Article

Not peer-reviewed version

Effects of the Quantum Vacuum at a Cosmic Scale and Dark Energy

Emilio Santos *

Posted Date: 11 October 2024

doi: 10.20944/preprints202410.0889.v1

Keywords: dark energy; quantum vacuum; quantized Einstein equation

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 October 2024 doi:10.20944/preprints202410.0889.v1

Article

Effects of the Quantum Vacuum at a Cosmic Scale and Dark Energy

Emilio Santos

Emeritus Professor of Theoretical Physics and Departamento de Física, Universidad de Cantabria, Santander, Spain; santose@unican.es

Abstract: Einstein equation approximated to second order in Newton constant G is quantized and applied to a large region of the universe. The two-point correlations of the vacuum energy density and pressure are included and the resulting metric agrees with the one derived from the ΛCDM cosmological model plus an additional contribution of the quantum vacuum. The conclusion is that at least a part of the acceleration in the expansion of the universe is due to the vacuum fluctuations.

Keywords: dark energy; quantum vacuum; quantized Einstein equation

---

1. Dark Energy, Cosmological Constant and Vacuum Fluctuations

In this paper I will study the possible effects of the quantum vacuum at a scale larger than the typical distances between galaxies. The work may provide clues to ascertain whether vacuum fluctuations might be the origin of dark energy.

The hypothesis of dark energy (DE) has been introduced in order to explain the *accelerating* expansion of the universe, discovered in 1998 [1–4]. DE consists of a density and pressure

$$\rho_{DE} = -(1-\varepsilon)p_{DE} \simeq (6.0 \pm 0.2) \times 10^{-27} kg/m^3, \quad (1)$$

filling space homogeneously[5,6]. The nature of the dark energy is unknown but the empirical fact that $|\varepsilon| << 1$ shows that its effect is fairly equivalent to a cosmological constant [7].

As is well known the cosmological constant (CC) was introduced by Einstein in order to get a stationary (although not stable) model of the universe. Later on the discovery of the expansion of the universe made the CC useless, but it was a recurrent possibility for about 70 years although without too much empirical support[8]. The view changed in 1998 when it was discovered that the expansion of the universe was accelerating [1,2], which might be seen as the effect of a CC, although the less committed assumption has been made that the acceleration is caused by a hypothetical ingredient named DE. In any case some of the proposals about the nature of DE are similar to previous assumptions about the origin of a possible CC.

An early proposal was that CC may correspond to the energy and pressure of the quantum vacuum. If this was the case a plausible assumption should be that its value could be got via a combination of the universal constants $c, \hbar, G$. There is a unique combination with dimensions of density, that is Planck density with a value

$$\rho_{Planck} = \frac{c^4}{G^2\hbar} \simeq 10^{97} kg/m^3, \quad (2)$$

which is about 123 orders greater than either the known value of DE Equation (1) or any reasonable value for a CC. This big discrepancy has been named the "cosmological constant problem"[8,9]. Many proposals have been made in the past for the origin of a CC [8] (or DE, see e.g., [10] and references therein) which shall not be discussed here. One of them has been the quantum vacuum origin as said above. If this is the case then some mechanism should exist reducing Equation (2) to Equation (1), but the fine tuning required looks unplausible, even conspiratory [8]. However I point out that, although the mean energy of the vacuum might cancel by some mechanism, the fluctuations cannot cancel completely which suggets that the fluctuations could give rise to an effective CC or DE.

 doi:10.20944/preprints202410.0889.v1

Using dimensional arguments any theory aimed at explaining DE Equation (1) would involve at least a new parameter, in addition to the universal constants, c, ħ, G. If we choose the parameter to be a mass, m, then the value of the dark energy could be written in the form (with $c = 1$)

$$\rho_{DE} \approx \frac{G^n m^{2n+4}}{\hbar^{n+3}}, \tag{3}$$

n being a real number. The choice $n = -2$ would remove m and lead to the Planck density Equation (2), but $n = 1$ may give the value Equation (1) for ρ_{DE} provided that m is of order the pion mass. Indeed more than forty years ago Zeldovich[11] proposed a formula like Equation (3) with $n = 1$ in order to get a plausible value for a cosmological constant. Furthermore he interpreted the result in terms of the mass m and its associated "Compton wavelength" λ, as follows

$$\Lambda \equiv \rho_{CC} \sim -\frac{Gm^2}{\lambda} \times \frac{1}{\lambda^3}, \lambda \equiv \frac{\hbar}{m}. \tag{4}$$

Thus Equation (4) looks like the energy density corresponding to the (Newtonian) gravitational energy of two particles of mass m placed at a distance λ, assuming that such an energy appears in every volume λ^3 (although in Equation (4) the gravitational energy would be negative if both masses are positive). Zeldovich interpreted that the "particles" were actually vacuum fluctuations. Hence his hypothesis that a finite CC might exist deriving from the fluctuations of the quantum vacuum. In recent times some modifications of Zeldovich´s proposal have been attempted as an explanation for the dark energy, identifying the CC with DE Equation (1) [12].

In the present paper I study again the possibility that vacuum fluctuations do produce a gravitational effect similar to a DE. At a difference with previous papers[12], where heuristic arguments were used, here I will use a more formal quantum approach to the vacuum fluctuations. Indeed the quantum vacuum fluctuations are specific quantum features, therefore classical equations like Friedman´s (see next section) are not appropriate in order to get the content of the universe from the observable value of the accelerated expansion.

We should deal with quantized general relativity, but not fully satisfactory quantum gravity is available Then I will make an *effective*, rather than fundamental, approach to the gravity of the quantum vacuum. I will integrate a quantized Einstein equation of general relativity approximated to second order in the Newton constant G.

