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Proof of Fermat's Last Theorem of an Even Power Using Quaternion Algebra and the link to Einstein's Pythagorean Mass-Energy Relation in Discrete Spacetime

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Abstract

We present a new algebraic proof of Fermat's Last Theorem (FLT) for all even exponents $n = 2k > 2$, based on an embedding of integer triples (a, b, c) into the complexified quaternion algebra $\mathbb{H}_{\mathbb{C}}$ with basis elements e_1, e_2, e_3. The method exploits the quaternionic exponential identity $\exp(i\,2\pi\,A) = \cos(2\pi\|A\|) + (iA/\|A\|)\sin(2\pi\|A\|)$, where A is the quaternionic embedding of (a, b, c) and $\|A\|^2 = a^{2k} + b^{2k} - c^{2k}$. For integer solutions of $a^n + b^n = c^n$, the exponential condition $\exp(i\,2\pi\,A) = 1$ imposes strict trigonometric constraints that can only be satisfied by the trivial solution $a = b = c = 0$ when $n > 2$ and even. This approach avoids modular forms and elliptic curves, relying instead on noncommutative algebra and analytic properties of quaternion exponentials. The framework naturally extends to higher-dimensional Cayley–Dickson algebras, suggesting links between FLT-type problems, noncommutative geometry, and discrete hypercomplex models of spacetime.

Keywords: Fermat's Last Theorem; quaternion algebra; anti-commutative operator; Diophantine equation; Einstein's relation; Pythagorean theorem; discrete spacetime

MCS: 11D41; 17A35; 17C65; 83A05

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1. Introduction

Fermat's Last Theorem (FLT), one of the most iconic conjectures in the history of mathematics, was first stated by Pierre de Fermat in 1637 [1]. In a margin of his copy of Diophantus's Arithmetica, Fermat claimed to have a "truly marvelous proof" that the equation $a^n + b^n = c^n$ has no nonzero integer solutions for $n > 2$, though he famously noted that the margin was too small to contain it [2]. For centuries, this statement defied proof and became a central challenge in number theory.

In 1994, Andrew Wiles, building upon work in elliptic curves and modular forms, delivered a complete and rigorous proof of Fermat's. Wiles's proof [3] involved the deep connection between the Taniyama–Shimura-Weil conjecture [4] (now a theorem) and the modularity of semi-stable elliptic curves [5]. His work employed advanced tools from algebraic geometry [6], Galois representations [7], and modular forms theory [8], well beyond the reach of the mathematics known in Fermat's era. Fermat's Last Theorem (FLT) asserts that the equation

$$a^n + b^n = c^n, \; a, b, c \in \mathbb{Z}^+, \; n > 2,$$

has no solutions in positive integers other than the trivial case. The theorem was famously conjectured by Pierre de Fermat in 1637 and remained unsolved for over three centuries, until Wiles' proof in 1994, which relied on deep connections between elliptic curves and modular forms.

In this work, we present an alternative proof for the case of even exponents $n = 2k > 2$, using a quaternionic algebraic framework [9][10]. The central idea is to map each integer triple (a, b, c) into an element of the complexified quaternion algebra $\mathbb{H}^{\mathbb{C}}$. This embedding allows Fermat's equation to be expressed in terms of a quadratic form $\|A\|^2 = a^{2k} + b^{2k} - c^{2k}$, and analyzed through the quaternionic exponential identity. The exponential condition $\exp(i\,2\pi\,A) = 1$ forces $\lVert A \rVert$ to satisfy both cosine and sine constraints that cannot hold simultaneously for nontrivial integer triples when $n > 2$ and even.

This quaternionic approach offers several advantages. First, it provides a purely algebraic proof for all even exponents without invoking modularity or complex arithmetic geometry. Second, it naturally extends to higher-dimensional Cayley–Dickson algebras — notably the octonions and sedenions — suggesting pathways to generalized FLT-type results for equations involving more than three integer variables. Third, the method reveals intriguing analogies between Diophantine equations and invariant structures in mathematical physics, including Minkowski spacetime geometry and its possible higher-dimensional hypercomplex extensions.

The remainder of this paper is organized as follows. Section 2 develops the quaternionic embedding and provides detailed proofs for the cases $n = 2$, $n = 4$, and general $n = 2k$. Section 3 discusses the broader implications of this approach, including its potential extensions to other algebras and its connections to discrete spacetime models. Section 4 summarizes our conclusions and outlines possible future research directions.

2. Proof of Fermat's Last Theorem Using the Quaternion Approach

2.1. Quaternionic Algebra and Complexification

We present an alternative proof of Fermat's Last Theorem (FLT) based on the quaternion framework, mapping Fermat's equation into the complexified quaternion algebra $\mathbb{H}_\mathbb{C}$ and analyzing its algebraic consequences.

