear system (15) is CRE solvable. For more examples of the CREM, the readers can refer to [37].

3.2. Exact Solutions by the CREM

By balancing the highest nonlinearity and dispersive term, we suppose that solutions of (14) are as follows

$$\begin{cases} u = \bar{a} + \bar{b}R(W) + \bar{c}R(W)^2, \\ v = \bar{l} + \bar{m}R(W) + \bar{n}R(W)^2, \end{cases} \quad (18)$$

where $\bar{a}, \bar{b}, \bar{c}, \bar{l}, \bar{m}, \bar{n}$ and W are functions of x, y and T, and $R(W)$ is a solution of the Riccati equation (17).

Substituting (18) with (17) into (14) and vanishing the coefficients of different powers of $R(W)$ yields

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$$
\begin{aligned}
\overline{l} = \frac{1}{-75\beta^2 W_x^{\ 3}} \Big(& -1800\beta^2 MBAW_x^{\ 5} W_{xx} + 80\beta^2 M^2 A^2 W_x^{\ 7} + 25\beta^2 B^2 W_y W_x^{\ 4} + 5\alpha\beta W_y W_x^{\ 2} \\
& - 300\beta^2 BW_x^{\ 3} W_{yx} + 125\beta^2 W_x W_y W_x^{\ 3} - 1350\beta^2 BW_x W_{xx}^3 + 275\beta^2 W_x W_{xx} W_{xxxx} \\
& + 125\beta^2 B^2 W_x^{\ 4} W_{xxx} + 450\beta^2 B^3 W_x^{\ 5} W_{xx} - 250\beta^2 W_x W_{xx} W_{yx} - 450\beta^2 BW_x^{\ 3} W_{xxxx} \\
& + 625\beta^2 B^2 W_x^{\ 3} W_{xx}^2 + 600\beta^2 W_x W_{xxx}^2 + 175\beta^2 W_y W_{xx}^2 - 725\beta^2 W_{xxx} W_{xx}^2 \\
& - 25\beta^2 W_x W_y^{\ 2} - 25\beta^2 W_x^{\ 2} W_{yxx} - 145\beta^2 W_x^{\ 2} W_{xxxxx} - 5\beta W_x^{\ 2} W_T - \alpha^2 W_x^{\ 3} \\
& - 40\beta^2 MB^2 AW_x^{\ 7} + 200\beta^2 MAW_x^{\ 4} W_y + 1800\beta^2 BW_x^{\ 2} W_{xx} W_{xxx} \\
& + 450\beta^2 BW_x^{\ 2} W_{xx} W_y - 500\beta^2 MAW_x^{\ 4} W_{xxx} - 2500\beta^2 MAW_x^{\ 3} W_{xx}^2 + 5\beta^2 B^4 W_x^{\ 7} \Big),
\end{aligned}
$$

$$
\begin{aligned}
\overline{a} &= \frac{1}{-15\beta W_x^{\ 2}} \Big(30\beta BW_x^{\ 2} W_{xx} - 15\beta W_{xx}^{\ 2} + 20\beta W_x W_{xxx} + 5\beta B^2 W_x^{\ 4} \\
&\quad + 40\beta MAW_x^{\ 4} + 5\beta BW_x W_y - \alpha W_x^{\ 2} \Big), \\
\overline{b} &= -4MW_{xx} - 4MBW_x^{\ 2}, \\
\overline{c} &= -4M^2 W_x^{\ 2}, \\
\overline{m} &= -4MBW_x W_y - 4MW_{xy}, \\
\overline{n} &= -4M^2 W_x W_y,
\end{aligned}
$$

and the function W needs to satisfy the following three equations

$$
W_{xy} = \frac{1}{-W_x^2} \left(-4W_x W_{xx} W_{xxx} - B^2 W_x^4 W_{xx} + 4AMW_x^4 W_{xx} - W_x W_y W_{xx} + 3W_{xx}^{\ 3} + W_x^{\ 2} W_{xxxx} \right), \quad \text{(19a)}
$$

$$
W_{xT} = \Xi, \quad \text{(19b)}
$$

$$
\left\{ \frac{1}{-W_x^2} \left(-4W_x W_{xx} W_{xxx} - B^2 W_x^4 W_{xx} + 4AMW_x^4 W_{xx} - W_x W_y W_{xx} + 3W_{xx}^{\ 3} + W_x^{\ 2} W_{xxxx} \right) \right\}_T = \Xi_y, \quad \text{(19c)}
$$

where

$$
\begin{aligned}
\Xi = \frac{1}{-W_x^{\ 4}} \Big(& -W_T W_{xx} W_x^{\ 3} + \alpha W_x^{\ 4} W_{xxxx} + 4\beta W_x^{\ 4} W_{xxxxxx} + 3\alpha W_{xx}^3 W_x^{\ 2} + 5\beta W_x^{\ 4} W_{yy} - \alpha B^2 W_x^{\ 6} W_{xx} \\
& + 40\beta MAW_x^{\ 5} W_{xx} W_{xxx} - 8\beta B^2 MAW_x^{\ 8} W_{xx} - 20\beta MAW_x^{\ 5} W_{xx} W_y + 135\beta W_{xx}^5 \\
& + 20\beta MAW_x^{\ 6} W_{xxxx} - 10\beta B^2 W_x^{\ 5} W_{xx} W_{xxx} + 20\beta W_{xx} W_x^{\ 2} W_{xxx} W_y + 5\beta B^2 W_x^{\ 5} W_{xx} W_y \\
& + 16\beta M^2 A^2 W_x^{\ 8} W_{xx} + 4\alpha MAW_x^{\ 6} W_{xx} + \beta B^4 W_x^{\ 8} W_{xx} - 5\beta W_x^{\ 2} W_y^{\ 2} W_{xx} - 4\alpha W_x^{\ 3} W_{xx} W_{xxx} \\
& - 5\beta B^2 W_x^{\ 6} W_{xxxx} - 50\beta W_x^{\ 3} W_{xxx} W_{xxxx} - 24\beta W_x^{\ 3} W_{xx} W_{xxxxx} - 5\beta W_x^{\ 3} W_{xxxx} W_y \\
& + 160\beta W_{xx} W_x^{\ 2} W_{xxx}^2 + 105\beta W_{xx}^2 W_x^{\ 2} W_{xxxx} - 330\beta W_{xx}^3 W_x W_{xxx} - 15\beta W_{xx}^3 W_x W_y \Big).
\end{aligned}
$$

