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Article

Performance Comparison of Techniques for Fractional Kawahara and Modified Kawahara Equations in Blood Flow Dynamics

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Abstract

The Kawahara equation and its modified form are nonlinear dispersive partial differential equations commonly used to model wave propagation in fluids and plasmas. In hemodynamic studies, these equations have been adapted to describe the propagation of pulse waves in large arteries, where dispersive wave effects play an important role in capturing the dynamics of blood flow. In this study, we focus on their time fractional generalizations within the Caputo sense and construct approximate analytical solutions by employing the optimal auxiliary function method (OAFM). This technique generates auxiliary functions with adjustable parameters, which enhances both accuracy and convergence speed. By applying OAFM to the time fractional Caputo Kawahara (TCFKE) and modified Kawahara (MTCFKE) equations, we obtain symmetric approximate solutions and confirm their efficiency through numerical tests and graphical illustrations. Moreover, in the context of fluid flow and shallow water wave propagation, comparative analysis with other existing methods including the homotopy analysis method, residual power series method, and natural transform decomposition technique highlights the robustness and higher accuracy of OAFM.

Keywords: Kawahara equation; Caputo operator; optimal auxiliary function

1. Introduction

The governing equations of wave motion play a crucial role in physics, fluid dynamics, and applied mathematics. Among these, the Kawahara equation, first introduced by Kawahara in 1970 [1], describes solitary wave propagation in different media [2,3]. This equation has been further generalized to explore conservation laws and symmetry properties, particularly in shallow water dynamics and magneto-acoustic wave theory [4]. Applications of the modified Kawahara equation have also been reported in plasma wave studies and capillary-gravity wave problems [5–7]. In recent decades, the use of fractional calculus has gained momentum as an effective tool for capturing memory effects and hereditary properties in diverse materials and processes. Fractional partial differential equations provide accurate models across multiple fields, including fluid mechanics, engineering, polymer science, physics, biomedical science, groundwater hydrology, and electrical networks [8]–[11]. Partial differential equations (PDEs) also underpin various mathematical concepts, such as the Calabi and Poincaré coefficients, and numerous approximate solution techniques have been developed to study

nonlinear PDEs. Researchers have applied different analytical and semi-analytical methods to investigate fractional models of physical and biological systems. Examples include: the homotopy analysis method [12]; and reproducing kernel Hilbert space methods for telegraph, Fornberg-Whitham, and non-local fractional equations [13–15]; monotone iterative methods for reaction-diffusion models [16]; direct algebraic mapping techniques for damped KdV equations [17]; the Adams-Bashforth method for Atangana-Baleanu Caputo equations [18]; the modified expansion function approach for nonlinear Schrödinger equations [19]; homotopy perturbation methods [20]; the expansion method for higher-order nonlinear Boussinesq equations [21]; variational iteration methods for nonlinear FPDEs [22]; the sine-Gordon expansion applied to Wu-Zhang systems [23]; the Laplace residual power series method [24]; fractional Newton methods for convergence analysis [25]; the Laplace transform for fuzzy PDEs [26]. Among these, the OAFM has emerged as a versatile tool for tackling nonlinear engineering and physical problems. Originally introduced by Hernández and Beléndez [27] for nonlinear Schrödinger equations, OAFM has since been extended to a variety of applications. For instance, Iqbal et al. [28] applied it to Caputo fractional nonlinear long-wave systems, while Alsheekh et al. [29] employed it for fractional Whitham-Broer-Kaup systems. Other contributions include extending OAFM to fractional PDEs using the complex transform approach [30], modeling synchronous generators [31], and solving generalized PDEs [32–34]. Furthermore, OAFM has been applied in epidemiological modeling, such as the nonlinear SEIR system for COVID-19 dynamics [35]. For the TCFKE and its modified form MTCFKE, many methods have been explored, numerical schemes such as the septic B-spline collocation method [36], generalized formulations using analytical techniques [37], the homotopy analysis transform [38], residual power series method (RPSM) [39], the natural transform decomposition method (NTDM) [40], iterative Laplace transforms [41], Laplace-Adomian decomposition [42]. Other contributions investigate Kawahara and modified Kawahara equations with variable coefficients using Lie group analysis [43] and the homotopy analysis method (HAM) [44]. As we know that fractional extensions of the Kawahara and modified Kawahara equations capture memory effects in nonlinear wave phenomena more effectively than classical models. However, obtaining accurate solutions for such equations remains challenging, as many existing methods face convergence or accuracy limitations. This work is motivated by the need for a reliable and efficient approach, where the OAFM is applied to derive precise approximate solutions for these fractional models. In this study, we examine the effectiveness of the OAFM for deriving symmetric solutions of the following TCFKE (1) and MTCFKE (2)

$$\begin{cases} {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) + \mathcal{U}(\eta, \kappa) \frac{\partial \mathcal{U}(\eta, \kappa)}{\partial \eta} + \frac{\partial^3 \mathcal{U}(\eta, \kappa)}{\partial \eta^3} - \frac{\partial^5 \mathcal{U}(\eta, \kappa)}{\partial \eta^5} = 0, & 0 < \alpha \leq 1 \\ \mathcal{U}(\eta, 0) = k_1(\eta) \end{cases} \tag{1}$$

and

$$\begin{cases} {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) + \mathcal{U}^2(\eta, \kappa) \frac{\partial \mathcal{U}(\eta, \kappa)}{\partial \eta} + \zeta \frac{\partial^3 \mathcal{U}(\eta, \kappa)}{\partial \eta^3} + \xi \frac{\partial^5 \mathcal{U}(\eta, \kappa)}{\partial \eta^5} = 0, & 0 < \alpha \leq 1 \\ \mathcal{U}(\eta, 0) = k_2(\eta), \end{cases} \tag{2}$$

respectively, where $\zeta > 0$, $\xi < 0$ are constants and ${}^C\mathcal{D}_{0,\kappa}^\alpha$ is the Caputo operator. Section 2 provides the essential definitions and preliminaries of the Caputo fractional operator. Section 3 outlines the fundamental steps of the OAFM. In Section 4, the OAFM is applied to obtain symmetric solutions of the TCFKE (1) and MTCFKE (2). Section 5 presents numerical simulations and graphical illustrations, where the obtained solutions are compared with those generated by the RPSM [39], the NTDM [40], and the HAM [44]. Finally, Section 6 provides a summary of the main findings along with concluding remarks.

