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Posted Date: 7 January 2025

doi: 10.20944/preprints202501.0519.v1

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Examples of Compact Simply Connected Holomorphic Symplectic Manifolds Which Are Not Formal

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Abstract: In this paper, we prove that the complex four dimensional compact holomorphic symplectic manifold we found earlier is not formal. This gives another strong consequence that it is not a topological Kähler manifold. We also conjecture that this is true for the higher dimensional ones.

Keywords: holomorphic symplectic structure; complex manifolds; formality; Lefschets condition; relative Massey products

MSC: MSC2020 53C10; 53C26; 53D05; 22E25; 32M12; 32Q55

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1. Introduction

Every Kähler structure on a smooth manifold also provides a symplectic structure on that manifold. On the other hand, examples have been known for some time of symplectic manifolds that do not admit any Kähler structure [Th]. Much work has been done probing this difference between Kähler versus symplectic manifolds. A major step was taken by D. McDuff [Mc], who gave the first examples of simply connected symplectic manifolds that do not admit any Kähler structure. Her example was of a real dimension ten. In both Thurston's and McDuff's examples, the criterion used to determine non-admittance of a Kähler structure was cohomological.

At a deeper level than cohomology, the rational homotopy structure of Kähler manifolds was elucidated in [DGMS], where it was shown that compact, simply connected Kähler manifolds are formal. In [DGMS], the authors apply the work of the fourth author on relating the real homotopy type of a compact manifold to its algebra of differential forms [Su1, 2, 3], to Kähler manifolds. Using the full strength of Hodge theory the authors show in particular that the real homotopy type of a simply-connected compact Kähler manifold M is entirely determined by its cohomology ring [We].

In [Gu1,2] we have found irreducible compact holomorphic symplectic manifolds of dimension $2n$ for $n \geq 2$ which do not admit any Kähler structure. Those manifolds are simply connected. Actually, on those manifolds, there is a quadratic form on the second cohomology such that the n-th power of it is proportional to the 2n-th product of the element (in [Gu1]). This quadratic form has a kernel generated by an element **b** (β in [Gu2], or the element γ in [Gu1]). Therefore, **b** is in the kernel of the 2-Lefschets map with any element (or symplectic structure). That is, the Lefschets condition always fails for any given real symplectic structure. In particular, when $n = 2$, our example actually gave the first example of compact simply connected real symplectic manfold of eight dimension which is not Kähler. Many four and six dimensional examples were eventually found later on by Gompf.

Motivated by this construction, M. Fernandez and V. Muñoz [FM], as mentioned in their paper, found an eight dimensional compact simply connected real symplectic manifold which is not formal after [BT] solved the same problem for manifolds with dimensions greater or equal to ten.

In this paper, we find that

Theorem A. *When $n = 2$, our original eight dimensional manifold is also not formal.*

For any n there are two other elements a and c such that ab^{n-1}, b^n, cb^{n-1} are exact. Therefore, we can define the b^{n-1} relative Massey product $d = (b^{n-1} : a, b, c)$ as in [CFM]. We proved that for

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$n = 2$, d is nonzero. Therefore, we proved that when $n = 2$ our original simply connected real eight dimensional holomorphic symplectic manifold is not formal.

Let M be a complex manifold of dimension $2n$. A holomorphic symplectic structure or form is a closed holomorphic 2-form ω on M with maximal rank (see [Kb p.47]).

People might ask that under what condition a compact holomorphic symplectic manifold is Kählerian, i.e., admits a positive closed (1,1) form. For example, by [Td2] and [Si], we know that every K-3 surface is Kählerian. In [Td1], Todorov asked if every irreducible compact holomorphic symplectic manifold of dimension more than 4 is Kählerian. Some counter-examples have been found in [Gu1,2]. Those are the manifolds we deal with in this paper.