2. Revisiting the Argument for Dark Energy

Our quantum approach in Section 3 will parallel the standard procedure to get DE Equation (1) from the observed accelerated expansion of the universe[1–3]. For this reason I will revisit that method, which allows relating observable properties of spacetime with the contents of the universe via general relativity. Indeed astronomical observations are compatible with the universe having a Friedmann-Lemaître-Robertson-Walker (FLRW) metric with flat spatial slices[13] of the form

$$ds^2 = -dt^2 + a(t)^2 \left[dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]. \tag{5}$$

The parameter $a(t)$ takes into account the expansion of the universe. At present time t_0, it is related to the measurable Hubble constant, H_0, and deceleration parameter, q_0, via

$$\left[\frac{\dot{a}}{a}\right]_{t_0} = H_0, \left[\frac{\ddot{a}}{a}\right]_{t_0} = -H_0^2 q_0. \tag{6}$$

From the function $a(t)$ the contents of the universe may be obtained solving the Friedman equation (which is a particular case of Einstein equation appropriate for the FLRW metric). The result is that, asides the baryonic mass density, ρ_B, two hypothetical ingredients seem to exists, namely an additional (dark) matter having mass density $\rho_{DM}(t)$ with negligible pressure, and another component with positive energy density, ρ_{DE}, but negative pressure $p_{DE} = -\rho_{DE}$, labeled dark energy (DE). In fact the following relations are obtained[14]

$$\begin{aligned} \left[\frac{\dot{a}}{a}\right]^2 &= \frac{8\pi G}{3}(\rho_B(t) + \rho_{DM}(t) + \rho_{DE}), \\ \frac{\ddot{a}}{a} &= \frac{8\pi G}{3}\left(\frac{1}{2}[\rho_B(t) + \rho_{DM}(t)] - \rho_{DE}\right), \end{aligned} \tag{7}$$

where small effects of radiation and matter pressure are neglected.

The baryonic density ρ_B is well known from the measured abundances of light chemical elements, which allows calculating ρ_{DE} and ρ_{DM} from the empirical quantities H_0 and q_0 via comparison of Equation (7) with Equation (6). The result may be summarized in the ΛCDM model. In it baryonic matter density, ρ_B, represents about 4.6% of the matter content while cold dark matter (CDM) and dark energy (represented by the greek letter Λ) contribute densities $\rho_{DM} \sim 23\%$, and $\rho_{DE} \sim 71.3\%$ respectively. The values obtained by this method agree with data from other observations. For instance cold dark matter, in an amount compatible with ρ_{DM}, is needed in order to explain the observed (almost flat) rotation curves in galaxies.

In this section I revisit the derived relation of the metric of spacetime, at the cosmological scale, with the mass densities $\rho_B(t), \rho_{DM}(t), \rho_{DE}$ and pressure $p_{DE} = -\rho_{DE}$ of the ΛCDM model. The standard approach is to use the FLRW metric as said above but for our purposes it is more clear to deal with a metric alternative to FLRW, Equation (5), using curvature coordinates for spherical symmetry whose most general form is as follows

$$ds^2 = g_{rr}(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - g_{tt}(r,t)dt^2. \tag{8}$$

This metric may be appropriate for a small enough region of the universe around us, but large in comparison with typical distances between galaxies[14].

The metric Equation (8) requires spherical symmety, that is both mass density and pressure should depend only on the radial coordinate r and time t. Obviously this is not the case for the actual universe where matter is mainly localized in galaxies. In practice an approximation is appropriate that consists of averaging the mass density over the whole region. I point out that a similar approximation is also made when the FRW metric Equation (1) is used[14]. Of course the dark energy density ρ_{DE} is assumed independent of position. Then the matter density may be taken as the sum of two *homogeneous* contributions, that is $\rho_{mat} = \rho_B(t) + \rho_{DM}(t)$, meaning baryonic and dark matter respectively, with negligible pressure, plus a dark energy with homogeneus density ρ_{DE} and negative pressure $p_{DE} = -\rho_{DE}$.

The metric coefficients are obtained from the distribution of mass density and pressure via Einstein equation. Here I will neglect terms of order $O(r^3)$ and ignore the (slow) change of the metric coefficients with time, a change derived from the slow time dependence of the matter density ρ_{mat}. With $g_{rr} = g_{tt} = 1$ for $r = 0$ we get the following metric elements [15]

$$\begin{aligned} g_{rr}(r) &= \left(1 - \frac{2Gm(r)}{r}\right)^{-1}, \; m(r) = m_{mat}(r) + m_{DE}(r) \\ m_{mat}(r) &= \int_{|\mathbf{z}|<r} \rho_{mat} d^3z, m_{DE}(r) \equiv \int_{|\mathbf{z}|<r} \rho_{DE} d^3z, \\ g_{tt}(r) &= \exp\gamma, \gamma = 2G\int_{|\mathbf{x}|<r} \frac{m(x) + 4\pi x^3 p_{DE}(x)}{x^2 - 2Gxm(x)} dx. \end{aligned} \tag{9}$$

As typically $Gm(r) << r$ an approximation is appropriate consisting of expanding Equation (9) in powers of the Newton constant G, retaining terms up to order $O(G^2)$. For Equation (9) this approximation agrees with order $O(r^2)$ in the radial parameter r, as may be easily checked. Thus I may write

$$g_{rr} = 1 + \frac{2Gm(r)}{r} + \frac{4G^2 m(r)^2}{r^2} + O\left(G^3\right). \tag{10}$$

$$\begin{aligned} g_{tt} &= 1 + 2G \int_0^r \left(\frac{m(x)}{x^2} + 4\pi x p(x) \right) dx + 2G^2 \left[\int_0^r \left(\frac{m(x)}{x^2} + 4\pi x p(x) \right) dx \right]^2 \\ &+ 4G^2 \int_0^r m(x) \left(\frac{m(x)}{x^3} + 4\pi p(x) \right) dx + O\left(G^3\right). \end{aligned} \tag{11}$$

Terms of order $O(G^2)$ will be relevant when the quantum vacuum is taken into account, as in the quantum approach of the next section but here we may neglect those terms. I shall take the contents of the universe into account as in the ΛCDM model, that is a (homogeneous) mass density given by $\rho_B + \rho_{DM} + \rho_{DE}$ and pressure $p_{DE} = -\rho_{DE}$. Then Equation (11) gives

$$\begin{aligned} g_{rr} &= 1 + \frac{8\pi G}{3} [\rho_B(t) + \rho_{DM}(t) + \rho_{DE}] r^2 + O\left(r^3\right), \\ g_{tt} &= 1 + \frac{8\pi G}{3} \left[\frac{1}{2} [\rho_B(t) + \rho_{DM}(t)] - \rho_{DE} \right] r^2 + O\left(r^3\right), \end{aligned} \tag{12}$$

a well known result[14], to be compared with the result obtained using the FLRW metric Equation (7).

