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Abstract

We show that Quantum Mechanical Hilbert space can be paradoxical under some group action and explore its physical consequences. (1) Is there a more natural way of resolving the paradox of Wigner's friend without invoking the Heisenberg's cut?; (2) We notice the qualitative similarities between the paradox and paradoxical sets and use it as a motivation to rigorously prove that the Hilbert space $\mathcal{H}$ of the harmonic oscillator is paradoxical under the group action induced by $SO(2,1)$; (3) This paradoxical nature of the Hilbert space $\mathcal{H}$ provides the natural resolution for the paradox by using the Axiom of Choice instead of the Heisenberg's cut; (4) Finally, we show that due to the very same paradoxical nature of $\mathcal{H}$, certain class of quantum gravities naturally emerge from Quantum Mechanics that mediates a self-decoherence of the system.

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1. Introduction

Quantum Mechanics (QM) is the only field in physics whose foundational and fundamental aspects continue to remain open and sensitive to interpretation. This openness to interpretation makes QM susceptible to many paradoxes including a paradox in particular called *Wigner's friend* [1]. According to the paradox, Wigner's friend (F) is investigating an electron which with respect to F is in the following state of superposition

$$|\psi_E\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \tag{1}$$

However, Wigner himself now views his own friend and the electron as a closed quantum system and assigns a state $|u\rangle$ to F if F observes the electron in $|\uparrow\rangle$ state and $|d\rangle$ if F observes the electron in $|\downarrow\rangle$ state which makes the combined state from Wigner's perspective to be

$$|\psi_{FE}\rangle = \frac{1}{\sqrt{2}}(|u\rangle|\uparrow\rangle + |d\rangle|\downarrow\rangle) \tag{2}$$

But $|\psi_{FE}\rangle$ is an entangled state while $|\psi_E\rangle$ is a pure state. The state of the electron is either pure or entangled depending on the observer, or more precisely, where the *Heisenberg cut*[1] is made, hence, the paradox. Of course, the above paradox makes some assumptions that can be seen as problematic - (i) QM is universal and applies to a macroscopic object like F, and (ii) QM is invariant under the change of the Heisenberg cut. (i) can simply be removed by introducing the Heisenberg cut between all macroscopic observers and quantum experiments while (ii) can simply be removed by arguing that QM is actually not invariant under change of the Heisenberg cut. Clearly, the transformation $|\psi_E\rangle \rightarrow |\psi_{FE}\rangle$ under the change of the cut is highly nontrivial and cannot be an isometry, hence, resolving the paradox. However, it is also worth noting that the Heisenberg cut is actually an external addition and is not necessarily natural. A more natural resolution to the paradox, therefore, should

[1] *Heisenberg cut* is the hypothetical separation between a quantum experiment and the observer. At the level of the experiment only QM applies but at the level of the observer only classical mechanics applies.

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not involve the Heisenberg cut.

To do that, let's revisit the paradox and note a few peculiarities. Notice that $|\psi_E\rangle \in \mathcal{H}$ while $|\psi_{FE}\rangle \in \mathcal{H} \otimes \mathcal{H}$. It seems that the presence of an extra observer apart from F, namely Wigner, has duplicated the Hilbert space $\mathcal{H}$. But the term 'observer' implicitly involves a Heisenberg cut, hence, in a cut-independent manner one may naively state that the Hilbert space $\mathcal{H}$ itself has 'spontaneously' duplicated. This way we enter a much more familiar and well-established territory of *paradoxical set theory* where *paradoxical sets* can undergo duplication via specific group actions. The most celebrated and well-known example of this is the *Banach-Tarski paradox* which states that a sphere S^2 can be duplicated via SO_3 group action [2]. Qualitatively, this is very similar to what is happening with $\mathcal{H}$ above except the underlying group action that causes this duplication is unidentified. In the next section, we will rigorously show that the Hilbert space $\mathcal{H}$ in QM can indeed be *paradoxical* with respect to some group action.