2. Preliminary

2.1. Compact Kähler Holomorphic Symplectic Manifolds

Here we collect some results on compact Kähler holomorphic symplectic manifolds, i.e., compact complex manifolds with both holomorphic symplectic structures and Kähler structures. From [Bg], [Bv] and [Fj], we have:

Proposition 1. *Let X be a compact Kähler holomorphic symplectic manifold. Then M admits a hyperkähler structure, i.e., there is a Kähler structure which is Ricci flat and the holomorphic symplectic structure is parallel with regard to this Kähler structure. And there exists a finite unramified Galois covering $\tilde{X} \longrightarrow X$ such that $\tilde{X}$ is isomorphic to a product $T \times Y_1 \times \cdots \times Y_r$, where T is a complex torus and Y_i, $1 \leq i \leq r$, are simply connected Kähler symplectic manifolds with $h^{2k,0}(Y_i) = 1$ and $h^{2k-1,0}(Y_i) = 0$ for any $k \geq 1$. Here the direct factors are uniquely determined by X up to permutation. Moreover, if for each i we let $g_i(x)$ be the homogeneous form of degree $2n_i$ ($n_i = \dim_{\mathbf{C}} Y_i$) on $H^2(Y_i, \mathbf{Q})$ defined by $g_i(x) = x^{2n_i}[Y_i]$, $x \in H^2(Y_i, \mathbf{Q})$ where $[Y_i]$ denotes the evaluation on Y_i, then there exists a constant $c \in \mathbf{Q}^+$ and a nondegenerate quadratic form f_i of signature $(3, b_i - 3)$ on $H^2(Y_i, \mathbf{Q})$ such that $g_i = cf_i^{n_i}$, where $b_i = b_2(Y_i)$. If we let φ_i be a holomorphic 2-form on Y_i such that $(\varphi_i\bar{\varphi}_i)^{n_i}[Y_i] = 1$, then f_i can be written as*

$$\frac{n_i}{2}(\varphi_i\bar{\varphi}_i)^{n_i-1}x^2[Y_i] + (1-n_i)\varphi_i^{n_i-1}\bar{\varphi}_i^{n_i}x[Y_i] \cdot \varphi_i^{n_i}\bar{\varphi}_i^{n_i-1}x[Y_i].$$

up to a multiple of a constant.

This, for example, comes from [Bg], Theorem 5 in [Bv] and Theorem 4.7 in [Fj] (see also [Kb], [Gu2]). We gave a simpler proof and a generalization for the last part (or Fujiki Theorem [Fj]) of this Proposition in the section 5 in [Gu1].

2.2. Compact Holomorphic Symplectic Surfaces

It is well known that every simply connected holomorphic symplectic surface is a K-3 surface and its second Betti number is 22. There are 3 different classes of holomorphic symplectic surfaces, i.e., K-3 surface K, complex torus A and Kodaira-Thurston surface S (see [BPV p.188]). The holomorphic symplectic 2-forms come from any trivial canonical sections. We are more interested in the surface S here. We know that $b_1(S) = 3$, $b_2(S) = 4$. Let $K^{[r]}$, $A^{[r]}$ and $S^{[r]}$ be the Hilbert scheme which parameterize the finite subsets Z with $|Z| = r$ (see [Bv], [Fg1], [Fg2] and [Ir]), then

Proposition 2. (Cf. [Bv], [Vr]) *$K^{[r]}$, $A^{[r]}$ and $S^{[r]}$ are compact holomorphic symplectic manifolds and $b_1(K^{[r]}) = 0$, $b_1(A^{[r]}) = 4$, $b_1(S^{[r]}) = 3$. Moreover, $K^{[r]}$, $A^{[r]}$ are Kähler and $S^{[r]}$ is not Kähler.*

$S^{[r]}$ is not Kähler since $b_1(S^{[r]})$ is not even.

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2.3. Compact Parallelizable Manifolds

A *parallelizable manifold* is the quotient of a real Lie group by a discrete subgroup. It is a *solv-manifold* or *nil-manifold* according to that the Lie group is either solvable or nilpotent. We have following Nomizu's Theorem:

Proposition 3. *Let $M = G/H$ be a compact parallelizable nil-manifold, then the deRham cohomology can be calculated by the complex of G-right invariant forms, which is isomorphic to the complex of the Lie algebra.*

Notice that if G has a G-right invariant complex structure, which induces a complex structure on M, there was a question on the Dolbeault cohomology part. But in [Gu1, 2], similarly in this paper, we only deal with the case in which we apply a similar version to this Proposition to S, which is a complex torus over a complex torus. That is, our complex structures on the nilmanifolds are rational in [CF]. Similarly for the nilmanifolds in this paper. Actually, we do not need the Dolbeault cohomology in this paper. Therefore, there is an advantage of the method in this paper from [Gu1, 2].