2. On Fractional Derivative

This section presents some definitions and conclusions regarding the Caputo fractional derivative, as adopted from [45]. The Riemann–Liouville fractional integral of order α for the function $\mathcal{U}(\eta, \kappa)$ is given by

$$\mathcal{I}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\kappa (\kappa - t)^{\alpha-1} \mathcal{U}(\eta, t) dt, & \alpha > 0 \\ \mathcal{U}(\eta, \kappa), & \alpha = 0. \end{cases} \quad (3)$$

The Caputo fractional derivative for $\mathcal{U}(\eta, \kappa)$ of order α , [45] is given by

$${}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) = \begin{cases} \mathcal{I}_{0,\kappa}^{n-\alpha} \left[\frac{\partial^n \mathcal{U}(\eta, \kappa)}{\partial \kappa^n} \right], & n-1 < \alpha < n \\ \frac{\partial^\alpha \mathcal{U}(\eta, \kappa)}{\partial \kappa^\alpha}, & \alpha \in \mathbb{N}, \end{cases} \quad (4)$$

where $n \in \mathbb{N}$.

Lemma 1. [46] For $\kappa \geq 0$ and $n-1 < \alpha < n$,

1. $\mathcal{D}_{0,\kappa}^\alpha \mathcal{I}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) = \mathcal{U}(\eta, \kappa);$
2. $\mathcal{I}_{0,\kappa}^\alpha \mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) = \mathcal{U}(\eta, \kappa) + \sum_{k=0}^{n-1} \frac{\kappa^k}{k!} \frac{\partial^k \mathcal{U}(\eta, \kappa)}{\partial \kappa^k} \Big|_{\kappa=0};$
3. $\mathcal{D}_{0,\kappa}^\alpha \kappa^\gamma = \frac{\Gamma(\alpha+1)}{\Gamma(\gamma-\alpha+1)} \kappa^{\gamma-\alpha};$
4. $\mathcal{D}_{0,\kappa}^\alpha c = 0,$

where $c \in \mathbb{R}$ is a constant, $n \in \mathbb{N}$ and $\gamma > -1$.

3. Steps of the OAFM

A numerical method or technique for solving a PDE or a system of PDEs is often a collocation method used to obtain approximate solutions. This is done by selecting appropriate collocation points such that the solution lies within a finite-dimensional space of polynomials up to a certain degree, using a finite number of collocation points.

Auxiliary functions are, at the very least, proven to exist, established, or otherwise utilized to achieve the desired outcome.

Here, we present the steps and basic concepts of the OAFM. Consider the general nonlinear Caputo fractional PDE

$$\begin{cases} {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) = \psi(\eta, \kappa) + \mathcal{N}(\mathcal{U}(\eta, \kappa)), \\ {}^C\mathcal{D}_{0,\kappa}^{\alpha-k} \mathcal{U}(\eta, \kappa) \Big|_{\kappa=0} = G_k(\eta), \quad k = 0, 1, 2, \dots, m-1, \\ {}^C\mathcal{D}_{0,\kappa}^{\alpha-m} \mathcal{U}(\eta, \kappa) \Big|_{\kappa=0} = 0, \\ {}^C\mathcal{D}_{0,\kappa}^k \mathcal{U}(\eta, \kappa) \Big|_{\kappa=0} = g_k(\eta), \quad k = 0, 1, 2, \dots, m-1, \\ {}^C\mathcal{D}_{0,\kappa}^m \mathcal{U}(\eta, \kappa) \Big|_{\kappa=0} = 0, \end{cases} \quad (5)$$

where $m = [\alpha]$ and $\mathcal{N}$ is the nonlinear term consisting $\mathcal{U}(\eta, \kappa)$.

Step 1. We take the two components in the first part of (5) to obtain an approximate solution consisting of two components, $\mathcal{U}_0$ and $\mathcal{U}_1$, that is,

$$\mathcal{U}(\eta, \kappa) = \mathcal{U}_0(\eta, \kappa) + \mathcal{U}_1(\eta, \kappa, c_k), \quad k = 1, 2, 3, \dots, j. \quad (6)$$

Step 2. For getting the zero and first-order solutions, we substitute (6) into (5) which results in:

$${}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_0(\eta, \kappa) + {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_1(\eta, \kappa) + \psi(\eta, \kappa) + \mathcal{N}\left[{}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_0(\eta, \kappa) + {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_1(\eta, \kappa, c_k)\right] = 0. \quad (7)$$

Step 3. The first approximate solution $\mathcal{U}_0(\eta, \kappa)$ depends on the linear equation

$${}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_0(\eta, \kappa) + \theta(\eta, \kappa) = 0. \quad (8)$$

By taking the inverse operator, the approximate solution $\mathcal{U}_0(\eta, \kappa)$ implies as follows:

$$\mathcal{U}_0(\eta, \kappa) = \psi(\eta, \kappa). \quad (9)$$

Step 4. The version of expansion of the nonlinear term in (7) is

$$\mathcal{N}\left[{}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_0(\eta, \kappa) + {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_1(\eta, \kappa, c_k)\right] = \mathcal{N}[\mathcal{U}_0(\eta, \kappa)] + \sum_{j=0}^{\infty} \frac{\mathcal{U}_1^j}{j!} \mathcal{N}^{(j)}[\mathcal{U}_0(\eta, \kappa)]. \quad (10)$$

Step 5. For solving the equation (10), we consider the following alternative equation to facilitate the solution $\mathcal{U}(\eta, \kappa)$ of (10) and make the quick convergence of the first-order approximate solution:

$${}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_1(\eta, \kappa, c_k) = -\mathcal{J}_1[\mathcal{U}_0(\eta, \kappa)]\mathcal{N}[\mathcal{U}_0(\eta, \kappa)] - \mathcal{J}_2[\mathcal{U}_0(\eta, \kappa); c_d] \quad (11)$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are two auxiliary functions which depend on $\mathcal{U}_0(\eta, \kappa)$ and the convergences parameters c_k and c_d such that $k = 1, 2, 3, \dots, t$ and $d = t + 1, t + 2, \dots, j$. Sometimes $\mathcal{J}_1$ and $\mathcal{J}_2$ are given as the type $\mathcal{U}_0(\eta, \kappa)$, $\mathcal{N}(\mathcal{U}_0(\eta, \kappa))$, or the mixture of both $\mathcal{U}_0(\eta, \kappa)$, $\mathcal{N}(\mathcal{U}_0(\eta, \kappa))$. Note that if $\mathcal{U}_0(\eta, \kappa)$ or $\mathcal{N}(\mathcal{U}_0(\eta, \kappa))$ are trigonometric functions, then $\mathcal{J}_1[\mathcal{U}_0(\eta, \kappa)]$ and $\mathcal{J}_2[\mathcal{U}_0(\eta, \kappa); c_d]$ will be the additional of a trigonometric functions. If $\mathcal{U}_0(\eta, \kappa)$ or $\mathcal{N}(\mathcal{U}_0(\eta, \kappa))$ are the component of a polynomials, then $\mathcal{J}_1[\mathcal{U}_0(\eta, \kappa)]$ and $\mathcal{J}_2[\mathcal{U}_0(\eta, \kappa); c_d]$ are taken as the summations. If $\mathcal{U}_0(\eta, \kappa)$ or $\mathcal{N}(\mathcal{U}_0(\eta, \kappa))$ are the exponential component, then $\mathcal{J}_1[\mathcal{U}_0(\eta, \kappa)]$ and $\mathcal{J}_2[\mathcal{U}_0(\eta, \kappa); c_d]$ are the components for sum of an exponential maps. The term $\mathcal{U}_0(\eta, \kappa)$ is an exact solution of (6) if $\mathcal{N}(\mathcal{U}_0(\eta, \kappa)) = 0$.

Step 6. Put the auxiliary functions in (11) and then take the inverse operator for (11) to obtain a first-order solutions $\mathcal{U}_1(\eta, \kappa)$.

Step 7. This method is one of methods that is used to determine the values of c_k and c_d and calculate the square of the residual error

$$\mathcal{J}(c_k, c_d) = \int_0^\kappa \int_\Omega \Psi^2(\eta, \kappa, c_k, c_d) d\eta d\kappa, \quad (12)$$

such that Ψ is the residual, which is given by

$$\Psi(\eta, \kappa, c_k, c_d) = {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa, c_k, c_d) + \psi(\eta, \kappa) + \mathcal{N}[\mathcal{U}(\eta, \kappa, c_k, c_d)] \quad (13)$$

where $d = t + 1, t + 2, \dots, j$ and $k = 1, 2, 3, \dots, t$.

4. Approximate Solutions

As noted earlier, the Kawahara equation and its modified form are nonlinear dispersive partial differential equations primarily used to model wave propagation in fluids and plasmas. In the study of blood flow in arteries, the Kawahara equation has been applied to describe the propagation of pulse waves in large vessels, where dispersive effects play a critical role. By incorporating higher-order dispersion, the equation effectively captures wave deformation and solitary pulse-like structures in arterial hemodynamics. Figure (1) illustrates a fractional-order model of blood flow in arteries [47].

Extending the Kawahara equation to fractional forms further enhances its applicability by accounting for memory and hereditary effects, thereby providing a more realistic mathematical framework for modeling wave propagation in complex biological systems. In this section, we use the OAFM to obtain some approximate solutions of TCFKE (1) and MTCFKE (2).

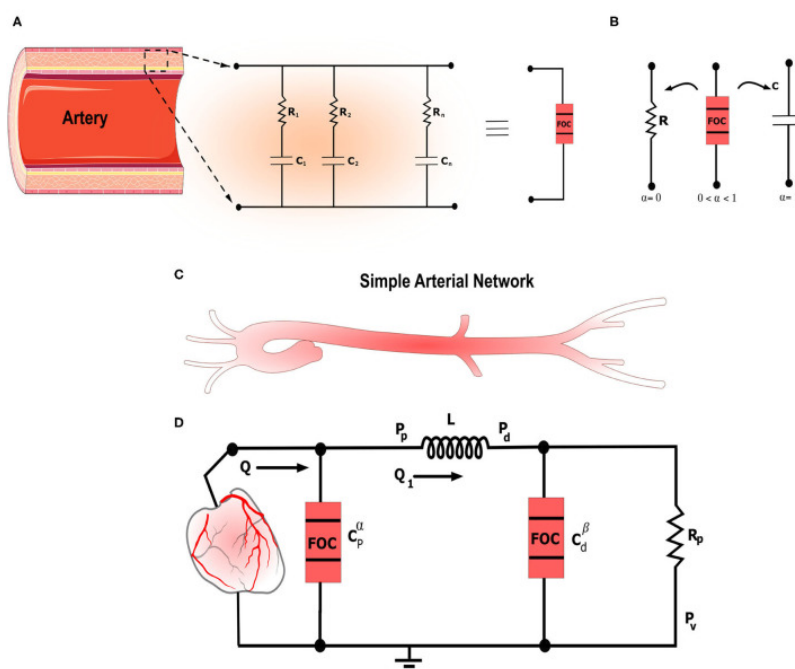


Figure 1. Model of blood flow in arteries.

Example 1. Consider the following TCFKE

$$\begin{cases} \mathcal{C} \mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) + \mathcal{U}(\eta, \kappa) \frac{\partial \mathcal{U}(\eta, \kappa)}{\partial \eta} + \frac{\partial^3 \mathcal{U}(\eta, \kappa)}{\partial \eta^3} - \frac{\partial^5 \mathcal{U}(\eta, \kappa)}{\partial \eta^5} = 0, & 0 < \alpha \leq 1 \\ \mathcal{U}(\eta, 0) = \frac{105}{169} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) \end{cases} \quad (14)$$

Recall [44] that if $\alpha = 1$ then the exact solution of the TCFKE (14) is given by

$$\mathcal{U}(\eta, \kappa) = \frac{105}{169} \operatorname{sech}^4 \left[\frac{1}{2\sqrt{13}} \left(\eta - \frac{36}{169} \kappa \right) \right]. \quad (15)$$

In (14), consider linear term

$$\mathcal{L}(\mathcal{U}(\eta, \kappa)) = {}^c\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) \quad (16)$$

and nonlinear term is given by

$$\mathcal{N}(\mathcal{U}(\eta, \kappa)) = \mathcal{U}(\eta, \kappa) \frac{\partial \mathcal{U}(\eta, \kappa)}{\partial \eta} + \frac{\partial^3 \mathcal{U}(\eta, \kappa)}{\partial \eta^3} - \frac{\partial^5 \mathcal{U}(\eta, \kappa)}{\partial \eta^5}. \quad (17)$$

The zero approximations of (14) is given by

$$\mathcal{U}_0(\eta, \kappa) = \mathcal{U}(\eta, 0) = \frac{105}{169} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right). \quad (18)$$