To sum up, the (2+1)-dimensional KdVSKR system (14) is CRE solvable, the following nonauto Backlund transform theorem has been proven.

Theorem 1. If W is a solution of (19) and R is a solution of (17), then

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$$u=\frac{1}{-15\beta W_x^{\,2}}\left(30\beta BW_x^{\,2}W_{xx}-15\beta W_{xx}^{\,2}+20\beta W_xW_{xxx}+5\beta B^2W_x^{\,4}+40\beta MAW_x^{\,4}\right.$$
$$\left.+5\beta BW_xW_y-\alpha W_x^{\,2}\right)+\left(-4MW_{xx}-4MBW_x^{\,2}\right)R-4M^2W_x^{\,2}R^2,$$
$$v=\overline{l}+\left(-4MBW_xW_y-4MW_{xy}\right)R-4M^2W_xW_yR^2,$$

is the solution of the (2+1)-dimensional KdVSKR system (14).

Taking $W=k_1x+k_2y+k_3T+k_0$ and applying Theorem 1, we can obtain the following two soliton solutions for (13). One is a dark soliton solution

$$u=\frac{8}{3}k_1^{\,2}-\frac{k_2}{3k_1}+\frac{\alpha}{15\beta}-4k_1^{\,2}\tanh\left(k_1x+k_2y+k_3T+k_0\right),$$

and another is a singular soliton solution

$$u=\frac{8}{3}k_1^{\,2}-\frac{k_2}{3k_1}+\frac{\alpha}{15\beta}-4k_1^{\,2}\coth\left(k_1x+k_2y+k_3T+k_0\right),$$

where k_0,k_1,k_2 and k_3 are constants.

By means of (12), fractional dark soliton and singular soliton solutions for (2) can be obtained and they are as follows:

$$u=\frac{8}{3}k_1^{\,2}-\frac{k_2}{3k_1}+\frac{\alpha}{15\beta}-4k_1^{\,2}\left(\tanh\left(k_1x+k_2y+k_3\frac{t^{\gamma}}{\Gamma(1+\gamma)}+k_0\right)\right)^2, \quad (20)$$

$$u=\frac{8}{3}k_1^{\,2}-\frac{k_2}{3k_1}+\frac{\alpha}{15\beta}-4k_1^{\,2}\left(\coth\left(k_1x+k_2y+k_3\frac{t^{\gamma}}{\Gamma(1+\gamma)}+k_0\right)\right)^2. \quad (21)$$

4. Interaction Solutions of the (2+1)-dimensional FKdVSKR equation by FSGM

The finite symmetry group method (FSGM) is an effective way to get new exact solutions from the known solutions. In addition, Lie point symmetry group can also be obtained from the finite symmetry group.

4.1. Finite Symmetry Group of (14)

According to the FSGM, we look for the following symmetry group for (14)

$$\begin{cases}u=\hat{a}+\hat{b}U(p,q,r)+\hat{c}V(p,q,r),\\ v=\hat{s}+\hat{m}U(p,q,r)+\hat{n}V(p,q,r),\end{cases} \quad (22)$$

where $p=p(x,y,T),q=q(x,y,T),r=r(x,y,T),\hat{a}=\hat{a}(x,y,T),\hat{b}=\hat{b}(x,y,T),\hat{c}=\hat{c}(x,y,T)$ $\hat{s}=\hat{s}(x,y,T),\hat{m}=\hat{m}(x,y,T)$ and $\hat{n}=\hat{n}(x,y,T)$ are functions to be determined by requiring that $U(p,q,r)$ and $V(p,q,r)$ satisfies the same (2+1)-dimensional PDEs as u and v with the transformation

$$\{x,y,T,u(x,y,T),v(x,y,T)\}\rightarrow\{p,q,r,U(p,q,r),V(p,q,r)\}. \quad (23)$$

That is to say, $U(p,q,r)$ is supposed to satisfy the following nonlinear PDE

$$U_{ppppp}=\frac{1}{\beta}\left(U_r+\alpha U_{ppp}+6\alpha UU_p-15\beta U_pU_{pp}-45\beta U^2U_P\right.$$
$$\left.-15\beta UU_q-15\beta UU_{ppp}-5\beta U_{ppq}+5\beta V_q-15\beta U_pV\right), \quad (24)$$
$$V_p=U_q.$$