A Kodaira-Thurston surface is a nil-manifold with a complex structure that comes from a right G-invariant complex structure on G. For example, we consider $S_r = G/H_r$ with

$$G = \left\{ (x,\ y) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & \bar{x} \\ 0 & 0 & 1 \end{pmatrix} |_{x,\ y \in \mathbf{C}} \right\}$$

and

$$H_r = \{(x,\ y) \in G|_{x \in r(\mathbf{Z}+\mathbf{Z}i),\ y \in r(\mathbf{Z}+\mathbf{Z}i)}\}$$

where $r \in \mathbf{Z}$. They admit holomorphic symplectic structure $\omega = dx \wedge dy$. There is a covering $M = G/T$ of S_r with $T = \{(0,\ y) \in H_r\}$. On M the function x define the hamiltonian vector field $X_x = \partial/\partial y$ by $dx = \omega(\ , X_x)$. This structure induces a holomorphic symplectic reduction of $S_r^{[r]}$ (see [Gu2]).

Proposition 4. (Cf. [Bv], [Gu2]) *Let R be either $A^{[r]}$, $A = T \times T$ with T a torus, or $S_r^{[r]}$, let L be either the torus which is the left torus in the definition of A, or the center of S (which is generated by y). Then the L action on R which is induced by the diagonal action of L on the product of L is a subgroup of the holomorphic symplectic automorphism group. If R_r is the symplectic quotient of R under L, then R_r is a holomorphic symplectic orbifold and there is a r^2 to 1 covering from a compact simply connected holomorphic symplectic manifold $R_{[r]}$ to R_r. And $R_{[2]}$ is a K-3 surface. If $r > 2$, we have that in the case of A, $b_2(R_{[r]}) = 7$ and $R_{[r]}$ is Kähler; in the case of S_r, $b_2(R_{[r]}) = 6$ and $R_{[r]}$ is not Kähler.*

3. General Construction from Parallelizible Manifolds

Now, following [Gu1], we start to construct some examples of simply connected compact holomorphic symplectic manifolds from parallelizible manifolds. We call a parallelizible manifold a *parallelizible holomorphic symplectic manifold* if it admits a holomorphic symplectic structure which is induced by a right invariant 2-form on G. As one can see from the last part of the proof of the Theorem A in [Gu3] that these manifolds must be solv-manifolds:

Proposition 5. *Every compact parallelizible symplectic manifold is a solv-manifold.*

The method which we use here (was from [Gu1]) generalizes the method in [Gu2]. Starting from a holomorphic symplectic nil-manifold (or solv-manifold) M (see [Gu1] for more examples of them), we try to find a faithful representation of a finite group $\mathcal{S}$ as a subgroup (we denote this group also by $\mathcal{S}$ if there is no confusion) of the automorphism of the Lie group which preserves the isotropic group and the complex structure as well as the holomorphic symplectic structure. One might find this easily by the condition that the complex structure equation is preserved by $\mathcal{S}$. Suppose that the subset $D = \{m \in M|_{s(m)=m \text{ for some } s \in \mathcal{S}}\}$ has only codimension 2 irreducible components and

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$(\pi_1(M))^{\mathcal{S}} = 1$, then a desingulation of the quotient $M/\mathcal{S}$ is probably a simply connected holomorphic symplectic manifold which is not Kählerian. For example, we can construct the examples $R_{[r]}$ in [Gu2] directly from the nil-manifold $Q^{[r]}$ there.

Instead of $Q^{[r]}$, we consider $M_{r,t}$ with structure equation:

$$dx_i = 0$$
$$dy_i = -(r-1)tx_i \wedge \bar{x}_i + 2t\sum_j x_j \wedge \bar{x}_j + t\sum_{j\neq k} x_j \wedge \bar{x}_k$$

where $(x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_r)$ are the coordinates of the Lie algebra of $M_{r,t}$, $r \in \mathbf{N}$ $t \in \mathbf{Z}$. It is not difficult to check that $Q^{[r]} = M_{r-1,1}$.