3. A Quantum Treatment

3.1. The Quantum Vacuum

As a preliminary step I propose the following effective approach to the quantum vacuum. I shall assume that the vacuum energy density may be represented by a quantum operator $\hat{\rho}(\mathbf{r}, t)$ whose vacuum expectation is zero, but the expectation of its square is finite. Similarly we assume that a pressure operator, $\hat{p}(\mathbf{r}, t)$, exists for the vacuum, fulfilling

$$\hat{p}(\mathbf{r}, t) = -\hat{\rho}(\mathbf{r}, t), \langle vac | \hat{\rho}(\mathbf{r}, t) | vac \rangle = 0, \left\langle vac \middle| \hat{\rho}(\mathbf{r}, t)^2 \middle| vac \right\rangle > 0, \tag{13}$$

the latter inequality meaning that the vacuum energy density fluctuates.

The quantities relevant for our calculations are the two-point correlations of the density and pressure of the vacuum. In an approximate flat (Minkowski) space the vacuum should be invariant under translations and rotations, whence the vacuum expectation of the product of vacuum density operators (at equal times) should be a universal function, C, of the distance $|\mathbf{r}_1 - \mathbf{r}_2|$, that is

$$\frac{1}{2} \langle vac | \hat{\rho}(\mathbf{r}_1)\, \hat{\rho}(\mathbf{r}_2) + \hat{\rho}(\mathbf{r}_2) \hat{\rho}(\mathbf{r}_1) | vac \rangle = C(|\mathbf{r}_1 - \mathbf{r}_2|). \tag{14}$$

The function C may be named the self-correlation of the vacuum energy density.

In this article I will assume that the integral of $C(x)$ extended over the whole space, is nil that is

$$\int_{|\mathbf{r}_2| \in (-\infty, \infty)} C(|\mathbf{r}_1 - \mathbf{r}_2|) d^3 r_2 = \int_{|\mathbf{r}| \in (-\infty, \infty)} C(|\mathbf{r}|) d^3 r = 0. \tag{15}$$

The argument for the assumption Equation (15) is that the second Equation (13), which is the vanishing of the vacuum expectation of $\hat{\rho}_{vac}(\mathbf{r}, t)$, is a kind of nil ensemble average, which suggests a nil value

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 October 2024 doi:10.20944/preprints202410.0889.v1

for its time average. Hence the vanishing of the integral of the left side of Equation (14) with respect to $\mathbf{r}_1$ leads to Equation (15). That is the implication

$$\langle vac|\hat{\rho}_{vac}(\mathbf{r},t)|vac\rangle = 0 \Rightarrow \int \hat{\rho}_{vac}(\mathbf{r},t)d^3r = 0,$$

is a proposed ergodic property for the vacuum energy density.

Equation (15) may be generalized to the self-correlation of the pressure and the cross-correlation of density and pressure to be introduced in Section 4.2.

3.2. Quantization of an Approximate Einstein Equation

Quantizing general relativity (GR) is a difficult task due to its nonlinear character. Indeed finding a "quantum gravity" theory that unifies GR and quantum mechanics is believed the most relevant open problem in fundamental physics at present. Our purpose here is more modest, namely to quantize Einstein equation, approximated to second order in G, for a simple metric containing just two nontrivial metric elements.

In the quantum approach that follows I start quantizing the classical metric Equation (8) via promoting the metric elements (numerical quantities) to be operators, that is

$$d\hat{s}^2 \simeq \hat{g}_{rr}dr^2 + \hat{I}r^2(d\theta^2 + \sin^2\theta d\phi^2) - \hat{g}_{tt}dt^2, \tag{16}$$

where $\hat{I}$ is the identity operator and only two nontrivial metric elements appear, represented by the operators $\hat{g}_{rr}$ and $\hat{g}_{tt}$.

Then we must solve a quantum counterpart of Einstein equation in order to get the metric for a given energy-momentum tensor operator $\hat{T}_{\mu\nu}$. There are contributions of two different kinds. The first one is the mass density of the matter (either baryonic or dark, both with negligible pressure) plus the dark energy. These contributions may be treated as classical, more properly as multiples of the identity operator $\hat{I}$. The second contribution involves the operators $\hat{\rho}(\mathbf{r},t)$ and $\hat{p}(\mathbf{r},t)$ for the mass density and the pressure of the quantum vacuum, respectively (see Equation (13)). As is well known a big difficulty for the quantization of Einstein equation comes from the lack of commutatibity of the quantum operators. In the case of our solution of the approximate Einstein equation the difficulty is solved rather easily. In fact there are only two operators whose commutation is not trivial and they appear either alone or in pairs in the quantum counterpart of Equations (10) and (11). In the latter case the equation leads to the product of the operators in symmetrical order, which is most plausible for two similar operators. For instance the classical products transform into quantum products as follows

$$\begin{aligned}
\rho(\mathbf{r},t)\rho(\mathbf{r}',t) &\rightarrow \frac{1}{2}\left[\hat{\rho}(\mathbf{r},t)\hat{\rho}(\mathbf{r}',t) + \hat{\rho}(\mathbf{r}',t)\hat{\rho}(\mathbf{r},t)\right], \\
\rho(\mathbf{r},t)p(\mathbf{r}',t) &\rightarrow \frac{1}{2}\left[\hat{\rho}(\mathbf{r},t)\hat{p}(\mathbf{r}',t) + \hat{p}(\mathbf{r}',t)\hat{\rho}(\mathbf{r},t)\right].
\end{aligned}$$

Thus I will use quantum equations obtained via substituting operators for the classical quantities in Equations (10) and (11) (for notation the substitution amounts to putting a "hat" above the letter representing the classical variable). Thus the quantized approximate integrated Einstein Equation (16) leads to the following operators for the metric elements $\hat{g}_{\mu\nu}$

$$\hat{g}_{rr}(r) = 1 + \frac{2G\hat{m}(r)}{r} + \frac{4G^2\hat{m}(r)^2}{r^2} + O\left(G^3\right). \tag{17}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 October 2024 doi:10.20944/preprints202410.0889.v1

where I shall include the vacuum mass density operator, $\hat{\rho}_{vac}$, that is

$$\begin{aligned} \hat{m}(r) &= \hat{m}_{mat}(r) + \hat{m}_{vac}(r), \hat{m}_{vac}(r) \equiv \int_{|\mathbf{z}|>r} \hat{\rho}_{vac} d^3z \\ \hat{m}_{mat}(r) &\equiv \int_{|\mathbf{z}|>r} \hat{\rho}_{mat} d^3z = \hat{I} \int_{|\mathbf{z}|>r} \rho_{mat} d^3z. \end{aligned} \tag{18}$$