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Substituting (22) into (14), ruling out U_{ppppp} and V_p via (24), and collecting the other coefficients of U,V and their derivatives, we have a set of overdetermined PDEs. From them, we have

$$\begin{aligned}
&\hat{a} = -\frac{2}{45}\frac{F^2\alpha}{\beta} + \frac{2}{45}\frac{\alpha}{\beta} + \frac{1}{15}\frac{yF_T}{\beta F} + \frac{1}{45}\frac{G_T}{\beta F},\\
&\hat{b} = F^2,\\
&\hat{c} = 0,\\
&p = Fx - \frac{1}{10}\frac{y^2F_T}{\beta} - \frac{1}{15}\frac{yG_T}{\beta} + \frac{1}{15}\frac{yF\alpha}{\beta} - \frac{1}{15}\frac{yF^3\alpha}{\beta} + H, \quad (25)\\
&q = yF^3 + \int G_T F^2 dT,\\
&r = \int F^5 dT,
\end{aligned}$$

$$\begin{aligned}
\hat{s} =& \frac{1}{675}\frac{\alpha G_T}{\beta^2 F} - \frac{1}{255}\frac{yG_{TT}}{\beta^2 F} + \frac{2}{255}\frac{yG_T F_T}{\beta^2 F^2} + \frac{1}{15}\frac{H_T}{\beta F} + \frac{1}{135}\frac{F\alpha G_T}{\beta^2} + \frac{1}{75}\frac{y^2 F_T^2}{\beta^2 F^2} - \frac{1}{150}\frac{y^2 F_{TT}}{\beta^2 F}\\
&+ \frac{2}{225}\frac{FF_T y\alpha}{\beta^2} + \frac{2}{225}\frac{F_T\alpha y}{\beta^2 F} + \frac{1}{15}\frac{F_T x}{\beta F} + \frac{1}{675}\frac{G_T^2}{\beta^2 F^2} - \frac{1}{135}\frac{F^4\alpha^2}{\beta^2} - \frac{2}{675}\frac{F^2\alpha^2}{\beta^2} + \frac{7}{675}\frac{\alpha^2}{\beta^2},\\
\hat{m} =& -\frac{1}{5}\frac{yFF_T}{\beta} - \frac{1}{15}\frac{FG_T}{\beta} + \frac{1}{15}\frac{F^2\alpha}{\beta} - \frac{1}{15}\frac{F^4\alpha}{\beta},\\
\hat{n} =& F^4,
\end{aligned}$$

where F, G and H are arbitrary functions of T. So we have got the finite symmetry groups for (14).

Theorem 2 If $U(x,y,T)$ and $V(x,y,T)$ is a solution of the (2+1)-dimensional KdVSKR equation equation (14), then so is u and v expressed by (22) with (25).

Applying Theorem 1, we can get new exact solutions for (14) or (13) from the known solutions.

To see the relation between the finite symmetry groups and the Lie point symmetry group obtained by the standard Lie group approach, we set

$$F = 1 + \varepsilon\frac{f_T}{5}, G = \varepsilon g, H = \varepsilon h,$$

where ε is an infinitesimal parameter, f, g and h are arbitrary functions of T. Then (22) can be written as

$$u = U + \varepsilon\sigma(U), \qquad v = V + \varepsilon\sigma(V),$$

then

$$\begin{aligned}
&p = x + \varepsilon\left(\frac{1}{5}f_T x - \frac{2\alpha}{75\beta}f_T y - \frac{1}{50\beta}f_{TT}y^2 - \frac{1}{15\beta}g_T y + h\right),\\
&q = y + \varepsilon\left(\frac{3}{5}f_T y + g\right), \qquad r = T + \varepsilon f,
\end{aligned}$$

and

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$$\sigma(U)=\left(\frac{1}{5}f_T x-\frac{2\alpha}{75\beta}f_T y-\frac{1}{50\beta}f_{TT}y^2-\frac{1}{15\beta}g_T y+h\right)U_x+\left(\frac{3}{5}f_T y+g\right)U_y+fU_T$$
$$-\left(-\frac{2}{5}f_T u+\frac{4\alpha}{225\beta}f_T-\frac{1}{75\beta}f_{TT}y-\frac{1}{45\beta}g_T\right),$$
$$\sigma(V)=\left(\frac{1}{5}f_T x-\frac{2\alpha}{75\beta}f_{TT}y-\frac{1}{50\beta}f_{TT}y^2-\frac{1}{15\beta}g_T y+h\right)V_x+\left(\frac{3}{5}f_T y+g\right)V_y+fV_T$$
$$-\left(-\frac{4}{5}f_T\omega+\frac{1}{25\beta}f_{TT}yu+\frac{2\alpha}{75\beta}f_T u-\frac{4\alpha}{1125\beta^2}f_{TT}y-\frac{1}{75\beta}f_{TT}x+\frac{1}{15\beta}g_T u\right.$$
$$\left.+\frac{8\alpha^2}{1125\beta^2}f_T+\frac{1}{750\beta^2}f_{TTT}y^2-\frac{2\alpha}{225\beta^2}g_T+\frac{1}{225\beta^2}g_{TT}y-\frac{1}{15\beta}h_T\right),$$

so the Lie symmetry generators of (14) are as follows

$$P(f)=f(T)\frac{\partial}{\partial T}+\left(\frac{1}{5}f_T x-\frac{2\alpha}{75\beta}f_T y-\frac{1}{50\beta}f_{TT}y^2\right)\frac{\partial}{\partial x}+\frac{3}{5}f_T y\frac{\partial}{\partial y}+\left(-\frac{2}{5}uf_T+\frac{4\alpha}{225\beta}f_T-\frac{1}{75\beta}f_{TT}y\right)\frac{\partial}{\partial u}$$
$$+\left(-\frac{4}{5}f_T\omega+\frac{1}{25\beta}f_{TT}uy+\frac{2\alpha}{75\beta}f_T u+\frac{8\alpha^2}{1125\beta^2}f_T-\frac{4\alpha}{1125\beta^2}f_{TT}y-\frac{1}{75\beta}f_{TT}x+\frac{1}{750\beta^2}f_{TTT}y^2\right)\frac{\partial}{\partial v},$$
$$Q(g)=-\frac{1}{15\beta}g_T\frac{\partial}{\partial x}+g(T)\frac{\partial}{\partial y}-\frac{1}{45\beta}g_T\frac{\partial}{\partial u}+\left(\frac{1}{15\beta}g_T u-\frac{2\alpha}{225\beta^2}g_T+\frac{1}{225\beta^2}g_{TT}y\right)\frac{\partial}{\partial v},$$
$$R(h)=h(T)\frac{\partial}{\partial x}-\frac{1}{15\beta}h_T\frac{\partial}{\partial v},$$

they are exactly the same with the results in [56], which is obtained by the standard Lie group method, and the corresponding Lie algebra constitutes Kac-Moody-Virasoro type algebra.