Put $\mathcal{U}_0(\eta, \kappa)$ in (17) to get

$$\begin{aligned}
 -\mathcal{N}(\mathcal{U}_0(\eta, \kappa)) &= -\mathcal{U}_0(\eta, \kappa) \frac{\partial \mathcal{U}_0(\eta, \kappa)}{\partial \eta} - \frac{\partial^3 \mathcal{U}_0(\eta, \kappa)}{\partial \eta^3} + \frac{\partial^5 \mathcal{U}_0(\eta, \kappa)}{\partial \eta^5} \\
 &= \frac{26985}{28561\sqrt{13}} \operatorname{sech}^8\left(\frac{\eta}{2\sqrt{13}}\right) \tanh\left(\frac{\eta}{2\sqrt{13}}\right) \\
 &\quad - \frac{1470}{4394\sqrt{13}} \operatorname{sech}^6\left(\frac{\eta}{2\sqrt{13}}\right) \tanh\left(\frac{\eta}{2\sqrt{13}}\right) \\
 &\quad + \frac{27510}{57122\sqrt{13}} \operatorname{sech}^6\left(\frac{\eta}{2\sqrt{13}}\right) \tanh^3\left(\frac{\eta}{2\sqrt{13}}\right) \\
 &\quad + \frac{840}{2197\sqrt{13}} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) \tanh^3\left(\frac{\eta}{2\sqrt{13}}\right) \\
 &\quad + \frac{6720}{57122\sqrt{13}} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) \tanh^5\left(\frac{\eta}{2\sqrt{13}}\right).
 \end{aligned} \tag{19}$$

Take the auxiliary functions as follows

$$\begin{cases} \mathcal{J}_1[\mathcal{U}_0(\eta, \kappa)] = c_1 \frac{105}{169} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) + c_2 \left(\frac{105}{169}\right)^3 \operatorname{sech}^{12}\left(\frac{\eta}{2\sqrt{13}}\right) \\ \mathcal{J}_2[\mathcal{U}_0(\eta, \kappa); c_d] = c_3 \left(\frac{105}{169}\right)^5 \operatorname{sech}^{20}\left(\frac{\eta}{2\sqrt{13}}\right). \end{cases} \tag{20}$$

By the procedure of the OAFM, the first approximation is

$${}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_1(\eta, \kappa) = -\mathcal{J}_1[\mathcal{U}_0(\eta, \kappa)]\mathcal{N}[\mathcal{U}_0(\eta, \kappa)] - \mathcal{J}_2[\mathcal{U}_0(\eta, \kappa); c_d] \tag{21}$$

Put $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{N}(\mathcal{U}_0(\eta, \kappa))$ in (21) to get

$$\begin{aligned}
 {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}_1(\eta, \kappa) &= \left\{ \frac{\operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) \tanh\left(\frac{\eta}{2\sqrt{13}}\right)}{28561\sqrt{13}} \right\} \\
 &\quad \times \left[16590 \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) - 13440 \operatorname{sech}^2\left(\frac{\eta}{2\sqrt{13}}\right) + 14280 \right] \\
 &\quad \times \left[c_1 \frac{105}{169} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) + c_2 \left(\frac{105}{169}\right)^3 \operatorname{sech}^{12}\left(\frac{\eta}{2\sqrt{13}}\right) \right] \\
 &\quad - c_3 \left(\frac{105}{169}\right)^5 \operatorname{sech}^{20}\left(\frac{\eta}{2\sqrt{13}}\right).
 \end{aligned} \tag{22}$$

Take the inverses operator of (22) to obtain

$$\begin{aligned}
 \mathcal{U}_1(\eta, \kappa) &= \frac{\kappa^\alpha}{\Gamma(\alpha+1)} \left\{ \frac{\operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) \tanh\left(\frac{\eta}{2\sqrt{13}}\right)}{28561\sqrt{13}} \right\} \\
 &\quad \times \left[16590 \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) - 13440 \operatorname{sech}^2\left(\frac{\eta}{2\sqrt{13}}\right) + 14280 \right] \\
 &\quad \times \left[c_1 \frac{105}{169} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) + c_2 \left(\frac{105}{169}\right)^3 \operatorname{sech}^{12}\left(\frac{\eta}{2\sqrt{13}}\right) \right] \\
 &\quad - \frac{\kappa^\alpha}{\Gamma(\alpha+1)} c_3 \left(\frac{105}{169}\right)^5 \operatorname{sech}^{20}\left(\frac{\eta}{2\sqrt{13}}\right).
 \end{aligned} \tag{23}$$

Hence the approximate solution is

$$\begin{aligned} \mathcal{U}(\eta, \kappa) = & \frac{\kappa^\alpha}{\Gamma(\alpha + 1)} \left\{ \frac{\operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) \tanh\left(\frac{\eta}{2\sqrt{13}}\right)}{28561\sqrt{13}} \right\} \\ & \times \left[16590 \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) - 13440 \operatorname{sech}^2\left(\frac{\eta}{2\sqrt{13}}\right) + 14280 \right] \\ & \times \left[c_1 \frac{105}{169} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right) + c_2 \left(\frac{105}{169}\right)^3 \operatorname{sech}^{12}\left(\frac{\eta}{2\sqrt{13}}\right) \right] \\ & - \frac{\kappa^\alpha}{\Gamma(\alpha + 1)} c_3 \left(\frac{105}{169}\right)^5 \operatorname{sech}^{20}\left(\frac{\eta}{2\sqrt{13}}\right) + \frac{105}{169} \operatorname{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right). \end{aligned} \tag{24}$$

Example 2. Consider the following MTCFKE

$$\begin{cases} {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) + \mathcal{U}^2(\eta, \kappa) \frac{\partial \mathcal{U}(\eta, \kappa)}{\partial \eta} + \zeta \frac{\partial^3 \mathcal{U}(\eta, \kappa)}{\partial \eta^3} + \xi \frac{\partial^5 \mathcal{U}(\eta, \kappa)}{\partial \eta^5} = 0, & 0 < \alpha \leq 1 \\ \mathcal{U}(\eta, 0) = \frac{3\zeta}{\sqrt{-10\xi}} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\xi}} \eta\right) \end{cases} \tag{25}$$

where $\zeta > 0, \xi < 0$.