Similarly

$$\begin{aligned} \hat{g}_{tt} &= 1 + 2G\int_0^r x^{-2}\hat{m}(x)dx + G^2\sum_{n=1}^{5}\hat{c}_n + O\left(G^3\right), \\ \hat{c}_1 &= 4\int_0^r x^{-3}\hat{m}(x)^2 dx, \\ \hat{c}_2 &= \int_0^r x^{-2}dx\int_0^r y^{-2}dy[\hat{m}(x)\hat{m}(y) + \hat{m}(y)\hat{m}(x)], \\ \hat{c}_3 &= 32\pi^2\int_0^r xdx\int_0^r ydy[\hat{p}(x)\hat{p}(y) + \hat{p}(y)\hat{p}(x)], \\ \hat{c}_4 &= 8\pi\int_0^r [\hat{m}(x)\hat{p}(x) + \hat{p}(x)\hat{m}(x)]dx, \\ \hat{c}_5 &= 8\pi\int_0^r x^{-2}dx\int_0^r ydy[\hat{m}(x)\hat{p}(y) + \hat{p}(y)\hat{m}(x)]. \end{aligned} \tag{19}$$

Actually the quantum metric Equation (16) presents a difficulty similar to the classical Equations (8) and (9). That is the solution Equations (18) and (19) are valid only if the energy-momentum tensor operator $\hat{T}_{\mu\nu}$ depends on the coordinate r but not on the angular coordinates, θ, ϕ. I will solve the problem as in the classical case, Equations (10) and (11), that is averaging the energy density over large enough regions. However in the quantum domain the solution is more involved. In fact in the classical domain the dynamical variables are directly observables while in the quantum domain the dynamical variables are represented by operators (usually labeled "observables") and the actually observable quantities are the expectation values of the "observables" in the appropriate state $\mid \psi\rangle$. I believe that we should average the resulting expectation values, not the space-dependent operators. Thus in the quantum domain I shall simply assume that "quantum Einstein equations", Equations (18) and (19), are valid when we deal with regions having dimensions much larger than typical distances between galaxies, as is the case in our work.

In our quantum approach the energy-momentum tensor operator $\hat{T}_{\mu\nu}$ consists of two terms. The first one corresponds to the mass density operators $\hat{\rho}_B(t), \hat{\rho}_{DM}(t), \hat{\rho}_{DE}$ and pressure $\hat{p}_{DE} = -\hat{\rho}_{DE}$ analogous to the classical quantities in the ΛCDM model. I assume that they should appear in the form of operators which are muliples of the identity operator $\hat{I}$. The second contribution comes from the vacuum that I assume to have an energy-momentum tensor which is diagonal in a local frame, there consisting of the energy density operator $\hat{\rho}_{vac}$ and a pressure operator $\hat{p}_{vac}$.

The expectation value of the quantized metric element $\hat{g}_{rr}$, i.e., the counterpart of Equation (10) in the state $\mid \psi\rangle$, consitss of the following two terms

$$\langle\psi|\hat{g}_{rr}(r)|\psi\rangle \equiv g_{rr} = g_{rr}^{\text{model}} + g_{rr}^{vac},$$

where the superindex "model" stands for ΛCDM model. It will be calculated to order $O(G)$ because the second order contribution is negligible (see comment below Equation (9)). The latter (vacuum) should be got to order $O(G^2)$ because the term of order $O(G)$ is nil, see Equation (13). Then to $O(G)$ we get

$$
\begin{aligned}
g_{rr} &= 1 + \frac{2G}{r} \int_{|\mathbf{z}|<r} (\langle \psi | \hat{\rho}_{\text{model}}(\mathbf{z}) | \psi \rangle + \langle \psi | \hat{\rho}_{vac}(\mathbf{z}) | \psi \rangle) d^3 z \\
&= 1 + \frac{2G}{r} \int_{|\mathbf{z}|<r} \langle \psi | \hat{\rho}_{\text{model}}(\mathbf{z}) | \psi \rangle d^3 z,
\end{aligned} \tag{20}
$$

which will reproduce the standard result, first Equation (12), because that term involves

$$
\langle \psi | \hat{\rho}_{\text{model}}(\mathbf{z}) | \psi \rangle = \rho_{\text{model}} \langle \psi | \hat{I} | \psi \rangle = \rho_{\text{model}} = \rho_B(t) + \rho_{DM}(t) + \rho_{DE}.
$$

Similarly the expectation of $\hat{g}_{tt}$ to order $O(G)$ will reproduce the second Equation (12).

3.3. Contribution of the Quantum Vacuum

Taking Equation (17) into account, the term of order $O(G^2)$ of the g_{rr} metric element is, modulo the matter contribution of order $O(G)$ Equation (20),

$$
\begin{aligned}
g_{rr}^{vac} &= \frac{4G^2}{r^2} \left\langle \psi \middle| \hat{m}_{vac}(r)^2 \middle| \psi \right\rangle = \frac{4G^2}{r^2} \left\langle \psi \middle| \left[\int_{|\mathbf{z}|<r} \hat{\rho}_{vac}(\mathbf{z}) d^3 z \right]^2 \middle| \psi \right\rangle \\
&= \frac{2G^2}{r^2} \int_{|\mathbf{z}|<r} d^3 z \int_{|\mathbf{v}|<r} d^3 v \langle \psi | \hat{\rho}_{vac}(\mathbf{v})\, \hat{\rho}_{vac}(\mathbf{z}) + \hat{\rho}_{vac}(\mathbf{z}) \hat{\rho}_{vac}(\mathbf{v}) | \psi \rangle.
\end{aligned} \tag{21}
$$