Remark 1. In [43], the finite symmetry group of (1+1)-dimensional KdVSKR equation (6) has been derived, but there are no arbitrary functions. The case in (2+1)-dimensional KdVSKR system (14) is quite different, since the finite symmetry group of (14) has three arbitrary functions. The arbitrary functions in the finite symmetry group is very useful in extracting new type interaction solutions.

4.2. Dark-Soliton-Sine INTERACTION Solution for (2)

Applying Theorem 2 and the given solution (20), we can get new solutions for (13). For example, taking $F=\sin(T), G=0, H=0$, a new solution for (13) is

$$u=-\frac{2\alpha}{45\beta}\sin^2(T)+\frac{2\alpha}{45\beta}+\frac{y\cos(T)}{15\beta\sin(T)}+\sin^2(T)\left(\frac{8}{3}k_1^2-\frac{k_2}{3k_1}+\frac{\alpha}{15\beta}\right)$$
$$+\sin^2(T)\left\{-4k_1^2\left[\tanh\left(k_1(x\sin(T)-\frac{y^2\cos(T)}{10\beta}+\frac{\alpha y}{15\beta}\sin(T)-\frac{\alpha y}{15\beta}\sin^3(T))\right.\right.\right. \tag{26}$$
$$\left.\left.\left.+k_2\left(y\sin^3(T)\right)+k_3\left(-\frac{\sin^4(T)\cos(T)}{5}-\frac{4\sin^2(T)\cos(T)}{15}-\frac{8\cos(T)}{15}\right)+k_0\right)^2\right]\right\}.$$

Substituting $T=\dfrac{t^\gamma}{\Gamma(1+\gamma)}$ into (26), a dark soliton and trigonometric sine periodic interaction solution for the (2+1)-dimensional FKdVSKR equation is obtained. To illustrate interactions of the dark soliton and the trigonometrical sine function, graphs of solution (26) with different fractional order parameters have been plotted. When taking $y=2, \alpha=1, \beta=2, k_1=1, k_2=1, k_3=1, k_0=2$, graghs in 3D are given in Figure 6.

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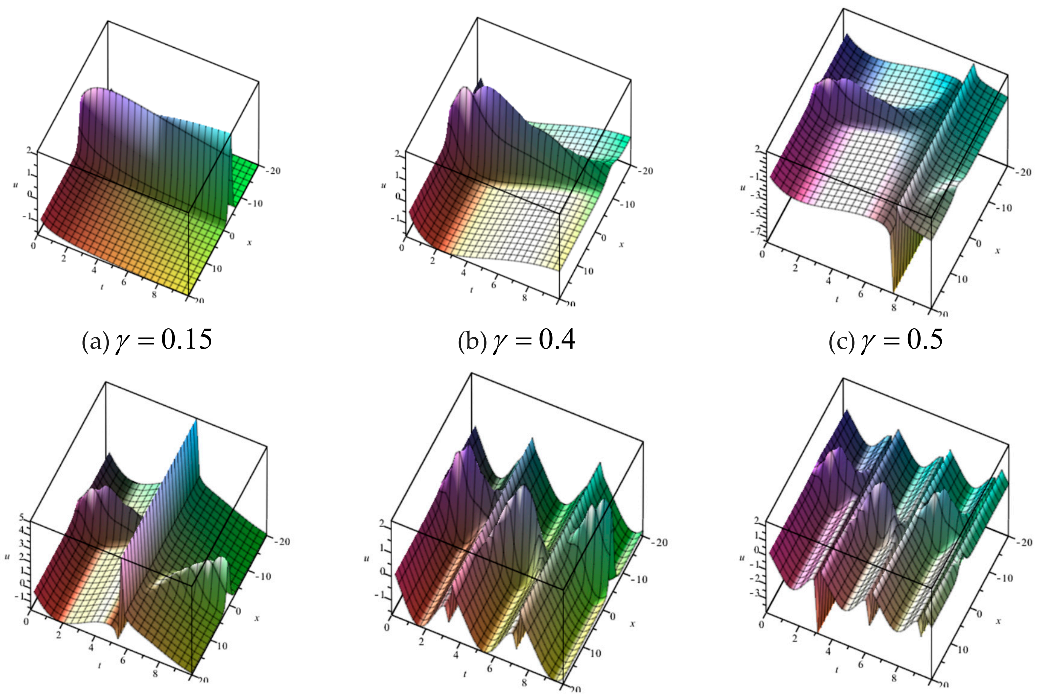
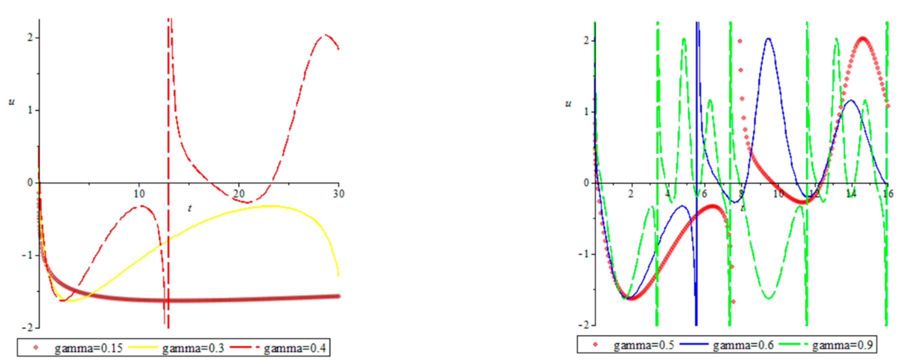
From graphs (a)-(f) in Figure 6, we find an interesting fact. When the fractional order $\gamma < 0.4$, the soliton dominates the solution. When $\gamma = 0.4$, periodicity of the solution begins to emerge. When $\gamma > 0.5$, periodicity of the solution become strong when the fractional order parameter grow larger. The conclusion can also be proven by Figure 7, which is at $x = 1, y = 2$ with the same parameters in Figure 6.