2. Hilbert Space as a Paradoxical Set

A paradoxical set is defined with respect to a group G that has the following (left) group action

$$\begin{aligned} G \times X &\to X \\ (g, x) &\to gx \end{aligned} \tag{3}$$

The group action is assumed to be associative. Then [2–4]

Definition 1. *Let G be a group acting on a set X and suppose $A, B \subseteq X$. A is G-*equidecomposable *to B (denoted $A \sim_G B$) if $A = \bigcup_{i=1}^m A_i$, $B = \bigcup_{i=1}^m B_i$ such that $A_i = g_i \cdot B_i$ and $A_i \cap A_j = B_i \cap B_j = \varnothing$.*

Definition 2. *Let G be a group acting on a set X and suppose $E \subseteq X$. E is G-*paradoxical *if for some positive integers m, n there exists pairwise disjoint subsets $A_1, ..., A_n$, $B_1, ..., B_m$ of E and $g_1, ..., g_n, h_1, ..., h_m \in G$ such that*

$$E = \bigcup_{k=1}^{n} g_k \cdot A_k = \bigcup_{k=1}^{m} h_k \cdot B_k$$

*Alternatively, a subset E is G-*paradoxical *if there exists two disjoint subsets A and B such that $A \sim_G E$ and $B \sim_G E$.*

Consider the ladder operators in the standard Quantum Harmonic Oscillator (QHO) given by

$$[a, a^\dagger] = 1 \tag{4}$$

Now, we build the following operator algebra from the above

$$J_2 \equiv \frac{a^\dagger a}{2} + \frac{1}{4} \quad K_1 \equiv \frac{a^2 + a^{\dagger 2}}{4} \quad K_3 \equiv i\left(\frac{a^2 - a^{\dagger 2}}{4}\right) \tag{5}$$

$$[J_2, K_3] = iK_1 \quad [J_2, K_1] = -iK_3 \quad [K_1, K_3] = iJ_2 \tag{6}$$

which is precisely the Lie algebra for $SO(2,1)$ [5]. The reason we need to do this is because the group generated from the above is not nilpotent. Using the above one can define

$$X_\alpha = e^{i\theta_\alpha \cdot L} \quad L = (K_1, J_2, K_3) \tag{7}$$

where X_α are elements $SO(2,1) \leq U(\mathcal{H}) \times U(1)$ where $\mathcal{H}$ is the *rigged* Hilbert space of the QHO and $U(\mathcal{H})$ is the space of all unitary operators on $\mathcal{H}$. We are also not restricting to the projective Hilbert

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space $\mathcal{PH}$ as well. We wish to show that $\mathcal{H}$ is $SO(2,1)$-paradoxical. The first step towards that is to show that it contains a free subgroup $\mathbb{F}_2$ which is defined as [4]

Definition 3. *A* free group $\mathbb{F}_2$ *is the set of all words generated by a generating set* $S = \{a, b\}$*. It follows [4] that* free group $\mathbb{F}_2$ *is also* $\mathbb{F}_2$*-paradoxical which can be demonstrated by the disjoint subsets* $P_1 \equiv A_1 \cup B_1, P_2 \equiv A_2 \cup B_2$ *that partition* $\mathbb{F}_2$ *where*

$$A_1 \equiv \Psi(a^{-1}) \cup \{e\} \cup \{a\} \cup \{a^2\} \cup \ldots \tag{8}$$

$$A_2 \equiv \Psi(a) \backslash \{a\} \cup \{a^2\} \cup \ldots \tag{9}$$

$$B_1 \equiv \Psi(b^{-1}) \quad B_2 \equiv \Psi(b) \tag{10}$$

where $\Psi(x)$ *represents a word starting from* $x \in \{b^{-1}, a^{-1}, a, b\}$ *such that* $P_1 \sim_{\mathbb{F}_2} \mathbb{F}_2, P_2 \sim_{\mathbb{F}_2} \mathbb{F}_2$*.*

Theorem 1. $S = \left\{e^{i\theta_1 K_1}, e^{i\theta_3 K_3}\right\}$ *form the generating set for the free subgroup* $\mathbb{F}_2$ *of* $SO(2,1)$*.*

Proof. Due to the similarity in the Lie algebra structure of $SO(2,1)$ and SO_3, one can simply follow the procedure of defining the free subgroups of SO_3 in $SO(2,1)$ i.e. two rotations whose axes are perpendicular to each other [2]. The choice of S above represents the $SO(2,1)$ analogue of the same. □

The above implies that $SO(2,1)$ has a free subgroup and therefore, guarantees that $\mathcal{H}$ is indeed $SO(2,1)$-paradoxical but for the sake of completion, we will follow through all the crucial steps of the proof closely following Kaseorg [3] and Buchhorn [4].