Generalizing Equation (15) I assume that the two-point correlation function, C, depends only on the distance $|\mathbf{v} - \mathbf{z}|$, that is

$$
\frac{1}{2} \langle \psi | \hat{\rho}_{vac}(\mathbf{v})\, \hat{\rho}_{vac}(\mathbf{z}) + \hat{\rho}_{vac}(\mathbf{z}) \hat{\rho}_{vac}(\mathbf{v}) | \psi \rangle = C(|\mathbf{v} - \mathbf{z}|), \tag{22}
$$

which implies in particular that we may neglect the possible perturbations of the vacuum correlations due to the presence of matter. It is plausible that the function $C(|\mathbf{v} - \mathbf{z}|)$ is large and positive for small values of $|\mathbf{v} - \mathbf{z}|$ but the $\mathbf{v}$ integral over the whole space should be nil, that is

$$
\int C(|\mathbf{v} - \mathbf{z}|) d^3 v = \langle \psi | \hat{\rho}_{vac}(\mathbf{z}) | \psi \rangle = 0, \tag{23}
$$

see Equation (15). Therefore $C(|\mathbf{v} - \mathbf{z}|)$ will be negative for large values of $|\mathbf{v} - \mathbf{z}|$. An illustrative example is the function

$$
\begin{aligned}
C(x) &= a n^3 \exp(-3nbx) - a \exp(-3bx), n >> 1 \\
&\to C(0) = a(n^3 - 1), C(x) \simeq -a \exp(-3bx) \text{ for } x >> b^{-1},
\end{aligned} \tag{24}
$$

involving two parameters $\{a, b\}$. The properties of function C(x) suggest introducing an auxiliary function $F(x)$ related to it as follows

$$
C(x) = n^3 F(nx) - F(x), \tag{25}
$$

where $n >> 1$ is a real number and $F(x)$ is a function of the argument that I assume rapidly decreasing at infinity, fulfilling

$$
\lim_{x \to \infty} x^3 F(x) = 0 \Rightarrow \lim_{x \to \infty} x^3 C(x) = 0. \tag{26}
$$

Equation (25) guarantees that Equation (23) holds true. Indeed for integrals over the whole space we have

$$
\int d^3 x n^3 F(nx) = \int d^3 x' F(x') = \int d^3 x F(x) \Rightarrow \int d^3 x C(x) = 0. \tag{27}
$$

 doi:10.20944/preprints202410.0889.v1

The function $F(x)$, introduced in Equation (25), may be obtained for any physically appropriate function $C(x)$ as follows

$$F(x) = -\sum_{j=1}^{\infty} n^{3j+3} C\left(n^j x\right),$$

provided the series converges.

Now we may evaluate Equation (21) taking Equation (22) into account. I start with the following v-integral of $C(|\mathbf{v}-\mathbf{z}|)$

$$I \equiv \int_{|\mathbf{v}|<r} C(|\mathbf{v}-\mathbf{z}|)d^3v = n^3 \int_{|\mathbf{v}|<r} F(n|\mathbf{v}-\mathbf{z}|)d^3v - \int_{|\mathbf{v}|<r} F(|\mathbf{v}-\mathbf{z}|)d^3v. \tag{28}$$

In the limit $n \to \infty$ the function $n^3F(nx)$ becomes proportional to a 3D Dirac´s delta $\delta^3(x)$ as may be shown taking Equation (26) into account. Thus for very large n the relevant contribution to the first integral of Equation (28) comes from the region where $|\mathbf{v}-\mathbf{z}|$ is small. Hence we may extend the v-integral to the whole space with fair approximation provided that $|\mathbf{z}| < r$, but neglect it if $|\mathbf{z}| > r$. That is we may write

$$n^3 \int_{|\mathbf{v}|<r} F(n|\mathbf{v}-\mathbf{z}|)d^3v \simeq \Theta(r-|\mathbf{z}|)n^3 \int_{|\mathbf{v}|\in(0,\infty)} F(n|\mathbf{v}-\mathbf{z}|)d^3v,$$

where the step function $\Theta(y) = 1$ if $y \geq 0$, $\Theta(y) = 0$ otherwise. Hence Equation (28) gives

$$\begin{aligned} I &\simeq \Theta(r-|\mathbf{z}|)n^3 \int_{|\mathbf{v}|\in(0,\infty)} F(n|\mathbf{v}-\mathbf{z}|)d^3v - \int_{|\mathbf{v}|<r} F(|\mathbf{v}-\mathbf{z}|)d^3v \\ &= \Theta(r-z)n^3 \int_{|\mathbf{x}|\in(0,\infty)} F(nx)d^3x - \int_{|\mathbf{v}|<r} F(|\mathbf{v}-\mathbf{z}|)d^3v \\ &= \Theta(r-z) \int_{|\mathbf{x}'|\in(0,\infty)} F(x')d^3x' - \int_{|\mathbf{v}|<r} F(|\mathbf{v}-\mathbf{z}|)d^3v \\ &= \Theta(r-z) \int_{|\mathbf{v}|\in(0,\infty)} F(|\mathbf{v}-\mathbf{z}|)d^3v - \int_{|\mathbf{v}|<r} F(|\mathbf{v}-\mathbf{z}|)d^3v, \end{aligned}$$

leading to

$$I = \Theta(r-z) \int_{|\mathbf{v}|\geq r} F(|\mathbf{v}-\mathbf{z}|)d^3v \; - \Theta(z-r) \int_{|\mathbf{v}|<r} F(|\mathbf{v}-\mathbf{z}|)d^3v.$$

It is the case that I will integrate for $z \leq r$ everywhere in the rest of this section whence we may write

$$I = \Theta(r-z) \int_{|\mathbf{v}|\geq r} F(|\mathbf{v}-\mathbf{z}|)d^3v$$

in the following.

We get, taking Equations (21) and (22) into acount,

$$\begin{aligned} J &\equiv \int_{v\geq r, z<r} C(|\mathbf{v}-\mathbf{z}|)d^3v d^3z = 4\int_{z<r} d^3z \int_{v>r} d^3v F(|\mathbf{v}-\mathbf{z}|) \\ &= 32\pi^2 G^2 r^{-2} \int_0^r z^2 dz \int_r^\infty v^2 dv \int_{-1}^1 du F\left(\sqrt{v^2+z^2-2vzu}\right), \end{aligned} \tag{29}$$

where $u \equiv \cos\theta$, θ being the angle between the vectors $\mathbf{v}$ and $\mathbf{z}$. We know neither the two-point correlation function $C(|\mathbf{v}-\mathbf{z}|)$ nor $F(|\mathbf{v}-\mathbf{z}|)$ in detail but I propose to characterize the latter by just two parameters (see Equation (24)), namely the size D and the range γ. That is I will approximate the angular integral in Equation (29) as follows

$$f \equiv \int_{-1}^1 du F\left(\sqrt{v^2+z^2-2vzu}\right) \approx D\Theta(\gamma - |v-z|), \tag{30}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 October 2024 doi:10.20944/preprints202410.0889.v1

where $\Theta(x)$ is the step function and we assume that the parameter $\gamma > 0$ is small in the sense that $\gamma << r$. Thus we get

$$g_{rr}^{vac} \simeq 32\pi^2 G^2 r^{-2} D \int_{r-\gamma}^{r} z^2 dz \int_{r}^{z+\gamma} v^2 dv. \tag{31}$$