(a) $\gamma = 0.15$ (b) $\gamma = 0.4$ (c) $\gamma = 0.5$

(d) $\gamma = 0.6$ (e) $\gamma = 0.9$ (f) $\gamma = 1$

Figure 6. 3D graphs of solution (26) with different fractional order γ .



(a) small influence of periodicity (b) strong influence of periodicity

Figure 7. Influence of fractional order on periodicity.

4.3. Bright-Soliton-Elliptical-Interaction Solution for (2)

From [52], we know the (2+1)-dimensional KdVSKR equation (13) has a bright soliton solution

$$u = 2k_1^2 \operatorname{sech}\left(\frac{k_1}{2}x + \frac{3\alpha y}{20\beta}\left(k_1 - \frac{1}{k_1}\right) + \left(\frac{k_1^5\beta}{2} + \frac{k_1^3\alpha}{4} - \frac{9k_1\alpha^2}{40\beta} - \frac{3k_1\alpha}{4} + \frac{9\alpha^2}{20k_1\beta} - \frac{9\alpha^2}{40k_1^3\beta}\right)t + k_0\right)^2, \tag{27}$$

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where k_0 and k_1 are constants.

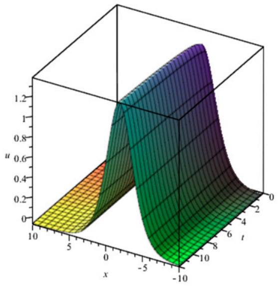
Applying Theorem 2 and the above solution (27), we can get new solutions for (13). For example, taking $F=\mathrm{Sn}, G=0, H=0$, a new solution for (13) is

$$\begin{aligned} u=&\frac{-2\alpha}{45\beta}\mathrm{Sn}^2+\frac{2\alpha}{45\beta}+\frac{y}{15\beta}\frac{\ln(\mathrm{Dn}-k\,\mathrm{Cn})}{k\,\mathrm{Sn}}+\\ &\mathrm{Sn}^2\left\{2k_1^{\,2}\operatorname{sech}\left(\frac{k_1}{2}\left(\mathrm{Sn}\,x-\frac{y^2\ln(\mathrm{Dn}-k\,\mathrm{Cn})}{10\beta k}+\frac{\alpha y\,\mathrm{Sn}}{15\beta}-\frac{\alpha y\,\mathrm{Sn}^3}{15\beta}\right)+\right.\right.\\ &\frac{3\alpha}{20\beta}\left(k_1-\frac{1}{k_1}\right)\mathrm{Sn}^3\,y+\left(\frac{k_1^{\,5}\beta}{2}+\frac{k_1^{\,3}\alpha}{4}-\frac{9k_1\alpha^2}{40\beta}-\frac{3k_1\alpha}{4}+\frac{9\alpha^2}{20k_1\beta}-\frac{9\alpha^2}{40k_1^{\,3}\beta}\right)\left(\frac{\mathrm{Sn}^2\,\mathrm{Cn}}{4k^2}\right.\\ &\left.\left.+\frac{\ln(\mathrm{Dn}-k\,\mathrm{Cn})}{4k^3}+\frac{3\ln(\mathrm{Dn}-k\,\mathrm{Cn})}{8k}+\frac{3\,\mathrm{Cn}\,\mathrm{Dn}}{8k^2}+\frac{3\,\mathrm{Cn}\,\mathrm{Dn}}{8k^4}+\frac{3\ln(\mathrm{Dn}-k\,\mathrm{Cn})}{8k^5}\right\},\end{aligned} \tag{28}$$

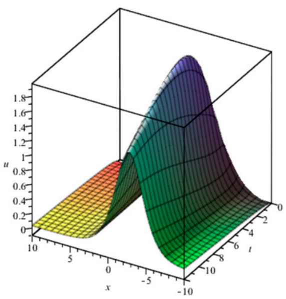
where $\mathrm{Sn},\mathrm{Dn},\mathrm{Cn}$ are short for the Jacobi elliptic function $\mathrm{JacobiSN}(T,k),\mathrm{JacobiDN}(T,k)$ and $\mathrm{JacobiCN}(T,k)$ respectively, $k\ (0<k<1)$ denotes the modulus of the Jacobi elliptic function.

Substituting $T=\dfrac{t^\gamma}{\Gamma(1+\gamma)}$ into (28), a two-wave interaction solution between bright soliton and elliptic periodic function for the (2+1)-dimensional FKdVSKR equation (2) is obtained. To illustrate interactions of the bright soliton and the elliptic periodic function, graphs of solution (28) with different fractional order parameters have been plotted. When taking $y=2,\alpha=1,\beta=2,k_1=1,k_0=2,$ and the modulus of the Jacobi elliptic function $k=0.5$, graghs in 3D are in Figure 8.