The exact solution [44] of (25) at $\alpha = 1$ is given by

$$\mathcal{U}(\eta, \kappa) = \frac{3\zeta}{\sqrt{-10\xi}} \operatorname{sech}^2\left[\frac{1}{2}\sqrt{\frac{-\zeta}{5\xi}}\left(\eta - \frac{25\xi - 4\zeta^2}{25\xi} \kappa\right)\right]. \tag{26}$$

In (25), consider linear term

$$\mathcal{L}(\mathcal{U}(\eta, \kappa)) = {}^C\mathcal{D}_{0,\kappa}^\alpha \mathcal{U}(\eta, \kappa) \tag{27}$$

and nonlinear term is given by

$$\mathcal{N}(\mathcal{U}(\eta, \kappa)) = \mathcal{U}^2(\eta, \kappa) \frac{\partial \mathcal{U}(\eta, \kappa)}{\partial \eta} + \zeta \frac{\partial^3 \mathcal{U}(\eta, \kappa)}{\partial \eta^3} + \xi \frac{\partial^5 \mathcal{U}(\eta, \kappa)}{\partial \eta^5}. \tag{28}$$

The zero approximations of (25) is given by

$$\mathcal{U}_0(\eta, \kappa) = \mathcal{U}(\eta, 0) = \frac{3\zeta}{\sqrt{-10\xi}} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\xi}} \eta\right). \tag{29}$$

Put $\mathcal{U}_0(\eta, \kappa)$ in (28) to get

$$\begin{aligned}
 \mathcal{N}(\mathcal{U}_0(\eta, \kappa)) &= \mathcal{U}_0^2(\eta, \kappa) \frac{\partial \mathcal{U}_0(\eta, \kappa)}{\partial \eta} + \zeta \frac{\partial^3 \mathcal{U}_0(\eta, \kappa)}{\partial \eta^3} + \zeta \frac{\partial^5 \mathcal{U}_0(\eta, \kappa)}{\partial \eta^5} \\
 &= \frac{9\zeta^2 \sqrt{5\zeta}}{50\zeta \sqrt{-\zeta}} \operatorname{sech}^4 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \tanh \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \\
 &\quad + \frac{3\zeta^3}{25\zeta^2} \sqrt{\frac{\zeta}{2}} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \tanh^3 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \\
 &\quad - \frac{6\zeta^3}{25\zeta^2} \sqrt{\frac{\zeta}{2}} \operatorname{sech}^4 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \tanh \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \\
 &\quad - \frac{3\zeta^3}{125\zeta^2} \sqrt{\frac{\zeta}{2}} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \tanh^5 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \\
 &\quad + \frac{39\zeta^3}{125\zeta^2} \sqrt{\frac{\zeta}{2}} \operatorname{sech}^4 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \tanh^3 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \\
 &\quad - \frac{51\zeta^3}{250\zeta^2} \sqrt{\frac{\zeta}{2}} \operatorname{sech}^6 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \tanh \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right). \tag{30}
 \end{aligned}$$

Take the auxiliary functions as follows

$$\begin{cases} \mathcal{J}_1[\mathcal{U}_0(\eta, \kappa)] = c_1 \frac{3\zeta}{\sqrt{-10\zeta}} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) - c_2 \frac{27\zeta^3}{10\zeta \sqrt{-10\zeta}} \operatorname{sech}^6 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \\ \mathcal{J}_2[\mathcal{U}_0(\eta, \kappa); c_d] = c_3 \frac{243\zeta^5}{100\zeta^2 \sqrt{-10\zeta}} \operatorname{sech}^{10} \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \end{cases} \tag{31}$$

By the procedure of the OAFM, the first approximation is

$${}^c D_{0,\eta}^\alpha \mathcal{U}_1(\eta, \kappa) = -\mathcal{J}_1[\mathcal{U}_0(\eta, \kappa)] \mathcal{N}[\mathcal{U}_0(\eta, \kappa)] - \mathcal{J}_2[\mathcal{U}_0(\eta, \kappa); c_d] \tag{32}$$

Put $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{N}(\mathcal{U}_0(\eta, \kappa))$ in (32) to get

$$\begin{aligned}
 {}^c D_{0,\eta}^\alpha \mathcal{U}_1(\eta, \kappa) &= \left\{ \frac{3\zeta^3 \sqrt{\zeta} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \tanh \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right)}{25\sqrt{2}\zeta^2} \right\} \\
 &\quad \times \left[\frac{43}{10} \operatorname{sech}^4 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) - \left(\frac{15\sqrt{-10\zeta} - 6a}{10a} \right) \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) - 1 \right] \\
 &\quad \times \left[c_1 \frac{3a}{\sqrt{-10\zeta}} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) - c_2 \frac{27\zeta^3}{10\zeta \sqrt{-10\zeta}} \operatorname{sech}^6 \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right) \right] \\
 &\quad - c_3 \frac{243\zeta^5}{100\zeta^2 \sqrt{-10\zeta}} \operatorname{sech}^{10} \left(\frac{1}{2} \sqrt{\frac{-\zeta}{5\zeta}} \eta \right). \tag{33}
 \end{aligned}$$

Take the inverses operator of (33) to obtain

$$\begin{aligned} \mathcal{U}_1(\eta, \kappa) = & \frac{\kappa^\alpha}{\Gamma(\alpha + 1)} \left\{ \frac{3\zeta^3 \sqrt{\zeta} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) \tanh\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right)}{25\sqrt{2}\zeta^2} \right\} \\ & \times \left[\frac{43}{10} \operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) - \left(\frac{15\sqrt{-10\zeta} - 6a}{10a}\right) \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) - 1 \right] \\ & \times \left[c_1 \frac{3a}{\sqrt{-10\zeta}} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) - c_2 \frac{27\zeta^3}{10\zeta\sqrt{-10\zeta}} \operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) \right] \\ & - \frac{\kappa^\alpha}{\Gamma(\alpha + 1)} c_3 \frac{243\zeta^5}{100\zeta^2\sqrt{-10\zeta}} \operatorname{sech}^{10}\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right). \end{aligned} \tag{34}$$

Hence the approximate solution is

$$\begin{aligned} \mathcal{U}(\eta, \kappa) = & \frac{\kappa^\alpha}{\Gamma(\alpha + 1)} \left\{ \frac{3\zeta^3 \sqrt{\zeta} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) \tanh\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right)}{25\sqrt{2}\zeta^2} \right\} \\ & \times \left[\frac{43}{10} \operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) - \left(\frac{15\sqrt{-10\zeta} - 6a}{10a}\right) \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) - 1 \right] \\ & \times \left[c_1 \frac{3a}{\sqrt{-10\zeta}} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) - c_2 \frac{27\zeta^3}{10\zeta\sqrt{-10\zeta}} \operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) \right] \\ & - \frac{\kappa^\alpha}{\Gamma(\alpha + 1)} c_3 \frac{243\zeta^5}{100\zeta^2\sqrt{-10\zeta}} \operatorname{sech}^{10}\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right) + \frac{3a}{\sqrt{-10\zeta}} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\zeta}{5\zeta}} \eta\right). \end{aligned} \tag{35}$$

Table 1. Absolute error of OAFM at $\alpha = 1$ and $\eta = 10$ with different values of κ in Example (1).