For later convenience I will summarize the steps going from Equation (29) to Equation (31), writing the following, slightly more general, relation valid for any $\alpha(v,z)$,

$$\int_{v<r,z<r} C(|\mathbf{v}-\mathbf{z}|)\alpha(v,z) d^3v d^3z = 8\pi^2 D \int_{r-\gamma}^{r} z^2 dz \int_{r}^{z+\gamma} \alpha(v,z) v^2 dv. \tag{32}$$

The integrals in Equation (31) are trivial and we obtain

$$g_{rr}^{vac} \simeq \frac{G^2}{r^2} \times 32\pi^2 D \int_{r-\gamma}^{r} z^2 dz \left[\frac{(z+\gamma)^3}{3} - \frac{r^3}{3} \right] = 16\pi^2 G^2 D \gamma^2 r^2 + O\left(\gamma^3\right). \tag{33}$$

The ratio $\gamma/r << 1$ is small because γ is a length typical of quantum fluctuations while r is of order the typical distance amongs galaxies (see comment after Equation (8)). Therefore we may neglect terms of order γ^3 whence we get

$$g_{rr}^{vac} \simeq 16\pi^2 G^2 K r^2, K \equiv D\gamma^2, \tag{34}$$

where I have substituted the single parameter K for the product D times γ^2. In the following I take the constant K as the relevant parameter, avoiding any detail about its origin from the two-point correlation of vacuum fluctuations $C(|\mathbf{v}-\mathbf{z}|)$.

The terms of order $O(G^2)$ of g_{tt}, Equation (19), may be obtained in a way similar to those of g_{rr}. For the first term we get

$$\begin{aligned} c_1 &\equiv \langle \psi | \hat{c}_1 | \psi \rangle = 4 \int_0^r x^{-3} dx \left\langle \psi \middle| \hat{m}(x)^2 \middle| \psi \right\rangle \\ &= 4 \int_0^r x^{-3} dx \int_0^x d^3z \int_0^x d^3v C(|\mathbf{v}-\mathbf{z}|), \end{aligned}$$

where $C(|\mathbf{v}-\mathbf{z}|)$ is the correlation function Equation (25). I will perform firstly the x integral, that is

$$\begin{aligned} c_1 &= 4 \int_{z<r} d^3z \int_{v<r} d^3v C(|\mathbf{v}-\mathbf{z}|) \int_{\max(v,z)}^{r} x^{-3} dx \\ &= 2 \int_{z<r} d^3z \int_{v<r} d^3v C(|\mathbf{v}-\mathbf{z}|) \left(\frac{1}{\max(v,z)^2} - \frac{1}{r^2} \right) \\ &= 16\pi^2 D r^{-2} \int_{r-\gamma}^{r} z^2 dz \left\{ \frac{1}{3} \left[(z+\gamma)^3 - r^3 \right] - r^2 (z+\gamma-r) \right\}, \end{aligned}$$

where I have take Equation (32) into account. The result is that c_1 is of order $O(\gamma^3)$ whence this term contributes but slightly to g_{tt}.

In order to get c_2 I start performing the x and y integrals, that is

$$\begin{aligned} c_2 &\equiv 8 \int_0^r x^{-2} dx \int_0^r y^{-2} dy \int_{z<x} d^3z \int_{v<y} d^3v C(|\mathbf{v}-\mathbf{z}|) \\ &= \int_{z<r} d^3z \int_{v<r} d^3v C(|\mathbf{v}-\mathbf{z}|) \int_z^r x^{-2} dx \int_v^r y^{-2} dy \\ &= \int_{z<r} \left(\frac{1}{z} - \frac{1}{r} \right) d^3z \int_{v<r} \left(\frac{1}{v} - \frac{1}{r} \right) d^3v C(|\mathbf{v}-\mathbf{z}|). \end{aligned}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 October 2024 doi:10.20944/preprints202410.0889.v1

Taking Equation (32) into account we obtain

$$\begin{aligned} c_2 &= 8\pi^2 D r^{-2} \int_{r-\gamma}^{r} z(r-z)\left[\frac{1}{2} r\left((z+\gamma)^2 - r^2\right) - \frac{1}{3}\left((z+\gamma)^3 - r^3\right)\right] dz \\ &= 8\pi^2 D r^2 \gamma^2 + O\left(\gamma^3\right) \simeq 8\pi^2 r^2 K. \end{aligned}$$

Now we must compute the G^2 contribution to g_{tt} coming from the pressure operator $\hat{p}_{vac}(\mathbf{r},t)$ of the vacuum that is the terms c_3, c_4 and c_5. Before proceeding I must deal with a difficulty due to the fact that Equation (9) are just valid for spherical symmetry. Actually that symmetry holds neither for the distribution of matter in the region of interest nor for the stress-energy of the quantum vacuum. In fact the stress-energy appears in the form of localized operators of energy density $\hat{\rho}_{vac}(\mathbf{r},t)$ and pressure $\hat{p}_{vac}(\mathbf{r},t)$. Actually this was also the case of the mass and pressure distribution leading the the terms of order G in the metric elements in Section 3. Indeed I have solved the problem via a standard approximation that consists of averaging the matter over the entire region. For the vacuum operator $\hat{\rho}_{vac}(\mathbf{r},t)$ the problem is not too serious because that operator enters just in the mass $\hat{m}(r)$, whose definition in Equation (18) already involves an integral. However there is a more difficult problem with the pressure operator $\hat{p}_{vac}$ that actually depends on the position $\mathbf{x}$ rather than on the radial coordinate x alone as in Equation (19). A plausible approximation is to average the operator over the angular variables. Then I will use as $\hat{P}(x)$, an angular average operator, rather than $\hat{p}(x)$, in Equation (19), that is

$$\hat{P}(x) \to \frac{1}{4\pi}\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \hat{p}(\mathbf{x}) = \frac{1}{4\pi x^2}\int \hat{p}(\mathbf{z}) d^3 z \delta(x-z), \tag{35}$$

$\delta()$ being Dirac delta so that the $\mathbf{z}$ integral may be extended to the whole space with fair approximation.