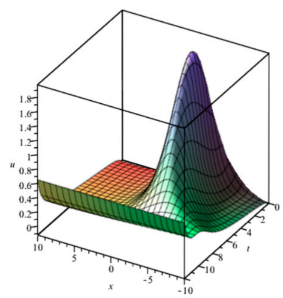
From graphs (a)-(h) in Figure 8, the impact of fractional order parameter γ on the shape of solutions can be observed. When the fractional order $\gamma=0.02$, it is the shape of soliton, and we can't see the influence of elliptic function. When the fractional order $\gamma=0.3$ and $\gamma=0.43$, the shape of soliton has changed. When $\gamma=0.45$, periodicity of the solution begins to emerge. When $\gamma>0.45$, periodicity of the solution becomes stronger when the fractional order parameter grows larger. The conclusion can also be proven by Figure 9. It is ploted at $x=2, y=2$ and other pamameters are the same with Figure 8.



(a) γ =0.02



(b) γ =0.3



(c) γ =0.43

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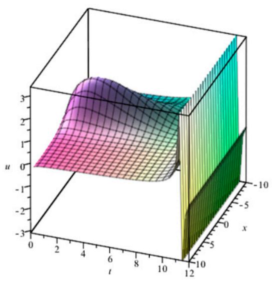
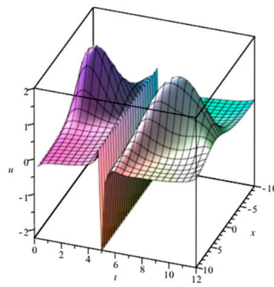
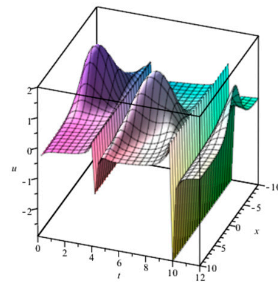


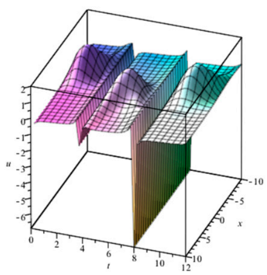
(d) γ =0.45



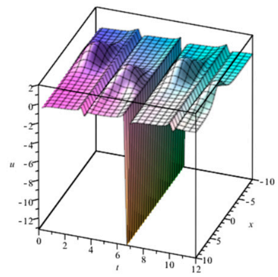
(e) γ =0.7



(f) γ =0.8

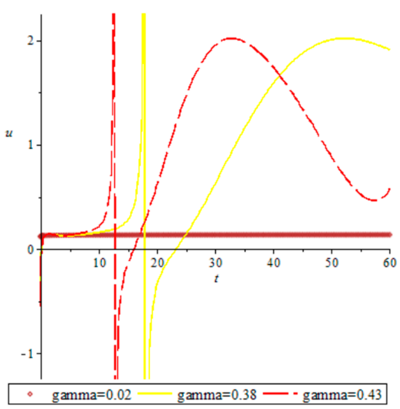


(g) γ =0.9

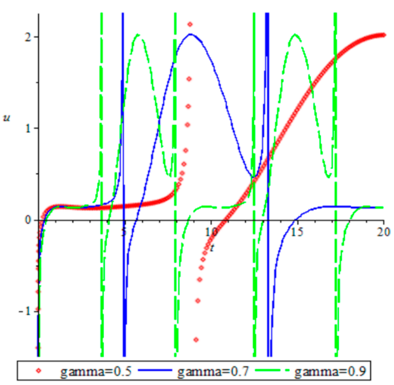


(h) γ =1

Figure 8. 3D graphs of solution (28) with different fractional order γ .



(a)



(b)

Figure 9. Influence of fractional order γ on periodicity.

5. Results and Discussion

In this paper, FKdVSKR equation in (1+1)-dimension and (2+1)-dimension have been studied from the point of analytical solutions. The CREM and FSGM are used to study exact solutions of KdVSKR equation (6) and (14). For the (1+1)-dimensional KdVSKR equation (6), the authors in [48] show that the equation is CRE solvable and a soliton-cnoidal interaction solution is obtained. From the known soliton-cnoidal interaction solution, we obtain a fractional soliton-cnoidal interaction solution (10) for the FKdVSKR equation (1). It is a fractional soliton lying on a fractional cnoidal periodic wave background. It the first time that we have got a fractional soliton-cnoidal interaction solution for a fractional differential system. The FSGM has been applied to (6) in [43], there aren't any arbitrary functions in the expression of the finite symmetry group, so we can't get new solutions from the FSGM. Fortunately, the finite symmetry group of the (2+1)-dimensional KdVSKR system (14) has three arbitrary functions, so that we can get a lot of new solutions making use of the known solutions of (13) and Theorem 2.

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From [55], we know the (2+1)-dimensional KdVSKR equation (13) has the following lump-hyperbolic solution

$$u=\frac{2e_1e_0^2\cosh(\eta_1)}{\eta_2^2-\frac{e_2^2\eta_3^2}{b_2^2}+e_1\cosh(\eta_1)}-\frac{2\left(\frac{2e_2b_0\eta_2}{b_2}-\frac{2e_2^2b_0\eta_3}{b_2^2}-e_1e_0\sinh(\eta_1)\right)^2}{\left(\eta_2^2-\frac{e_2^2\eta_3^2}{b_2^2}+e_1\cosh(\eta_1)\right)^2}, \quad (29)$$