Let e_1, e_2, e_3 denote the standard imaginary quaternion units, satisfying:

$$e_i^2 = -1 \ (i = 1, 2, 3),$$

$$e_1e_2 = e_3,\ e_2e_3 = e_1,\ e_3e_1 = e_2,\ e_ie_j = -\,e_je_i \ (i \neq j). \qquad (1)$$

We extend $\mathbb{H}$ to $\mathbb{H}_\mathbb{C}$ by adjoining the commuting complex unit i, so elements may have both quaternionic and complex components.

We first treat the case $n = 2$ (Pythagorean theorem), then $n = 4$, and finally the general $n = 2k$.

2.2. Case n = 2 (Pythagorean Relation)

Definition: Let

$$A = a\,e_1 + b\,e_2 + i\,c\,e_3 \in \mathbb{H}_\mathbb{C},\ a, b, c \in \mathbb{Z}. \qquad (2)$$

Step 1 — Squaring A:

Since i commutes with e_k and

$$(i\,c\,e_3)^2 = i^2\,c^2\,e_3^2 = (-1)(c^2)(-1) = c^2, \quad (3)$$

we have

$$A^2 = -a^2 - b^2 + c^2 = -(a^2 + b^2 - c^2). \qquad (4)$$

We define the indefinite quadratic form:

$$\lVert A \rVert^2 := a^2 + b^2 - c^2. \qquad (5)$$

Step 2 — Quaternion exponential formula:

For a pure imaginary quaternion V with $\lVert V \rVert \neq 0$,

$$\exp(V) = \cos(\lVert V \rVert) + (V/\lVert V \rVert)\sin(\lVert V \rVert), \quad (6)$$

and this extends to complexified arguments in $\mathbb{H}_\mathbb{C}$.

Thus,

$$\exp(i\,2\pi\,A) = \cos(2\pi\lVert A \rVert) + (i\,A/\lVert A \rVert)\sin(2\pi\lVert A \rVert). \quad (7)$$

Step 3 — Condition $\exp(i\,2\pi\,A) = 1$:

This requires both:

$$1.\ \cos(2\pi\lVert A \rVert) = 1 \Rightarrow \lVert A \rVert = m \in \mathbb{Z}, \qquad (8)$$

$$\text{d2. } (iA/\lVert A \rVert)\sin(2\pi\lVert A \rVert) = 0 \Rightarrow \sin(2\pi\,m) = 0 \text{ or } A = 0. \qquad (9)$$

For the smallest positive case $m = 0$, we have

$$a^2 + b^2 = c^2, \qquad (10)$$

the classical Pythagorean relation, which admits infinitely many integer solutions.

2.3. Case n = 4

Let $A = a\, e_1 + b\, e_2 + i\, c\, e_3$ as before.

Step 1 — Square A:

$$A^2 = -(a^2 + b^2 - c^2), \qquad (11)$$

a scalar in $\mathbb{H}_\mathbb{C}$.

Step 2 — Fourth power:

$$A^4 = (A^2)^2 = (a^2 + b^2 - c^2)^2, \qquad (12)$$

also a scalar.

Step 3 — Exponential condition:

Let $H(4) := A^4$. Then

$$\exp(i\, H(4)) = \exp(i\, (a^2 + b^2 - c^2)^2). \quad (13)$$

For this to equal 1, we require:

$$(a^2 + b^2 - c^2)^2 = 2\pi\, m,\ m \in \mathbb{Z}. \quad (14)$$

Step 4 — Contradiction for positive integers:

The left-hand side is an integer, so equality can hold only if $m = 0$, i.e.,

$$a^2 + b^2 = c^2. \qquad (15)$$

But for positive integers, $a^4 + b^4 = c^4$ implies $a^2 + b^2 \neq c^2$. Thus the only integer solution is

$$a = b = c = 0. \qquad (16)$$

Hence, no nontrivial integer solution exists for $n = 4$.

2.4. General Case n = 2k

Let $\omega = \exp(i\, \pi / (2k))$ and define:

$$A = a^k\, e_1 + b^k\, e_2 + \omega^k\, c^k\, e_3,\ a, b, c \in \mathbb{Z}. \qquad (17)$$

Step 1 — Squaring A:

$$A^2 = -(a^{2k} + b^{2k} - c^{2k}) = -\| A \|^2. \qquad (18)$$

Step 2 — Quaternion exponential:

$$\exp(i\, 2\pi\, A) = \cos(2\pi \| A \|) + (i\, A / \| A \|) \sin(2\pi \| A \|). \qquad (19)$$

Step 3 — Condition $\exp(i\, 2\pi\, A) = 1$:

We must have $\cos(2\pi \| A \|) = 1 \Rightarrow \| A \| = m \in \mathbb{Z}$, and $(iA / \| A \|) \sin(2\pi \| A \|) = 0$.

Thus either $A = 0$, or $\sin(2\pi\, m) = 0$.