Theorem 2. *Let* $\Delta = \{|\psi\rangle : \hat{x}|\psi\rangle = |\psi\rangle,\ \hat{x} \in \mathbb{F}_2 \backslash e\}$*.*

1. Δ *is countably infinite.*
2. *For any* $f \in \mathbb{F}_2$*,* $|\phi\rangle \in \Delta$ *implies* $f|\phi\rangle \in \Delta$*.*

Proof. 1. Consider

$$\Delta_n = \{|\psi\rangle_n : \hat{x}_n|\psi\rangle_n = |\psi\rangle_n,\ \hat{x}_n \in \chi_n \subset \mathbb{F}_2\} \tag{11}$$

where $|\psi\rangle_i \in \mathcal{H}$ and

$$\chi_n = \{\hat{x}_n : \hat{x}_n \text{ is an irreducible word of length } n\} \tag{12}$$

such that

$$\Delta = \bigcup_{n=1}^{\infty} \Delta_n \tag{13}$$

From the Cayley's graph $\Gamma(\mathbb{F}_2, S)$ in Figure 1, we can see that

$$|\chi_n| = 4 \times 3^{n-1} \tag{14}$$

Since, we have

$$\hat{x}_n|\psi\rangle_n = |\psi\rangle_n \implies \hat{x}_n^{-1}|\psi\rangle_n = |\psi\rangle_n \tag{15}$$

this leads to

$$|\Delta_n| = \frac{|\chi_n|}{2} \tag{16}$$

Since, each Δ_n is countable, therefore, Δ is countably infinite.

2. Consider $|\phi\rangle \in \Delta$ then $\exists\, h \in \mathbb{F}_2$ such that $h|\phi\rangle = |\phi\rangle$. Now, consider

$$f|\phi\rangle = fh|\phi\rangle = (fhf^{-1})f|\phi\rangle \tag{17}$$

Since, $fhf^{-1} \in \mathbb{F}_2$ as both $f, h \in \mathbb{F}_2 \implies f|\phi\rangle \in \Delta$.