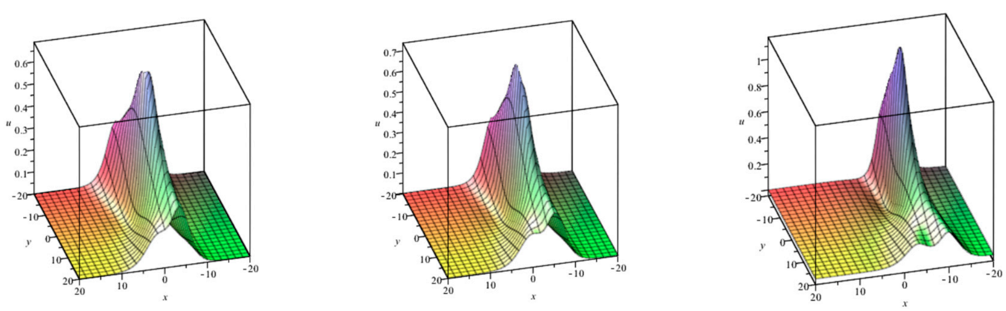
with

$$\eta_1=-e_0x+4e_0^3(-4\beta e_0^2+\alpha)T-e_3,$$
$$\eta_2=\frac{e_2b_0}{b_2}x+\frac{e_2b_0}{b_2}y+e_2T+\frac{e_2b_3}{b_2},$$
$$\eta_3=b_0x+b_1y+b_2T+b_3,$$

where b_i and e_i $(i=0,1,2,3)$ are constants.

Substituting $T=\frac{t^\gamma}{\Gamma(1+\gamma)}$ into (29), a lump-hyperbolic interaction solution for the (2+1)-dimensional FKdVSKR equation (2) is obtained. When taking $y=2,\alpha=\beta=1$, $e_0=e_1=0.6, e_2=0.1, e_3=0.4, b_0=0.6, b_1=0.1, b_2=1, b_3=0.1,$ graphs are as follows.

From graphs (a)-(c) in Figure 10, we find the impact of fractional order parameter γ on the shape of solutions is not obvious.



(a) γ =0.01,t=10 (b) γ =0.48,t=10 (c) γ =0.98,t=10

Figure 10. Lump-hyperbolic solution with different fractional order γ.

Applying Theorem 2 and the above solution (29), we can get new solutions for (2). In Theorem 2 , we take $F=\sin(T), G=0, H=0$, a three-wave interaction solution for (13) is as follows.

$$u=-\frac{2\alpha}{45\beta}\sin^2(T)+\frac{2\alpha}{45\beta}+\frac{y\cos(T)}{15\beta\sin(T)}+$$
$$\sin^2(T)\left\{\frac{2e_1e_0^2\cosh(\bar\eta_1)}{\bar\eta_2^2-\frac{e_2^2\bar\eta_3^2}{b_2^2}+e_1\cosh(\bar\eta_1)}-\frac{2\left(\frac{2e_2b_0\bar\eta_2}{b_2}-\frac{2e_2^2b_0\bar\eta_3}{b_2^2}-e_1e_0\sinh(\bar\eta_1)\right)^2}{\left(\bar\eta_2^2-\frac{e_2^2\bar\eta_3^2}{b_2^2}+e_1\cosh(\bar\eta_1)\right)^2}\right\}, \quad (30)$$

with

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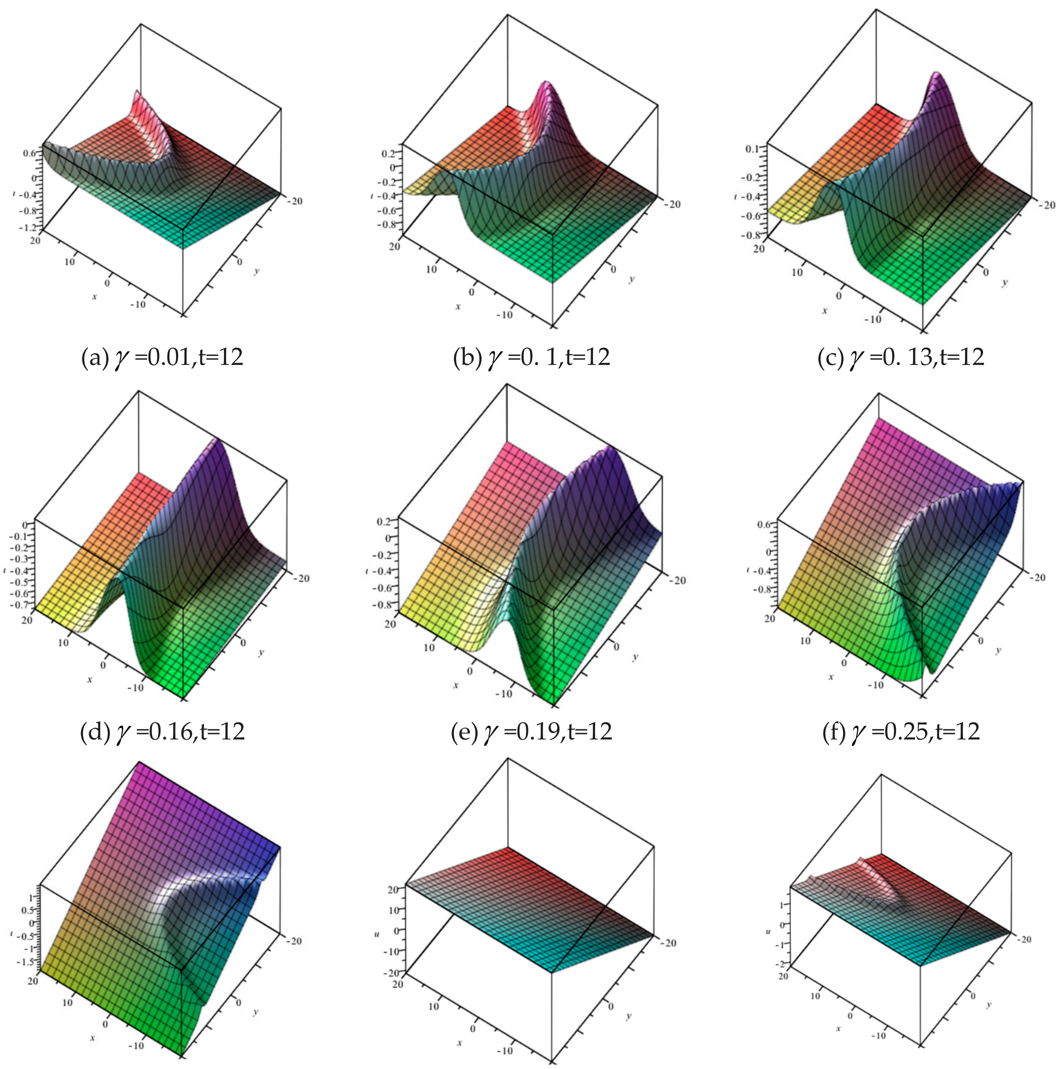
$$\overline{\eta}_1 = -e_0 p + 4e_0^{\,3}(-4\beta e_0^{\,2} + \alpha)r - e_3,\ \overline{\eta}_2 = \frac{e_2 b_0}{b_2} p + \frac{e_2 b_0}{b_2} q + e_2 r + \frac{e_2 b_3}{b_2},$$