κ	AE(OAFM)	AE(NTDM)[40]	AE(RPSM) [39]
0.0	0	0	0
0.1	1.41552×10^{-15}	1.41553×10^{-15}	1.41553×10^{-15}
0.2	4.68064×10^{-14}	4.68063×10^{-14}	4.68063×10^{-14}
0.3	3.63911×10^{-13}	3.63910×10^{-13}	3.63910×10^{-13}
0.4	1.56885×10^{-12}	1.56886×10^{-12}	1.56886×10^{-12}
0.5	4.89616×10^{-12}	4.89617×10^{-12}	4.89617×10^{-12}
0.6	1.24541×10^{-11}	1.24542×10^{-11}	1.24542×10^{-11}
0.7	2.75067×10^{-11}	2.75069×10^{-11}	2.75069×10^{-11}
0.8	5.47830×10^{-11}	5.47829×10^{-11}	5.47829×10^{-11}
0.9	1.00811×10^{-10}	1.00810×10^{-10}	1.00810×10^{-10}
1.0	1.74281×10^{-10}	1.74280×10^{-10}	1.74280×10^{-10}

Table 2. Approximate solutions for TCFKE at $\eta = 10$ with different values of κ and α in Example (1).

α	κ	OAFM	NTDM [40]	RPSM [39]
1	0.2	3.10610×10^{-2}	3.10612×10^{-2}	3.10612×10^{-2}
	0.4	3.17143×10^{-2}	3.17143×10^{-2}	3.17143×10^{-2}
	0.6	3.23799×10^{-2}	3.23800×10^{-2}	3.23800×10^{-2}
	0.8	3.30587×10^{-2}	3.30588×10^{-2}	3.30588×10^{-2}
	1.0	3.37506×10^{-2}	3.37506×10^{-2}	3.37506×10^{-2}
0.75	0.2	3.14736×10^{-2}	3.14737×10^{-2}	3.14737×10^{-2}
	0.4	3.22161×10^{-2}	3.22162×10^{-2}	3.22163×10^{-2}
	0.6	3.28840×10^{-2}	3.28839×10^{-2}	3.28842×10^{-2}
	0.8	3.35113×10^{-2}	3.35111×10^{-2}	3.35117×10^{-2}
	1.0	3.41131×10^{-2}	3.41124×10^{-2}	3.41135×10^{-2}
0.5	0.2	3.20842×10^{-2}	3.20842×10^{-2}	3.20845×10^{-2}
	0.4	3.28116×10^{-2}	3.28114×10^{-2}	3.28122×10^{-2}
	0.6	3.33857×10^{-2}	3.33850×10^{-2}	3.33867×10^{-2}
	0.8	3.38815×10^{-2}	3.38794×10^{-2}	3.38822×10^{-2}
	1.0	3.43254×10^{-2}	3.43233×10^{-2}	3.43273×10^{-2}
0.25	0.2	3.29232×10^{-2}	3.29225×10^{-2}	3.29247×10^{-2}
	0.4	3.34353×10^{-2}	3.34339×10^{-2}	3.34378×10^{-2}
	0.6	3.37861×10^{-2}	3.37839×10^{-2}	3.37893×10^{-2}
	0.8	3.40622×10^{-2}	3.40585×10^{-2}	3.40655×10^{-2}
	1.0	3.42923×10^{-2}	3.42882×10^{-2}	3.42966×10^{-2}

Table 3. Some solutions of MTCFKE at $\alpha = 0.25$, $\zeta = 0.001$, $\xi = -1$, and $\alpha = 0.5$ with different values of η and κ in Example (2).

		$\alpha = 0.25$			$\alpha = 0.50$		
κ	η	OAFM	NTDM [40]	HAM [44]	OAFM	NTDM [40]	HAM [44]
20	0.2	$9.2994E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.4	$9.2994E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.6	$9.2994E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.8	$9.2994E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	1.0	$9.2994E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
10	0.2	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.4	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.6	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.8	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	1.0	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
0	0.2	$9.4863E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	0.4	$9.4863E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	0.6	$9.4863E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	0.8	$9.4863E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	1.0	$9.4863E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
−10	0.2	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.4	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.6	$9.4863E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	0.8	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	1.0	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
−20	0.2	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.4	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.6	$9.2993E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.8	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	1.0	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$

Table 4. Some solutions for Example (2) at $\zeta = 0.001$, $\xi = -1$, $\alpha = 0.75$ and $\alpha = 1.0$ with different values of η and κ .

		$\alpha = 0.75$			$\alpha = 1.0$		
κ	η	OAFM	NTDM [40]	HAM [44]	OAFM	NTDM [40]	HAM [44]
20	0.2	9.2991E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
	0.4	9.2990E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
	0.6	9.2993E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
	0.8	9.2992E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
	1.0	9.2991E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
10	0.2	9.4391E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
	0.4	9.4395E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
	0.6	9.4392E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
	0.8	9.4395E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
	1.0	9.4394E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
0	0.2	9.4861E-4	9.4868E-4	9.486E-4	9.48623E-4	9.4868E-4	9.486E-4
	0.4	9.4861E-4	9.4868E-4	9.486E-4	9.48623E-4	9.4868E-4	9.486E-4
	0.6	9.4863E-4	9.4868E-4	9.486E-4	9.48623E-4	9.4868E-4	9.486E-4
	0.8	9.4860E-4	9.4868E-4	9.486E-4	9.4862E-4	9.4868E-4	9.486E-4
	1.0	9.4861E-4	9.4868E-4	9.486E-4	9.4862E-4	9.4868E-4	9.486E-4
-10	0.2	9.4395E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
	0.4	9.4395E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
	0.6	9.4391E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
	0.8	9.4395E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
	1.0	9.4390E-4	9.4396E-4	9.439E-4	9.4393E-4	9.4396E-4	9.439E-4
-20	0.2	9.2990E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
	0.4	9.2991E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
	0.6	9.2992E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
	0.8	9.2993E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4
	1.0	9.2994E-4	9.2996E-4	9.299E-4	9.2993E-4	9.2996E-4	9.299E-4