Now we define the symmetric group $\mathcal{S}_{r+1}$ action on $M_{r,t}$. $\mathcal{S}_{r+1}$ is generated by $\tau_{(1,\ r+1)} = (1,\ r+1)$ and $\mathcal{S}_r$. The group $\mathcal{S}_r$ acts on $M_{r,t}$ just as $\mathcal{S}_r$ and

$$\begin{aligned}\tau_{(1,\ r+1)}(x_1,\ x_2,\ \cdots,\ x_r,\ \ y_1,\ y_2,\ \cdots,\ y_r) =\\ (-x_1 - x_2 - \cdots - x_r.\ x_2,\ \cdots,\ x_r,\\ -y_1 - y_2 - \cdots - y_r,\ y_2,\ \cdots,\ y_r).\end{aligned}$$

It is not difficult to check that $\mathcal{S}_{r+1}$ preserves the structure equations and the set $D_{r,t}$ has only codimension 2 irreducible components. Hence we can desingular $M_{r,t}/\mathcal{S}_{r+1}$ as in [Gu2], which is achieved through the resolution provided by the Hilbert schemes for the singularities of the symmetry product, and we denote the desingulation by $Q_{r,t}$, then we have:

Proposition 6. (Cf [Gu2 Theorem 1]) *The desingulation $Q_{r,t}$ of $M_{r,t}/\mathcal{S}_{r+1}$ is a compact simply connected holomorphic symplectic manifold which is a K-3 surface if $r = 1$ and is not Kählerian with $b_2 = 6$ if $r > 1$.*

The simply connectedness basically comes from

$$\pi_1(Q_{r,t}) = (\pi_1(M_{r,t}))^{\mathcal{S}_{r+1}} = 0.$$

See also [Bg1].

An alternative proof of the nonkählerness comes from the fact that the pair of $H^2(M_{r,t})$ and $H^2(M_{r,t}) \wedge \omega^{r-1} \wedge \bar{\omega}^{r-1}$ (or $H^2 \wedge \omega^{2r-2}$ for any given real symplectic structure ω) is not a perfect pair.

The possible closed $(1,1)$ $\mathcal{S}_{r+1}$-invariant classes are linear combinations of $\alpha = a\sum_j y_j \wedge \bar{y}_j + b\sum_{j\neq k} y_j \wedge \bar{y}_k$ and $\beta = c\sum_j x_j \wedge \bar{y}_j + d\sum_{j\neq k} x_j \wedge \bar{y}_k$, $\delta = g\sum_j y_j \wedge \bar{x}_j + h\sum_{j\neq k} y_j \wedge \bar{x}_k$, $\gamma = e\sum_j x_j \wedge \bar{x}_j + f\sum_{j\neq k} x_j \wedge \bar{x}_k$. Since they are invariant under the action of $\tau_{(1,r+1)}$, we have $a = 2b$, $c = 2d$, $e = 2f$, $g = 2h$. And from

$$\bar{\partial}\alpha = \bar{\partial}\beta = \bar{\partial}\gamma = \bar{\partial}\delta = 0$$

we get $a = 0$, that is, $\alpha = 0$. There is no $\mathcal{S}_{r+1}$ invariant $(1,0)$ form, we get that $(H^{1,1}(M_{r,t}))^{\mathcal{S}_{r+1}}$ is generated by β, δ, γ with $c = e = g = 2$. We see that $H^{1,1}(Q_{r,t})$ has dimension $3+1 = 4$. In the same way we find that $H^{2,0}(Q^{r,t})$ is generated by $\omega = 2\sum_j x_j \wedge y_j + \sum_{j\neq k} x_j \wedge y_k$ and $H^{0,2}(Q_{r,t})$ is generated by $\bar{\omega}$. We see that $H^2(Q_{r,t},\ \mathbf{C}) = H^{2,0} + H^{1,1} + H^{0,2}$.

We let $2b = i\gamma$, then b is the β in [Gu2]. By [Gu1], even in the nonkähler case with the property that $H^2 = H^{0,2} + H^{1,1} + H^{0,2}$ we still have the topological quadratic and b is in the kernel.

In the rest of this paper, *we assume that $t = 1$.*

We let $4a = \beta + \omega - \delta + \bar{\omega}$. Then a is the same as the $\sum_j z_j \wedge u_j$ in [Gu2].