$$\overline{\eta}_3 = b_0 p + b_1 q + b_2 r + b_3,\ p = x\sin(T) - \frac{y^2\cos(T)}{10\beta} + \frac{\alpha y}{15\beta}\sin(T) - \frac{\alpha y}{15\beta}\sin^3(T),$$

$$q = y\sin^3(T),\ \ r = -\frac{\sin^4(T)\cos(T)}{5} - \frac{4\sin^2(T)\cos(T)}{15} - \frac{8\cos(T)}{15}.$$

Substituting $T = \dfrac{t^{\gamma}}{\Gamma(1+\gamma)}$ into (30), an lump-hyperbolic-sine interaction solution for the (2+1)-dimensional FKdVSKR equation (2) is obtained. When taking $y = 2, \alpha = \beta = 1, e_0 = e_1 = 0.6, e_2 = 0.1, e_3 = 0.4, b_0 = 0.6, b_1 = 0.1, b_2 = 1, b_3 = 0.1,$ graphs are as follows.

From graphs (a)-(f) in Figure 11, we find that as fractional order γ increases, the shape of the solution has changed regularly. From $\gamma = 0.01$ to $\gamma = 0.16$, it goes from periodic solution to soliton solution gradually. From $\gamma = 0.16$ to $\gamma = 0.32$, the soliton becomes periodic solution gradually. From $\gamma = 0.32$ to $\gamma = 0.48$, the wave is from disappearing to emerging.



(a) γ =0.01,t=12 (b) γ =0. 1,t=12 (c) γ =0. 13,t=12

(d) γ =0.16,t=12 (e) γ =0.19,t=12 (f) γ =0.25,t=12

(g) γ =0.32,t=12 (h) γ =0.42,t=12 (i) γ =0.48,t=12

Figure 11. Lump-hyperbolic-periodic solution with different fractional order γ.

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Compared graphs in Figure 10 with those in Figure 11, we find the shape of the lump-hyperbolic interaction solution is very different with that of lump-hyperbolic-sine interaction solution. With the change of fractional order γ, the three-wave interaction solution changes regularly while two-wave interaction solution changes a little.

Remark 2. We should point out that making use of Theorem 2, lots of new solutions with unexpected dynamic behavior and properties can be obtained. It is necessary to study further.

6. Conclusions

Recently, a new (2+1)-dimensional PDE has been proposed, which is called KdVSKR equation and can model the resonances of solitons in shallow water. The available results show that this equation possesses rich local excitation patterns. Since fractional KdVSKR equation in (1+1) -dimension and (2+1) -dimension have not been studied so far, we perform in-depth study on their exact solutions by means of HBM, CREM and FSGM. For the (1+1)-dimensional FKdVSKR equation, influence of fractional order on the locations of fractional one-soliton, fractional two-soliton and fractional three-soliton have been illustrated by graphs. We find that one-soliton, two-soliton and three-soliton have the same variation tendency: the location of the fractional soliton is more backward when the fractional order γ becomes larger. In addition, we also get fractional lump-periodic and fractional soliton-cnoidal interaction wave solutions. To our best knowledge, soliton-cnoidal interaction wave solutions of fractional PDE have not been reported before.

For the (2+1)-dimensional KdVSKR equation, we get the finite symmetry group and Theorem 2. From the finite symmetry group, Lie point symmetry can be obtained and is exactly the same with the results in [56]. Applying Theorem 2, a lot of novel solutions can be derived for the (2+1)-dimensional KdVSKR equation. Making use of the fractional complex transform (12), new types of interaction solutions for the (2+1)-dimensional FKdVSKR equation (2) have been extracted. The solutions include dark soliton with trigonometric sine function interaction solution (26), bright soliton with elliptic function interaction solution (28), and lump-hyperbolic-sine three-wave interaction solution (29) under the condition $T=\dfrac{t^{\gamma}}{\Gamma(1+\gamma)}$. Impact of fractional order γ on the shapes of these solutions has been illustrated by plenty of graphs. As far as we know, lump-hyperbolic-sine three-wave interaction solutions for fractional PDEs have not been reported before. We have shown that the combination of the fractional complex transform and FSGM or CREM can be used to obtain new interaction solutions for fractional differential systems. In the future, we will study the fractional higher-order beam equation by applying the research idea in this paper.

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