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Posted Date: 14 May 2024

doi: 10.20944/preprints202405.0952.v1

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Article

Hyperbolic Primality Test and Catalan–Mersenne Number Conjecture

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Abstract: We develop a primality test for Mersenne numbers and investigate the primality of the sequence of Catalan-Mersenne numbers. We use induction as a proof to show that the 5th Catalan-Mersenne number is a prime number, thus demonstrating the infinitude of Mersenne primes.

Keywords: Catalan-Mersenne conjecture; Mersenne prime; double Mersenne prime

1. Introduction

Many primality tests have been developed, such as the Lucas–Lehmer primality [1] test, Proth's test [2], etc., which are useful prime test to specific use cases. In this note, we develop a primality test specified for Mersenne numbers. One of the interesting prime numbers is the Mersenne prime, M_p , which has the form $2^p - 1$.

Lemma 1 (Hyperbolic Primality Test). *If there exists a non-trivial x satisfying:*

$$2^{p-1} + x \equiv 0 \pmod{(2x + 1)} \quad (1)$$

Then $2^p - 1$ is prime. Here, the trivial solutions are $x = 0$ and $2^{p-1} - 1$.

Proof. When p is prime, for an arbitrary Mersenne number M_p :

$$M_p = 2^p - 1 = \left(\sum_{i=1}^k a_i 2^i + 1 \right) \left(\sum_{i=1}^q b_i 2^i + 1 \right) \quad (2)$$

And when M_p is prime, we call it a **Mersenne prime**. [3] When all a_i (or b_i) are zero, then M_p is prime.

$$\begin{aligned} 2^p - 1 &= 2(2^{p-1} - 1) + 1 \\ &= (2x + 1)(2y + 1) \\ &= 2(2xy + x + y) + 1 \end{aligned} \quad (3)$$

Therefore we can summarize the result as:

$$2^{p-1} - 1 = 2xy + x + y \quad (4)$$

This can be rewritten as:

$$2^{p-1} = xy + (x + 1)(y + 1) \quad (5)$$

which represents a **hyperbola**. So, if there exists any integer lattice point on the hyperbola, M_p is not prime. Thus, determining the primality of M_p is equivalent to checking for lattice points on the hyperbola.

In modulo forms, we can rewrite it as:

$$2^{p-1} + x \equiv 0 \pmod{(2x+1)} \quad (6)$$

For the computational test, the range of x is:

$$1 \leq x < \left\lfloor \frac{\sqrt{2^p-1}+1}{2} \right\rfloor = \left\lfloor 2^{\frac{p}{2}-1} \right\rfloor \quad (7)$$

□

2. Theorems and Lemmas

One interesting problem involves the primality test for Catalan-Mersenne numbers. The Catalan-Mersenne number is a recursively defined sequence [4]:

$$c_0 = 2, \quad c_{n+1} = 2^{c_n} - 1 = M_{c_n} \quad (8)$$

Therefore, the Catalan-Mersenne numbers are a subset of the double Mersenne numbers, which are numbers of the form:

$$M_{M_n} = 2^{2^n-1} - 1 \quad (9)$$

And we can choose a specific sequence where:

$$a_1 = 2^2 - 1, \quad a_{n+1} = 2^{a_n} - 1 \quad (10)$$

If $a_n = M_{a_{n-1}}$ is prime, then by Lemma 1, there is no non-trivial x , other than $x = 0, A_{n-1} - 1$, that satisfies:

$$2^{a_{n-1}-1} + x = A_{n-1} + x \equiv 0 \pmod{(2x+1)} \quad (11)$$

Then in the case of $a_{n+1} = M_{a_n}$, let us try to find a non-trivial x that satisfies:

$$2^{2^{a_n-1}-2} + x \equiv 0 \pmod{(2x+1)} \quad (12)$$

This equation can be rewritten as:

$$4^{2^{a_n-1}-1} + x = 4^{A_{n-1}-1} + x \equiv 0 \pmod{(2x+1)} \quad (13)$$

However, for some k_n , we can rewrite $4^{A_{n-1}-1} = A_{n-1}^{k_n}$:

$$k_n = \frac{A_{n-1}-1}{\log_4 A_{n-1}}, \quad \prod_{i=2}^n k_i = A_{n-1} - 1 \quad (14)$$

Thus, all k_n are integers, and Equation (15) holds for the primality test of a_n .

$$k_n = \frac{A_{n-1}-1}{A_{n-2}-1} = \sum_{i=0}^{k_{n-1}-1} 4^{(A_{n-2})i} \quad (15)$$

Equation (13) becomes as:

$$A_{n-1}^{k_n} + x \equiv 0 \pmod{(2x+1)} \quad (16)$$

Theorem 1. When \mathcal{CM} is the set of all Catalan-Mersenne numbers, if $a_n \in \mathcal{CM}$ is a double Mersenne prime, then M_{a_n} is also a prime number.