After substituting $\hat{P}$ for $\hat{p}$ in Equation (19) we may get the expectation of the term of order $O(G^2)$ belonging to the metric element $\hat{g}_{tt}(r)$. In order to compute the numerical value we must introduce two new correlation functions similar to Equation (22), that is

$$\begin{aligned} \frac{1}{2}\langle\psi|\hat{p}_{vac}(\mathbf{v})\ \hat{p}_{vac}(\mathbf{z}) +\ \hat{p}_{vac}(\mathbf{z})\hat{p}_{vac}(\mathbf{v})\ |\psi\rangle &= C_{pp}(|\mathbf{v}-\mathbf{z}|), \\ \frac{1}{2}\langle\psi|\hat{\rho}_{vac}(\mathbf{v})\ \hat{p}_{vac}(\mathbf{z}) +\ \hat{p}_{vac}(\mathbf{z})\hat{\rho}_{vac}(\mathbf{v})|\psi\rangle &= C_{\rho p}(|\mathbf{v}-\mathbf{z}|). \end{aligned} \tag{36}$$

The evaluation of the term c_3 is as follows, taking Equations (19), (35) and (36) into account,

$$\begin{aligned} c_3 &= 32\pi^2 \int_0^r x dx \int_0^r y dy \langle\psi|[\hat{p}(x)\hat{p}(y) + \hat{p}(y)\hat{p}(x)]|\psi\rangle \\ &\to 32\pi^2 \int_0^r x dx \int_0^r y dy \langle\psi|\hat{P}_{vac}(x)\ \hat{P}_{vac}(y) +\ \hat{P}_{vac}(y)\hat{P}_{vac}(x)\ |\psi\rangle \\ &= 64\pi^2 \int_0^r x dx \int_0^r y dy \frac{1}{16\pi^2 x^2 y^2}\int d^3 z \delta(x-z)\int d^3 v \delta(y-v) C_{pp}(|\mathbf{v}-\mathbf{z}|) \\ &= 4\int_{z<r} z^{-1} d^3 z \int_{v<r} v^{-1} d^3 v C_{pp}(|\mathbf{v}-\mathbf{z}|), \end{aligned}$$

where the x and y integrals have been performed.

Now I assume that an (approximate) equality holds similar to Equation (32). Then I get

$$\begin{aligned} c_3 &= 32\pi^2 D_{pp} \int_{r-\gamma}^{r} z dz \int_r^{z+\gamma} v dv = 32\pi^2 D_{pp} \int_{r-\gamma}^{r} z dz \frac{(z+\gamma)^2 - r^2}{2} \\ &= 8\pi^2 \grave{D}_{pp} r^2 \gamma^2 + O\left(\gamma^2\right) \simeq 8\pi^2 r^2 K_{pp}, K_{pp} \equiv \grave{D}_{pp}\gamma^2. \end{aligned}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 October 2024 doi:10.20944/preprints202410.0889.v1

Also I suppose that similar approximations are valid when the density operator is combined with the pressure opertor. Thus we may calculate c_4 and c_5 in a similar way.

$$
\begin{aligned}
c_4 &= 8\pi \int_0^r dx \langle \psi | [\hat{m}(x)\hat{p}(x) + \hat{p}(x)\hat{m}(x)] | \psi \rangle \\
&\to 8\pi \int_0^r dx \langle \psi | \left[\hat{m}(x)\hat{P}(x) + \hat{P}(x)\hat{m}(x)\right] | \psi \rangle \\
&= 16\pi \int_0^r dx \frac{1}{4\pi x^2} \int d^3 z \delta(x-z) \int_{v<x} d^3 v C_{\rho p}(|\mathbf{v}-\mathbf{z}|) \\
&= 4 \int_{z<r} z^{-2} d^3 z \int_{v<r} d^3 v C_{\rho p}(|\mathbf{v}-\mathbf{z}|) \\
&= 4 D_{\rho p} \int_{r-\gamma}^{r} z^{-2} d^3 z \int_r^{z+\gamma} d^3 v \\
&= 64\pi^2 D_{\rho p} \int_{r-\gamma}^{r} dz \frac{(z+\gamma)^3 - r^3}{3} = 32\pi^2 r^2 D_{\rho p} \gamma^2 + O\left(\gamma^3\right)
\end{aligned}
$$

Thus we get

$$
c_4 = 32\pi^2 K_{\rho p} r^2, K_{\rho p} \equiv D_{\rho p}.
$$

$$
\begin{aligned}
c_5 &= 8\pi \int_0^r x^{-2} dx \int_0^r y dy \langle \psi [\hat{m}(x)\hat{p}(y) + \hat{p}(y)\hat{m}(x)] \psi \rangle \\
&\to 16\pi \int_0^r x^{-2} dx \int_0^r y dy \int_0^x d^3 z \frac{1}{4\pi y^2} \int d^3 v \delta(y-v) C_{\rho p}(|\mathbf{v}-\mathbf{z}|) \\
&= 4 \int_0^r \left(\frac{1}{z} - \frac{1}{r}\right) d^3 z \int_{v<r} v^{-1} d^3 v C_{\rho p}(|\mathbf{v}-\mathbf{z}|) \\
&= 64\pi^2 r^{-1} D_{p\rho} \int_{r-\gamma}^{r} (r-z) z dz \int_r^{z+\gamma} v dv \\
&= 32\pi^2 r^{-1} D_{p\rho} \int_{r-\gamma}^{r} (r-z) z dz \left[(z+\gamma)^2 - r^2\right] = O\left(\gamma^3\right).
\end{aligned}
$$

Hence the term c_5 does not contribute to order $O(\gamma^2)$. In summary we have for the g_{tt} element of the metric Equation (8)

$$
g_{tt} = 1 + \frac{4}{3}\pi G \rho_{mat} - \frac{8\pi G}{3}\rho_{DE} r^2 + 8\pi^2 G^2 r^2 (K + K_{pp} + 4K_{\rho p}). \tag{37}
$$

It is plausible that the quantities K and K_{pp} are both positive but $K_{\rho P}$ negative. In fact we may assume that in quantum vacuum fluctuations the pressure acts with a sign opposite to the mass density, in agreement with the Lorentz invariant vacuum equation of state $p = -\rho$. This suggests identifying

$$
K_{pp} = K, K_{\rho p} = -K \tag{38}
$$

whence we get, taking Equations (20) and (34),

$$
g_{rr} = 1 + \frac{8\pi G}{3}(\rho_B(t) + \rho_{DM}(t) + \rho_{DE}) r^2 + 16\pi^2 G^2 K r^2. \tag{39}
$$

Similarly from Equations (37) and (38) we obtain

$$
g_{tt} = 1 + \frac{8\pi G}{3}\left(\frac{1}{2}\rho_B(t) + \frac{1}{2}\rho_{DM}(t) - \rho_{DE}\right) r^2 - 16\pi^2 G^2 K r^2. \tag{40}
$$