□

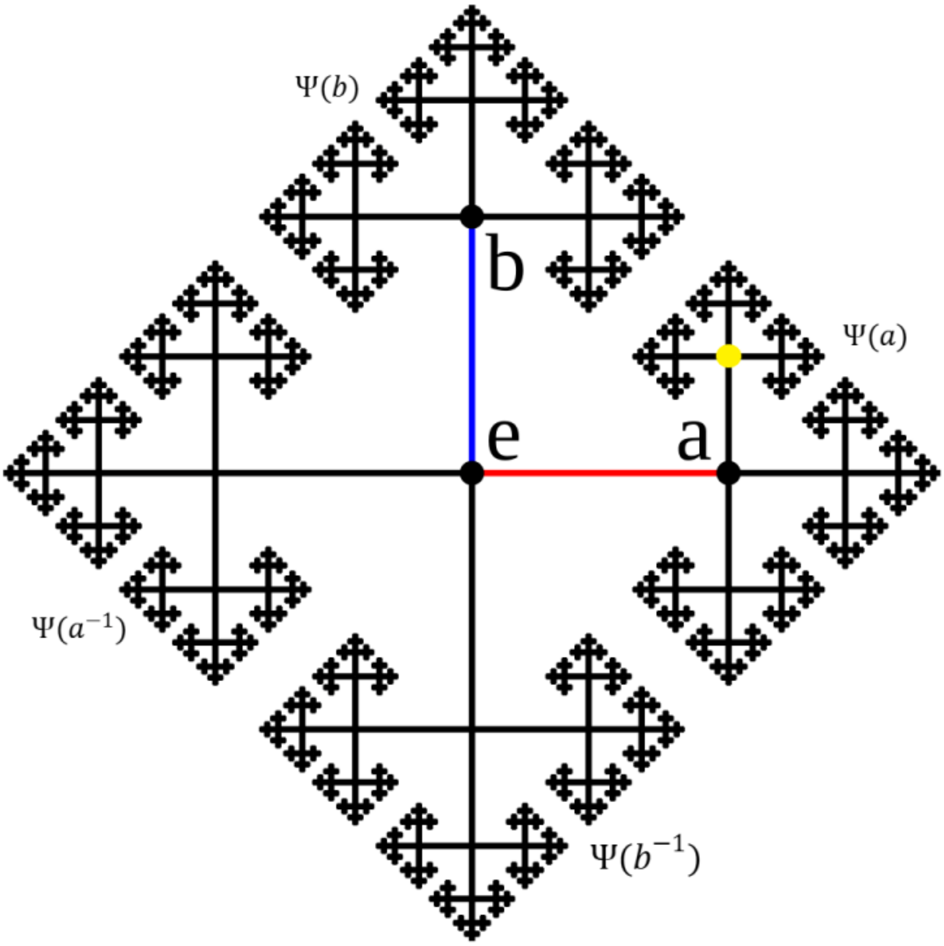

Figure 1. Cayley graph $\Gamma(\mathbb{F}_2, S)$ where $S = \{a, b\}$. Each vertex is a reduced word in $\mathbb{F}_2$ and the edges are elements of S. Each edge is a multiplication on the vertex it starts from. Edges that are going *up* multiply with b, the ones going *down* multiply with b^{-1}. Edges that go *right* multiply with a, the ones that go *left* multiply with a^{-1}. For instance, the word corresponding to the *yellow vertex* reads ab. To reach that vertex, we first go right which multiply a with e (identity) and then go up which acts with b to give $e \cdot a \cdot b = ab$.

Theorem 3. *$\mathcal{H}$ is $SO(2,1)$-equidecomposable to $\mathcal{H}\backslash\Delta$ i.e. $\mathcal{H} \sim_{SO(2,1)} \mathcal{H}\backslash\Delta$.*

Proof. Let $\mu = X_\gamma$ such that $X_\gamma \notin \mathbb{F}_2$ which due to Theorem 2 implies that $X^k_\gamma|\phi\rangle \neq |\phi'\rangle$ for any $|\phi\rangle, |\phi'\rangle \in \Delta$ and for any $k \in \mathbb{Z}\backslash\{0\}$. Hence, we have $\mu^k\Delta \cap \Delta = \varnothing$. Then for some $k = m - n$ without loss of generality, implies that $\mu^m\Delta \cap \mu^n\Delta = \varnothing$ where $m \neq n$. Now, we define

$$P_1 = \Delta \cup \mu\Delta \cup \mu^2\Delta \cdots \quad Q_1 = \mathcal{H}\backslash P_1 \tag{18}$$

$$P_2 = \mu\Delta \cup \mu^2\Delta \cup \mu^3\Delta \cdots \quad Q_2 = \mathcal{H}\backslash P_1 \tag{19}$$

where the above satisfy

$$P_1 \cap Q_1 = P_2 \cap Q_2 = \varnothing \tag{20}$$

$$P_1 \cup Q_1 = \mathcal{H} \quad P_2 \cup Q_2 = \mathcal{H}\backslash\Delta \tag{21}$$

$$P_2 = \mu P_1 \quad Q_2 = Q_1 \tag{22}$$

The above demonstrates that $\mathcal{H} \sim_{SO(2,1)} \mathcal{H}\backslash\Delta$. □

Definition 4. *$F_s = \{f|s\rangle : |s\rangle \in \mathcal{H}\backslash\Delta\ \forall f \in \mathbb{F}_2\}$ is called an $\mathbb{F}$-orbit around $|s\rangle$. The following property of the $\mathbb{F}$-orbit follows from its definition [4]*

1. *The $\mathbb{F}$-orbits partition the set $\mathcal{H}\backslash\Delta$.*
2. *The union of all the $\mathbb{F}$-orbits is $\mathcal{H}\backslash\Delta$ i.e. $\bigcup_{i\in I} F_{s_i} = \mathcal{H}\backslash\Delta$*

Theorem 4. *$\mathcal{H}$ is $SO(2,1)$-paradoxical i.e. there exists two disjoint subsets H_1 and H_2 of $\mathcal{H}$ satisfying $H_1 \sim_{SO(2,1)} \mathcal{H}$, $H_2 \sim_{SO(2,1)} \mathcal{H}$.*