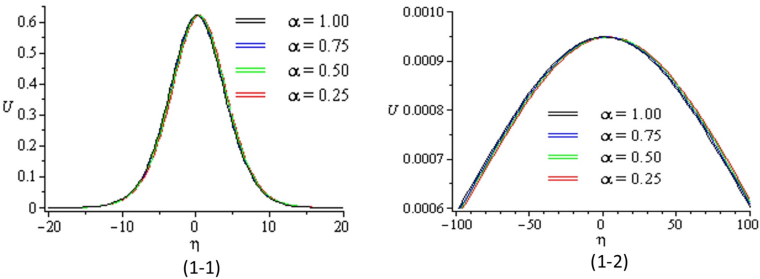


Figure 2. Approximate solutions for: (1-1) TCFKE in Example (1) with $\kappa = 0$ and different values of α ; (1-2) MTCFKE in Example (2) with $\kappa = 0$, $\zeta = 0.001$, $\xi = -1$.

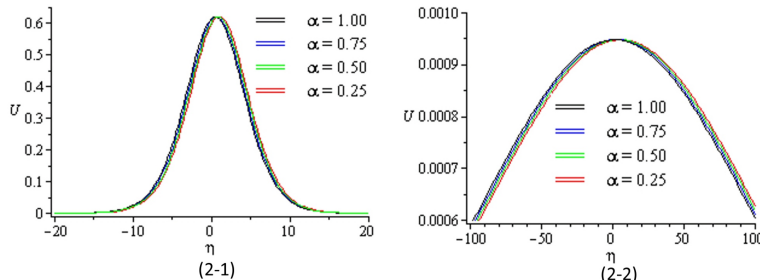


Figure 3. OAFM solutions for: (2-1) TCFKE in Example (1) with $\kappa = 2$ and different values of α ; (2-2) MTCFKE in Example (2) with $\kappa = 0$, $\zeta = 0.001$, $\xi = -1$.

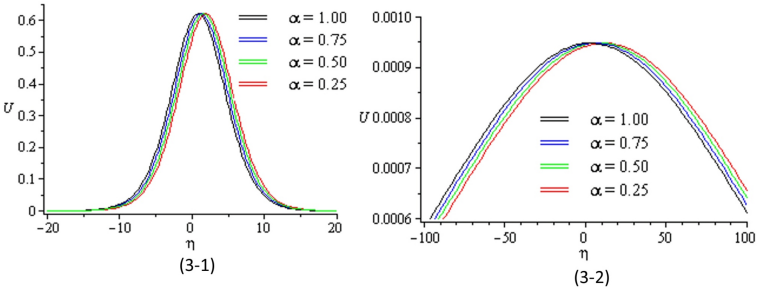


Figure 4. OAFM solutions for: (3-1) TCFKE in Example (1) with $\kappa = 4$ and different values of α ; (3-2) MTCFKE in Example (2) with $\kappa = 0$, $\zeta = 0.001$, $\xi = -1$.

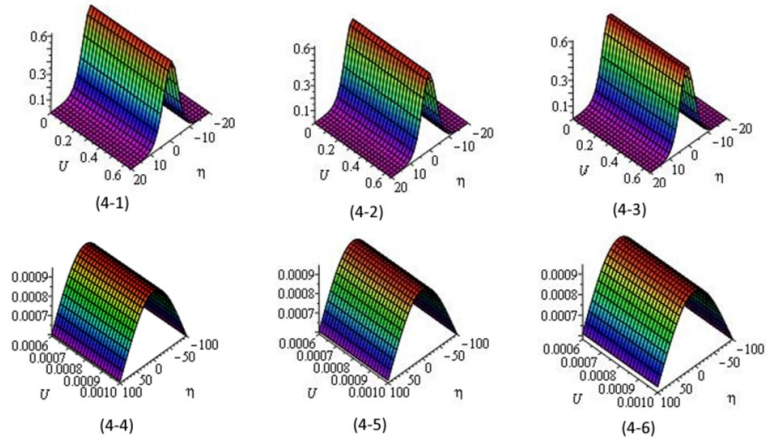


Figure 5. Plot of OAFM solutions for TCFKE in Example (1) and MTCFKE in Example (2) with $\alpha = 1.00$, $\zeta = 0.001$, $\xi = -1$.

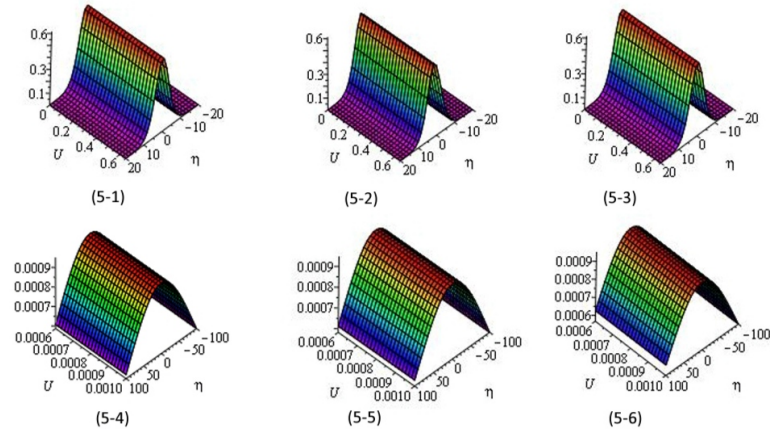


Figure 6. Plot of OAFM solutions for TCFKE in Example (1) and MTCFKE in Example (2) with $\alpha = 0.75$, $\zeta = 0.001$, $\xi = -1$.