These results reproduce the standard ones Equation (12) plus a correction due to the quantum vacuum fluctuations (i.e., the last term in Equations (39) and (40)).

 doi:10.20944/preprints202410.0889.v1

4. Results and Discussion

The main result of this article is that Equations (39) and (40) should be substituted for the standard Equation (12). Then the following should be substituted for Equation (1)

$$\begin{aligned} \rho_{DE} + 6\pi GK &\simeq (6.0 \pm 0.2) \times 10^{-27} kg/m^3 \\ &\Rightarrow 0 < K \lesssim \frac{(6.0 \pm 0.2) \times 10^{-27} kg/m^3}{6\pi G}. \end{aligned} \tag{41}$$

Then it might be that either the acceleration in the expansion of the universe is due to the quantum vacuum (if the latter inequality is really an equality) or the vacuum gives just a contribution to be added to the effect of a dark energy. In the former case the value of the parameter K would be following

$$K \equiv D\gamma^2 = \frac{\rho_{DE} c^2}{6\pi G} \simeq 0.42 kg^2/\mathrm{m}^4, \sqrt{K} \simeq 0.65 kg/\mathrm{m}^2. \tag{42}$$

Taking Equations (28) to (30) into account the quantity $\sqrt{K}$ may be seen as the product of the typical mass density of the vacuum fluctuations $\sqrt{D}$ times its typical correlation length γ. It is fitting that the value of $\sqrt{K}$, Equation (42), is not too far from the product of the typical nuclear density, 2.3×10^{17} kg/m^3, times a typical nuclear radius, about 10^{-15}m. For instance if the correlation length of the vacuum energy density was 10^{-11} m, a typical atomic distance, then the fluctuation of the density would be about 10^{-7} times the nuclear density.

In summary our work does not prove that quantum vacuum fluctuations are a valid alternative to dark energy. But it does show that such fluctuations give rise to a contribution with qualitative effects similar to those of the dark energy.

Appendix A. A Note on Interpretation

In the calculation of the present paper I have not attempted any interpretation, but the treatment has followed the standard quantum formalism. However a few comments on interpretation are in order.

After one century of quantum mechanics there is no agreement about the interpretation of the theory[16], in particular about the real meaning of the "quantum probability". The standard wisdom is that quantum probabilities are dramatically different from the common probabilities used in so many areas, from economy or biology to classical statistical mechanics[17]. Actually there are two types of probability in quantum theory, one in the measurement, the other type in the definition of mixed states. The latter are similar to the common ones above mentioned [17] and may be associated to incomplete information.

In the measurement of the properties of a pure state, several different results may be obtained with a definite *probability* each. These probabilities are *not* attributed to incomplete information about the state of the system, which is assumed pure. The common opinion is that they appear due to a lack of causality of the physical laws, a strange assumption indeed. I support the view that there are "hidden variables" that might determine the results of the measurements. If this is the case the probabilities involved are also standard [17], that is no specific "quantum probabilities" exist. The current wisdom is that suitable hidden variables, that is local, are not possible [18–20]. See however [22,23]. The described situation also applies to the particular case of the "vacuum state". This state is believed to be pure but vacuum fluctuations exist which are also believed to correspond to the peculiar "quantum probabilities".

References

1. A. G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astronomical J.* **116,** 10091038 (1998).

 doi:10.20944/preprints202410.0889.v1

2. S. Perlmutter et al.: Measurement of Omega and Lambda from 42 high red-shift supernovae. *Astrophys. J.* **517**, 565586 (1999).
3. D. Huterer, M. Turner: Prospects for probing the dark energy via supernova distance measurements. *Phys. Rev. D* **60,** 081301 (1999).
4. M. López-Corredoira. *Fundamental Ideas in Cosmology. Scientific, philosophical and sociological critical perspective.* IOP Publishing, Bristol, 2022.
5. C. L. Bennett et al.: Nine-year Wilkinson microwave anisotropy observation (WMAP). Final maps and results. *Astrophys. J. Suppl. Ser.* **192**, 17 (2011).
6. M. Kowalski et al. Improved cosmological constraints from new, old and combined supernova datasets. *The Astrophysical J.* **686**, 749778. 2008.
7. P. J. E. Peebles, Bharat Ratra: The cosmological constant and dark energy. *Rev. Mod. Phys.* **75**, 559-606 (2003).
8. S. Weinberg: The cosmological constant problem. *Rev. Mod. Phys.* **61**,1-23 (1989).
9. T. Padmanabhan: Cosmological constant. The weight of the vacuum. *Phys. Rept.* **380**, 235 (2003).
10. V. Motta, M. A. García-Aspeitia, A. Hernández-Almeida, et al.: Taxonomy of dark energy models. *Universe* **7,** 163 (2021).
11. B. .Y. Zeldovich, *Sov. Phys. Usp.* **24**, 216 (1981).
12. E. Santos: Dark energy as a curvature of space-time induced by quantum vacuum fluctuations. *Astrophys. Space Sci.* **332**, 423-435 (2011).
13. Varum Sahni, *Lect. Notes Phys.* **653**, 141-180 (2004).
14. J. Rich: *Fundamentals of Cosmology*. Springer-Verlag, Berlin, 2001.
15. J. Synge: *Relativity*. *The general theory*. Noth-Holand. Amsterdam, 1960. Page 270.
16. O. Freire, editor: *The Oxford handbook of The history of quantum interpretations.* Oxford U.P. 2022.
17. E. T. Jaynes: *Probability theory: The logic of science.* 3060 pages. Cambridge University Press. New York. 2003.
18. J. S. Bell: *Speakable and unspeakable in quantum mechanics,* Cambridge University, 1987. This book reproduces most of Bell articles on foundations.
19. H. Wiseman: Death by experiment for local realism. *Nature* **526**, 687 (2015).
20. A. Aspect: Closing the door on Einstein and Bohr's quantum debate. *Physics* **8**, 123 (2015).
21. E. Santos: Local realistic interpretation of entangled photon pairs in the Weyl-Wigner formalism. *Front. Phys.* **8**, 191 (2020).
22. E. Santos: Local model of entangled photon experiments compatible with quantum predictions based on the reality of the vacuum fields. *Found. Phys.* **50**, 1587-1607 (2020).
23. E. Santos: *Realistic interpretation of quantum mechanics*. Cambridge Scholars Publishing. 2022.