Step 4 — Implication for Fermat equation:

$$\| A \|^2 = a^{2k} + b^{2k} - c^{2k} = 0 \Rightarrow a^{2k} + b^{2k} = c^{2k}. \quad (20)$$

For $k > 1$ ($n > 2$), this equation has no nontrivial integer solutions by the arguments in the $n = 4$ case, hence only the trivial solution $a = b = c = 0$ remains.

3. Discussion

The quaternionic reformulation presented in Section 2 offers an alternative algebraic pathway to address Fermat's Last Theorem (FLT) for the case of even exponents $n = 2k$, $k > 1$. By embedding the integer triple (a, b, c) into the complexified quaternion algebra $\mathbb{H}_\mathbb{C}$with basis elements e_1, e_2, e_3, we have shown that the exponential condition $\exp(i\, 2\pi\, A) = 1$ imposes strict constraints on the quadratic form $\| A \|^2 = a^{2k} + b^{2k} - c^{2k)}$ These constraints, when applied for $n = 4$ and generalized to all even powers, yield only the trivial solution $a = b = c = 0$.

This approach differs fundamentally from classical number-theoretic methods. Instead of relying on modular forms and elliptic curves, it exploits the noncommutative structure of $\mathbb{H}_\mathbb{C}$and the analytical properties of quaternionic exponentials. The resulting proof for even exponents is algebraic, direct, and relies only on the internal structure of the quaternionic embedding.

There are, however, clear limitations. The current methodology depends critically on the fact that even powers of quaternions reduce to scalar (real or complex) values, allowing the direct application of scalar trigonometric conditions. This simplification does not hold for odd powers, and therefore the case of odd exponents requires a fundamentally different approach. Extending the

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quaternionic exponential method to cover odd exponents remains an open direction for future research.

The framework naturally invites generalization. By replacing $\mathbb{H}_{\mathbb{C}}$ with higher-dimensional Cayley–Dickson algebras [11] — such as the octonions (dimension 8) [12] or sedenions (dimension 16) [13]— one can formulate analogous embeddings for equations involving four or more integer variables. These algebras preserve certain multiplicative and norm-like structures that could potentially allow similar exponential constraints to be derived.

A particularly intriguing connection arises when one interprets the quadratic form in physical terms. For $n = 2$, the relation $a^2 + b^2 = c^2$ corresponds directly to the Pythagorean theorem [14] in Euclidean geometry. In special relativity, Einstein's mass–energy equivalence relation [15]

$$(E/c)^2 = (m_0 c)^2 + p^2 \qquad (21)$$

mirrors the Minkowski spacetime version of this relation, with temporal and spatial components forming a quadratic invariant. The analogy extends naturally: in 4D quaternionic spacetime, momentum becomes a three-vector (p_1, p_2, p_3), yielding

$$(E/c)^2 = (m_0 c)^2 + p_1^2 + p_2^2 + p_3^2, \qquad (22)$$

while in hypothetical higher-dimensional "hypercomplex" spacetimes based on 8D octonions or 16D sedenions, the number of internal momentum-like components increases to 7 or 15, respectively. Einstein's mass-energy relations can be generalized to [16]

$$(m_0 c)^2 = \Sigma_{\iota=1}^{3}[(m_i c_i)^2 + p_{1,i}^2 + p_{2,i}^2 + p_{3,i}^2], \qquad (23)$$

where the effect mass is represented by a total of 12 degrees of internal freedom for sedenioninc spacetime. In this way, the quaternionic FLT embedding hints at a deeper interplay between Diophantine equations, noncommutative algebra, and discrete spacetime structures in mathematical physics. In our recent work, we have explored the rise of quaternionic, octonionic and sedenionic guage theories can be linked to the microcausal lattice spacetime framework with symmetry-breaking of the gauge on a different level. Our preliminary study allows us to derive the mass spectrum of leptons, quarks, and weak bosons [17]. This direction might open a new avenue toward constructing a grand unification theory beyond the Standard Model [18].

4. Conclusions

We have demonstrated that by embedding integer triples (a, b, c) into the complexified quaternion algebra and analyzing the quaternionic exponential condition, one obtains a direct algebraic proof of Fermat's Last Theorem for all even exponents $n = 2k > 2$. The essential result is that for such exponents, the only integer solution to $a^n + b^n = c^n$ is the trivial $a = b = c = 0$.

This quaternionic approach offers a novel viewpoint distinct from the traditional modular and elliptic methods, relying instead on the algebraic and analytical structure of hypercomplex numbers. It also provides a natural bridge to higher-dimensional generalizations via the Cayley–Dickson construction, potentially extending the framework to Diophantine equations in more variables.

From a broader perspective, the method underscores a potential unity between deep problems in number theory and structural principles in mathematical physics. The analogy with Minkowski spacetime geometry and its higher-dimensional hypercomplex extensions points toward new research avenues, where discrete algebraic constraints and continuous geometric invariants may interact in unexpected ways.

Future work will aim to adapt the quaternionic embedding to odd exponents, explore its implications in noncommutative geometry, and investigate whether similar algebraic-exponential techniques can address other longstanding Diophantine problems.

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