Proof. Using Axiom of Choice (AOC), one can construct $M \subset \mathcal{H}\backslash\Delta$ such that $|M \cap F_{s_i}| = 1$. Then one can construct subsets $H_1 := P_1 M \cup \Delta$, $H_2 := P_2 M$ such that [4]

$$H_1 \cup H_2 = \mathbb{F}_2 M \cup \Delta = \mathcal{H}\backslash\Delta \cup \Delta = \mathcal{H} \tag{23}$$

$$H_1 \cap H_2 = \varnothing \tag{24}$$

where in the above we have used $\mathbb{F}_2 M = \mathcal{H}\backslash\Delta$ which follows from Definition 4. Now, since, we have

$$H_1 \sim_{SO(2,1)} \mathbb{F}_2 M \cup \Delta = \mathcal{H}\backslash\Delta \cup \Delta = \mathcal{H} \tag{25}$$

$$H_2 \sim_{SO(2,1)} \mathbb{F}_2 M = \mathcal{H}\backslash\Delta \sim_{SO(2,1)} \mathcal{H} \tag{26}$$

where in the above we used Theorem 3. Hence, proved. □

In other words, the Hilbert space can be partitioned into subsets, each of which can be transformed into the full $\mathcal{H}$ via $SO(2,1)$-transformations! Since, the elements of $SO(2,1)$ are time-independent transformations these duplications are indeed spontaneous.

3. Results

The paradoxical nature of the Hilbert space $\mathcal{H}$ allows it to be duplicated arbitrarily by applying Theorem 4 repeatedly as follows

$$\mathcal{H} \sim_{SO(2,1)} \mathcal{H} \otimes \mathcal{H}_1 \sim_{SO(2,1)} \mathcal{H} \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \sim_{SO(2,1)} \cdots \sim_{SO(2,1)} \mathcal{H} \otimes \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_n$$
$$\implies \left(\sum_i b_i|e_i\rangle \equiv\right)|\psi\rangle \xrightarrow[Tarski]{Banach} \sum_{i,j} c_{ij}|e_i\rangle \otimes |\chi_j\rangle, \quad |\chi_i\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \quad \mathcal{H} = \mathrm{span}(\{|e_i\rangle\}) \tag{27}$$

where the numerical subscripts on $\mathcal{H}$ are simply to label the duplicates. Also, it is worth noting how the final step of the duplication above resembles a quantum mechanical system in entanglement with an environment. In terms of their respective density matrices, Eq. (27) becomes

$$\rho_O \xrightarrow[\text{Tarski}]{\text{Banach}} \rho_{OD} \tag{28}$$

where O refers to the 'original' and D refers to the 'duplicates'. Now, we partially trace out the duplicates to obtain

$$\begin{aligned} \mathrm{Tr}_D(\rho_{OD}) &= \mathrm{Tr}_D\left(\sum_{i,j,a,b} c_{ij}c^*_{ab}|e_i\rangle \otimes |\chi_j\rangle\langle e_a| \otimes \langle\chi_b|\right) = \sum_{i,j,a,b} c_{ij}c^*_{ab}\langle\chi_b|\chi_j\rangle|e_i\rangle\langle e_a| \\ &= \left(\sum_{\substack{i,j,a,b\\ b=j}} + \sum_{\substack{i,j,a,b\\ b\neq j}}\right) c_{ij}c^*_{ab}\langle\chi_b|\chi_j\rangle|e_i\rangle\langle e_a| = \sum_i |b_i|^2|e_i\rangle\langle e_i| + \sum_{i,j} M_{ij}(n)|e_i\rangle\langle e_j| \end{aligned} \tag{29}$$

where we have used

$$m_{ia} \equiv \sum_j c_{ij}c^*_{aj} - \delta_{ia}|b_i|^2 \quad M_{ij}(n) \equiv \sum_{a,b}(c_{ia}c^*_{jb} + \delta_{ij}m_{ab})\langle\chi_b|\chi_a\rangle \tag{30}$$

which can be defined without loss of generality. Notice that if $M_{ij}(n) = b_i b^*_j$ then we will get back ρ_O. However, for a given n, $M_{ij}(n)$ in general is a random Hermitian matrix of infinite size and therefore, is governed by some Random Matrix Theory (RMT) in the $N \to \infty$ limit where N is the size of the matrix in RMT. Depending on the mean, the RMTs are of two types