We also let $4ic = \omega - \beta - \delta - \bar{\omega}$. Then c is the same as the $\sum_j z_j \wedge v_j$ in [Gu2].

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Similarly, when $r = 2$, one could calculate that the orbifold contribution of the de Rham cohomology in H^3 only comes from

$$\sum_{i\neq j} x_i \wedge ((y_j + \bar{y}_j) \wedge \bar{x}_i + (y_1 + y_2 + \bar{y}_1 + \bar{y}_2) \wedge \bar{x}_j).$$

4. Several Massey Products

4.1. Our Main Result

THEOREM 1. *Let (M, ω) be the compact simply connected holomorphic symplectic manifold constructed above of complex dimension $2n$, for the technic reason we assume that $t = 1$, then 1. $b^{n-1}a, b^n, b^{n-1}c$ are exact; 2. When $n = 2$ the quadruple Massey product (a, b, b, c) is nonzero; 3. When $n = 2$, the relative Massey product $(b : a, b, c)$ is nonzero; 4. M^2 is nonformal.*

We notice that our notation of the both Massey quadruple product and the relative b Massey product is a little bit different from those in [FM] and [CFM]. The reason is that our definitions (see later in this section) is more of a form than a class although they do represent a cohomology class.

4.2. Some Calculations

For the definitions of Massey quadruple products and the relative Massey products, one could check with [FM]. We shall also provide some simple definitions later on in the process of our proof of this Theorem.

Since we work on H^2 instead of the Hodge cohomology, we shall retain the notation from [Gu2] instead of that from [Gu1] in this section.

To start with, we have

$$b = 2\sum_j z_j \wedge w_j + \sum_{j\neq k} z_j \wedge w_k$$

$$a = \sum_j z_j \wedge u_j,\ c = \sum_j w_j \wedge u_j.$$

$$dz_i = dw_i = du_i = 0,\ dv_i = -2(\sum_{j\neq i} z_j \wedge w_j + \sum_{j\neq k} z_j \wedge w_k).$$

The symmetry group S_{n+1} is generated by S_n and $(1, n+1)$ with $(1, n+1)$:

$$(z_1, w_1) \to (-z_1 - z_2 - \cdots - z_n, w_1 - w_2 - \cdots - w_n)$$

$$(z_i, w_i) \to (z_i, w_i) \quad i > 1$$

$$(u_1, v_1) \to (-u_1, -v_1)$$

$$(u_i, v_i) \to (u_i - u_1, v_i - v_1) \quad i > 1.$$

We first check that

$$b^n = (n+1)! \wedge_i (z_i \wedge w_i)$$

$$b^{n-1} = (n-1)!(n\sum_j(\wedge_{i\neq j}(z_i \wedge w_i)) - \sum_{j\neq k}(\wedge_{i\notin(j,k)}(z_i \wedge w_i)) \wedge z_j \wedge w_k)$$

$$b^{n-1}a = (n-1)!\sum_j(\wedge_{i\neq j}(z_i \wedge w_i)) \wedge z_j \wedge (nu_j - \sum_{k\neq j} u_k)$$

Let $\delta = \sum_j(\wedge_{i\neq j}(z_i \wedge w_i)) \wedge \sum_{k\neq j} v_k$, then

$$d\delta = -2n(n-1) \wedge_j (z_j \wedge w_j) = A$$

let $\gamma = \sum_{k\neq j}(\wedge_{i\notin(j,k)}(z_i \wedge w_i)) \wedge z_j \wedge u_j \wedge ((n-2)v_k - nv_j)$, then

$$d\gamma = 2(n-1)\sum_j(\wedge_{i\neq j}(z_i \wedge w_i)) \wedge z_j \wedge (nu_j - \sum_{k\neq j} u_k) = B$$

Combine these calculations we have

$$d\bar{\delta} = A,\ d\bar{\gamma} = B$$

with $\bar{\delta}$ (or $\bar{\gamma}$ the average of δ (or γ). This concludes the proof of the exact property of b^n and $b^{n-1}a$. Similarly for $b^{n-1}c$ by exchanging z and w. Then we have 1. in our Theorem.