Proof. A double Mersenne prime, denoted as X , is a prime number that satisfies $X = M_{M_p}$, where M_p is also prime. According to Equation (11), $A_n = 2^{a_n - 1}$

1. When $n = 2$, we can perform the hyperbolic primality test:

$$A_1 + x \equiv 0 \pmod{(2x + 1)}$$

which yields:

$$4 + x \equiv 0 \pmod{(2x + 1)}$$

There are no non-trivial solutions for x .

Thus, a_2 is a double Mersenne prime.

2. When $n = 3$, we can perform the hyperbolic primality test:

$$A_2 + x \equiv 0 \pmod{(2x + 1)}$$

which yields:

$$64 + x \equiv 0 \pmod{(2x + 1)}$$

There are no non-trivial solutions for x .

Thus, a_3 is a double Mersenne prime.

For arbitrary x , we can express:

$$A_{n-1} = \mathcal{O}(2x + 1) + ax + b \quad (17)$$

where $a = 0$ or 1 .

Then let us use induction for the general case of $n \geq 3$. Let us assume that a_n (defined in Equation (10)) is a double Mersenne prime. According to Lemma 1 and Equation (17), there is no non-trivial x that satisfies:

$$A_{n-1} \equiv ax + b \equiv x + 1 \pmod{(2x + 1)} \quad (18)$$

This implies that $(a, b) \neq (1, 1)$ because $x \neq 0$.

Next, let us show that a_{n+1} is prime.

$$A_{n-1}^k = \mathcal{O}(2x + 1) + (ax + b)^k \quad (19)$$

We can use Equation (16) for the primality test of a_{n+1} , with the result of Equation (18) to check whether there is a non-trivial x that satisfies:

$$A_{n-1}^k \equiv (ax + b)^k \equiv x + 1 \pmod{(2x + 1)} \quad (20)$$

To make Equation (20) hold, b should be 1. However, Equation (18) shows that $(a, b) \neq (1, 1)$, so the available value of a is 0.

$$A_{n-1}^k \equiv 1 \equiv x + 1 \pmod{(2x + 1)} \quad (21)$$

So, we have:

$$0 \equiv x \pmod{(2x + 1)} \quad (22)$$

This implies that x should be 0, but this is not allowed. Hence, there is no non-trivial x that satisfies Equation (20). Therefore, a_{n+1} is prime. Consequently, if a_{n-1} is a double Mersenne prime, then a_n is prime. \square

Theorem 2. *There are infinitely many Mersenne primes.*

Proof. Theorem 1 shows that if there is one double Mersenne prime, it generates an infinite series of Mersenne primes. Since M_7 is a double Mersenne prime, there are infinitely many Mersenne primes. \square

3. Application

In this section, we investigate the generation of Catalan-Mersenne-like sequences using different initial values and examine the applicability of the recursive hyperbolic primality test.

3.1. $a_1 = 3$ Case

For $a_1 = 3$ Catalan-Mersenne number sequence, we can calculate k_2 to k_5 , such as

$$k_2 = \frac{4^{2-1} - 1}{2 - 1} = 3, \quad k_3 = \frac{4^{4-1} - 1}{4 - 1} = 21$$

$$k_4 = \frac{4^{63} - 1}{4^3 - 1} = 1350326852860866918505454791395905601$$

$$k_5 = \frac{A_4 - 1}{\log_4 A_4} = \frac{4^{4^{63}-1} - 1}{4^{63} - 1} = \sum_{i=0}^{k_4-1} 4^{63i}$$

Until a_4 it is showed that it is a prime by Lucas–Lehmer primality test. Based on Theorem. 1, a_5 is also prime.

$$a_5 = 2^{2^{127}-1} - 1 = M_{2^{127}-1} \quad (23)$$

which has $\approx 5.12 \times 10^{37}$ digits.

3.2. $a_1 = 2^{11} - 1$ Case

For the $a_1 = 2^{11} - 1$ Catalan-Mersene number sequence,

$$k_2 = \frac{2^{2^{11}-2} - 1}{2^{10} - 1} = (\text{non integer})$$

And a_1 is also not prime. So there is no infinite numbers of primes in this sequence.

3.3. $a_1 = 2^{13} - 1$ Case

For the $a_1 = 2^{13} - 1$ Catalan-Mersene number sequence,

$$k_2 = \frac{2^{2^{13}-2} - 1}{2^{12} - 1} = (\text{non integer})$$

And a_1 is a prime, but a_2 is not a prime. So there is no infinite numbers of primes in this sequence.

4. Conclusions and Outlook

In this research, we introduced a hyperbolic primality test specified for Mersenne numbers and demonstrated that a_5 is prime. Another intriguing observation is that there is an infinite number of primes that can be generated from the $a_1 = 3$ Catalan-Mersenne sequence. However, it's noteworthy that other infinite prime sequences with different initial values are not readily apparent.

For the next step, we can investigate the general relations between lattice points on the hyperbola and the initial values of Catalan-Mersenne numbers.

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