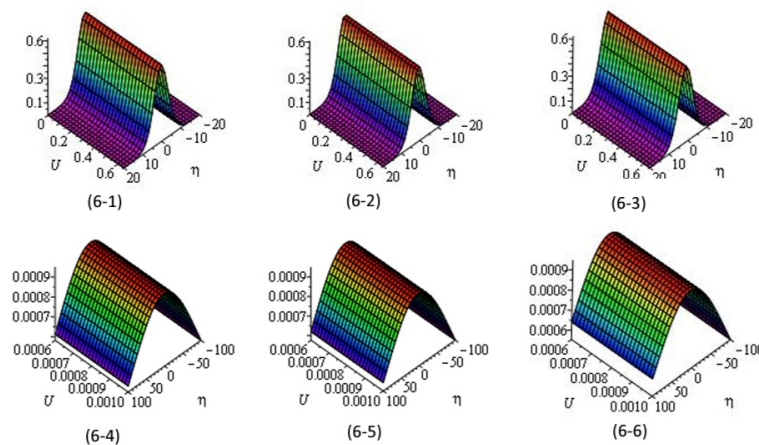


Figure 7. Surface plot of approximate solutions for TCFKE in Example (1) and MTCFKE in Example (2) with $\alpha = 0.50$, $\zeta = 0.001$, $\xi = -1$.

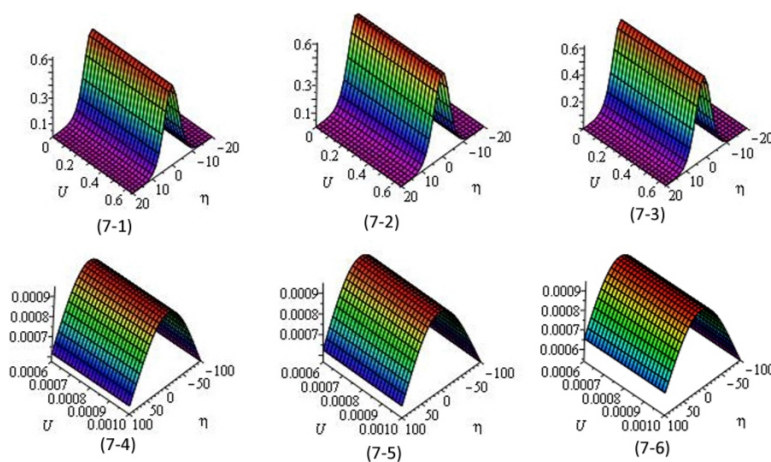


Figure 8. Plot of OAFM solutions for TCFKE in Example (1) and MTCFKE in Example (2) with $\alpha = 0.25$, $\zeta = 0.001$, $\xi = -1$.

5. Numerical Results

This section presents the approximate solutions of the TCFKE (14) and MTCFKE (25) for different fractional orders α . The results are summarized in Tables (1)–(4), while the corresponding dynamical behaviors are illustrated through numerical simulations in Figures (2)–(8).

Table (1) reports a comparison of the absolute error (AE) between the exact solution of TCFKE (14) at $\alpha = 1$, $\eta = 10$, and the approximate solutions obtained using three approaches: the present OAFM-based method, the RPSM [39], and the NTDM [40].

In Table (2), approximate solutions of TCFKE (14) are provided for $\eta = 10$ with different fractional orders α and values of κ , computed via OAFM and compared against the RPSM [39] and NTDM [40]. The numerical constants used for the solution $U(\eta, \kappa)$ in (24) are $c_1 = 0.00760324$, $c_2 = -3.75027$, and $c_3 = 3.87781114$.

For the MTCFKE, Table (3) highlights comparisons at $\zeta = 0.001$, $\xi = -1$, and $\alpha = 0.25, 0.50$, with varying values of η and κ . The OAFM results are contrasted with those obtained using RPSM [39] and HAM [44]. Table (4) further examines the MTCFKE for $\alpha = 0.75$ and $\alpha = 1.0$, again comparing OAFM with RPSM, NTDM [40], and HAM [44]. The constants in the solution $U(\eta, \kappa)$ of (35) are $c_1 = 4434650000$, $c_2 = -4946673909962748$, and $c_3 = 1754471350$.

The graphical outcomes supplement these tabulated comparisons. Figure (2) displays the solutions for Example (1) and Example (2) at $\kappa = 0$, $\zeta = 0.001$, $\xi = -1$ under different α values. Figures (3) and (4) extend this analysis to $\kappa = 2$ and $\kappa = 4$, respectively. Figure (5) provides surface plots for

the TCFKE at $\kappa = 0, 2, 4$ with $\alpha = 1.0$, alongside MTCFKE plots at the same κ values with $\zeta = 0.001$, $\zeta = -1$, and $\alpha = 1.0$. Figures (6) and (7) depict surface plots of both equations at fractional orders $\alpha = 0.75$ and $\alpha = 0.50$, respectively, while Figure (8) corresponds to $\alpha = 0.25$.

Overall, the presented tables and figures confirm the accuracy and reliability of OAFM. The tabulated data demonstrate its competitiveness with other established approaches, while the graphical results highlight the symmetry and consistency of the approximate solutions across different fractional orders.

The findings confirm that OAFM produces approximate solutions that closely align with both exact and numerical results, thereby validating its reliability. The method is systematic and straightforward, avoiding the need for linearization, discretization, or perturbation procedures, which simplifies its application. Furthermore, OAFM provides solutions in the form of rapidly convergent series, ensuring both stability and high accuracy.

A notable advantage of OAFM is its wide applicability: it can be employed for both linear and nonlinear fractional differential equations with varying fractional orders. From a computational standpoint, it is more efficient than many conventional numerical approaches, requiring fewer operations to achieve similar levels of accuracy. In addition, the method naturally integrates fractional derivatives and integral operators, making it well-suited for modern fractional models. Finally, the series-based representation of the solutions offers flexibility in accuracy control, as additional terms can be incorporated to enhance the approximation near the exact solution, thereby striking a balance between computational cost and precision.

6. Conclusions

In this study, we applied the OAFM to derive approximate analytical solutions of the TCFKE and its modified form, the MTCFKE. By optimizing auxiliary parameters, the OAFM produced highly accurate and rapidly convergent solutions, which were validated through numerical simulations and graphical analysis. Comparative results with established methods, including the HAM, RPSM, and NTDM, confirmed the reliability, efficiency, and superiority of the OAFM in handling fractional nonlinear wave equations.

The findings highlight the OAFM as a versatile and systematic tool for solving fractional differential models arising in applied sciences, physics, and engineering.

For future work, several promising directions arise from this study. The proposed methodology can be extended to other nonlinear dispersive equations, higher-dimensional systems, and models with variable coefficients. Incorporating different fractional operators—such as the Atangana–Baleanu or Caputo–Fabrizio derivatives—would broaden its applicability and allow for the investigation of nonlocal dynamics with non-singular kernels.

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