In practice, to calculate the average, we notice that $S_{n+1} = \cup_{i=1}^{n+1} S(i)$ with $S(i) = (n+1, i)S_n$.

4.3. Massey Quadruple Products

We define the Massey products by: If $da_{i,j} = \sum_{k=0}^{j-i-1} \bar{a}_{i,i+k} \wedge a_{i+k+1,j}$ for all $1 \leq i < j \leq 4$ with $(i,j) \neq (1,4)$ and $\bar{s} = (-1)^{\deg s+1}s$, then

$$(a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}) = \bar{a}_{1,1} \wedge a_{2,4} + \bar{a}_{1,2} \wedge a_{3,4} + \bar{a}_{1,3} \wedge a_{4,4}$$

is a cohomology class. Notice that our definition here is a form although it represent a cohomology class.

Now, we prove 2.: Let $a_{1,1} = 2a, a_{2,2} = a_{3,3} = 2b, a_{4,4} = 2c$, and

$$\delta = \sum_{i=1}^{2} z_i \wedge (u_i \wedge v_i - u_i \wedge v_{i^*} - u_{i^*} \wedge v_i)$$

$$\omega = \sum_{i=1}^{2} z_i \wedge (w_i \wedge (v_i - 2v_{i^*}) - w_{i^*} \wedge (v_i + v_{i^*}))$$

$$\gamma = \sum_{i=1}^{2} w_i \wedge (u_i \wedge v_i - u_i \wedge v_{i^*} - u_{i^*} \wedge v_i)$$

where $\{i, i^*\} = \{1, 2\}$. Then $a_{1,2} = -2\delta, a_{2,3} = -2\omega, a_{3,4} = -2\gamma$. $a_{1,3} = a_{2,4} = 0$.

$$16(a, b, b, c) = \bar{a}_{1,2} \wedge a_{3,4} = 4\gamma\delta = V_{uv}b \neq 0.$$

where $V_{uv} = du_1 \wedge du_2 \wedge dv_1 \wedge dv_2$. We notice that the nonzero property of our class does imply the nonformal property of the manifold. The reason is that 1. $a_{1,3} = a_{2,4} = 0$, i.e., only the middle term in our formula is nonzero (as a form); 2. The property of the only nonzero invariant H^3 cohomology class from the orbifold in the last sentence of last section is in a different format from γ and δ—it only involves z, w and u but not v, i.e., this is similar to the case in [FM] in which $H^3 = 0$.

4.4. Relative Massey Product

We define the relative Massey product (following [CFM] page 580): If $db_i = b \wedge a_i$ with $1 \leq i \leq 3$, then

$$(b : a_1, a_2, a_3) = b_1 \wedge b_2 \wedge a_3 + b_2 \wedge b_3 \wedge a_1 + b_3 \wedge b_1 \wedge a_2.$$

Now we prove 3. We notice that from [CFM] page 581 Lemma 2.6, if $(a_1, b, a_2) = (a_2, b, a_3) = (a_1, b, a_3) = 0$ as cohomology classes, then the given relative Massey product as a cohomology class does not depend on the choice of b_i.

Let $a_1 = 2a, a_2 = 2b, a_3 = 2c$, then $b_1 = -\delta, b_2 = -\omega, b_3 = -\gamma$. Other than what we have above we have $(a_1, b, a_3) = d(V_{uv})$.

$$(b : 2a, 2b, 2c) = -12V_{zw}V_{uv} = -12V \neq 0,$$

where $V_{zw} = dz_1 \wedge dz_2 \wedge dw_1 \wedge dw_2$.

From 2. or 3. we get 4.

Acknowledgments: Here I like to take this chance to express my heartly thanks to Marisa Fernandez, for her interest in this direction that eventually pushing me to work on this topic and pressing me in writting down this paper. I also thank Professors S. Kobayashi and M. Gromov, F. A. Bogomolov as well as Professor Y. T. Siu, etc., for their encouragements in the direction of holomorphic symplectic manifolds. I would also like to take this chance to thank Professor Feng, S. X. from Henan University and the School of Mathematics and Statistics for their hospitalities when I am publishing this result.

Funding: Partially Supported by National Nature Science Foundation of China (Grant No. 12171140).

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