$$\langle M\rangle = 0, \quad \langle M\rangle \neq 0 \tag{31}$$

The RMTs where $\langle M\rangle = 0$ are precisely the theories that correspond to the $N \times N$ matrix models that result in string theories in the $N \to \infty$ limit [6]. In fact, for finite N, $N \times N$ matrix models represent the discretized version of the string theories themselves [6]. The RMTs where $\langle M\rangle \neq 0$ represents interacting condensed matter systems [7,8]. Since, the quantum system under consideration is free and isolated, we must have $\langle M\rangle = 0$ for the above which is also the condition for decoherence [9]! This represents the most striking consequence of the Banach-Tarski paradox where quantum states can entangle with its duplicates and self-decohere even before a measurement is made. Since, the matrix models with zero mean are quantum theories of gravity, the decoherence is gravitationally mediated. The source of this gravity must be the particle itself as the system is isolated. This computation, therefore, also demonstrates how gravity can emerge from QM owing to the paradoxical nature of $\mathcal{H}$.

Coming back to our original motivation i.e. in the paradox of *Wigner's friend*, notice how the *choice* in the placement of the Heisenberg's cut lead to the duplication of $\mathcal{H}$ exactly like how the AOC in the above allows group theoretic duplications of $\mathcal{H}$. In other words, the Heisenberg's cut is a Quantum Mechanical analogue of the AOC. However, as mentioned earlier, compared to the AOC the underlying group action is unknown and the duplication doesn't follow *naturally*. But by demonstrating that such a duplication is indeed natural and built into the mathematical foundations of QM itself, the paradox is then resolved. The resolution being that all description of the QM system whether pure or entangled are equally valid owing to the paradoxical nature of $\mathcal{H}$.

4. Discussion and Conclusion

In this exercise, we explored a rare demonstration of how foundational abstract mathematics can have physical consequences. Usually, only the applied aspects of mathematics are relevant to physics as physical phenomena are not expected to be so counter-intuitive that it warrants a more abstract treatment. This is the reason paradoxical phenomena in QM is handled philosophically. But as demonstrated, working with the abstract foundations of mathematics is more natural. We also demonstrated how quantum gravity emerges from QM as a result of the paradoxical nature of $\mathcal{H}$. The entanglement of the quantum system with its own duplicates encoded via $M_{ij}(n)$ can be seen as an emergence of self-interaction. Since, this self-interaction is best modelled by RMTs that have a zero mean which are known to be quantum theories of gravity, the emergent self-interaction is, therefore, of gravitational nature. This result is qualitatively consistent with various models of gravitationally mediated collapse in the literature [10]. However, in this letter, this is made more natural by demonstrating gravity as emergent from QM due to the paradoxical nature of $\mathcal{H}$ which then mediates the collapse. Another interesting thing to note is that the Hilbert space of QHO was shown to be $SO(2,1)$-paradoxical which is not very different from the SO_3 group i.e. the Euler's rotation theorem applies to both of them. This similarity was crucial for us to be able to import the well-established mathematical results into quantum physics.

However, one may interpret the number of duplicates n to be some sort of a "hidden variable" that affects the outcome of measurements. If so, such a paradox may impose a Bell-like inequality on the quantum mechanical system which can lead to serious conflict with experiments. To check that, let A and B represent dichotomic measurement outcomes [11]

$$A(a,n) = \pm 1 \quad B(b,n) = \pm 1 \quad n \in \mathbb{N} \tag{32}$$

where $p(n)$ representing the probability of the state being entangled with n of its duplicates. The correlation between the measurements A and B is then given by

$$\langle A(a)B(b)\rangle = \sum_{n\in\mathbb{N}} p(n)A(a,n)B(b,n) \tag{33}$$

But probability theory deals with measurable sets while Banach-Tarski duplications are consequences of non-measurable sets [2,4], therefore, there are no means to define $p(n)$. Hence, no well-defined Bell-like inequality is enforced on the quantum mechanical state as it should be to be consistent with experiments.

It is also worth noting that the applications of Banach-Tarski paradox itself has been explored in Hadronic physics previously [12]. The Hadrons were likened to spheres that are then subjected to standard Banach-Tarski duplications which were noted to correspond to well-known Hadronic decay and tree-level processes. This means that paradoxical sets do play some roles in physics all the way upto the Standard Model. Therefore, working with paradoxical sets may help us develop new physics and formalism beyond the current one as well. This is something we intend to explore in the future.

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Abbreviations

The following abbreviations are used in this manuscript:

AOC	Axiom of Choice
QM	Quantum Mechanics

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