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Article

Every K_4-Minor-Free Graph Is Ramsey Size-Linear

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Abstract

A graph G is Ramsey size-linear if $R(G,H) = O(e(H))$ for every graph H without isolated vertices. Erdős, Faudree, Rousseau and Schelp asked whether every graph in which each k-vertex subgraph spans at most $2k-3$ edges is Ramsey size-linear (Erdős Problem #566). We prove that every graph with no K_4-minor is Ramsey size-linear, with $R(G,H) \le 624\, v(G)\, e(H)$. This covers all 2-trees, which attain the density $2k-3$, and extends results of Erdős et al. for graphs $K_1 + T$ with T a tree and for connected graphs with at most $v(G)+1$ edges. The proof combines the triangle-elimination process and an averaging argument of Bradač, Gishboliner and Sudakov. As consequences, a connected graph with at most $v(G)+2$ edges is Ramsey size-linear unless it contains K_4 or K_4^*, and each of the pairs $\{K_4^*, W_4\}$ and $\{W_5^-, W_5\}$ contains a minimally non-Ramsey size-linear graph. We also make explicit further structures produced by the proof of Bradač, Gishboliner and Sudakov and, by a computer search over graphs on at most eight vertices, reduce the open cases of the question to 49 minimal graphs governed by eleven cores.

Keywords: Ramsey size-linear graphs; graph Ramsey numbers; treewidth; series–parallel graphs; Erdős problems

MSC: 05C55; 05D10; 05C35; 05C83

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1. Introduction

For graphs G and H, the Ramsey number $R(G,H)$ is the smallest integer N such that for every graph Γ on N vertices, Γ contains a copy of G or its complement $\overline{\Gamma}$ contains a copy of H. Harary conjectured that $R(K_3,H) \le 2e(H)+1$ for every graph H without isolated vertices; this was proved by Sidorenko [18] and by Goddard and Kleitman [14]. Erdős, Faudree, Rousseau and Schelp [12] initiated a systematic study of this phenomenon and called a graph G *Ramsey size-linear* if there is a constant C_G such that

$$R(G,H) \le C_G\, e(H) \qquad \text{for every graph } H \text{ without isolated vertices.}$$

Among other results, they proved the following.

(i) Every connected graph G with $e(G) \le v(G)+1$ is Ramsey size-linear [12, Theorem 4].

(ii) If G contains a subgraph with k vertices and at least $2k-2$ edges, then G is not Ramsey size-linear [12, Corollary 1].

(iii) If T is a tree, then $K_1 + T$ is Ramsey size-linear; in fact $R(K_1+T,H) \le 2\, v(T)\, e(H)$ [12, Theorem 3 and Corollary 2].

(iv) Every graph with Turán number $O(n^{3/2})$ is Ramsey size-linear [12, Theorem 5]; examples mentioned there include $K_{3,3}-e$ and Q_3-e.

Fact (ii) comes from the lower bound $R(G,K_n) = \Omega\big((n/\log n)^{m_2(G)}\big)$, obtained by the probabilistic method [19] (see also [12,20]), where $m_2(G) = \max\{(e(F)-1)/(v(F)-2) : F \subseteq G,\ v(F) \ge 3\}$: a subgraph with k vertices and $2k-2$ edges forces $m_2(G) > 2$, and then $R(G,K_n)$ grows faster than $e(K_n)$. Erdős et al. asked whether this is the only obstruction.

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Question 1.1 ([12, Question 1]; Erdős Problem #566 [3]). *Let G be a graph such that every subgraph of G on $k \geq 2$ vertices has at most $2k-3$ edges, that is, $m_2(G) \leq 2$. Is G Ramsey size-linear?*

They also asked whether $K_{3,3}$, the cube Q_3 and $K_4^* = K_5 - (K_{1,2} \cup K_2)$, the graph obtained from K_4 by subdividing one edge, are Ramsey size-linear [12, Question 2], noting that each of them would be *minimally* non-Ramsey size-linear otherwise; see also [11], and [2,5] for K_4^*. Here a graph is minimally non-Ramsey size-linear if it is not Ramsey size-linear but all its proper subgraphs are. Erdős et al. observed that K_4 is such a graph and asked for others [12, Question 6] (Erdős Problem #79). Related questions are collected as Erdős Problems #566–#570 [3].

Facts (ii) and (iii) show that the density $2k-3$ in Question 1.1 cannot be raised and is attained by Ramsey size-linear graphs. Balister, Schelp and Simonovits [2] proved that joining two vertices at distance d of a Ramsey size-linear graph by a new path of length at least $d+3$ preserves the property. They asked whether every 2-connected graph of minimum degree at least three fails to be Ramsey size-linear; this was answered negatively by combining (iv) with a construction of Janzer [16], as observed in [5]. Bradač, Gishboliner and Sudakov [5] proved that every subdivision of K_4 with at least six vertices is Ramsey size-linear, that $R(K_4^*, F) = O(e(F))$ for every bipartite F, and that $R(K_4^*, K_n) = O(n^{5/2})$, and remarked that Question 1.1 seems to be out of reach. Wigderson [20] proved that there are infinitely many minimally non-Ramsey size-linear graphs; his argument is non-constructive, and he asked for an explicit example other than K_4 [20, Open problem 5]. For cycles, the optimal linear constant was recently determined by Cambie, Freschi, Morawski, Petrova and Pokrovskiy [7]; see also [15].

Main Result

Theorem 1.2. *Let G be a graph with no K_4-minor, equivalently a graph of treewidth at most two, and let $h = \max\{v(G), 3\}$. Then for every graph H with $m \geq 1$ edges and no isolated vertices,*

$$R(G, H) \leq (624h - 1816)\, m.$$

In particular, every K_4-minor-free graph is Ramsey size-linear.

A graph of treewidth at most two on $k \geq 2$ vertices has at most $2k-3$ edges, with equality for 2-trees, and its subgraphs again have treewidth at most two. Thus Theorem 1.2 answers Question 1.1 affirmatively for a minor-closed class which contains graphs of the extremal density $2k-3$ allowed there, including all 2-trees, all outerplanar graphs and all series–parallel graphs. It extends facts (i) and (iii): the graphs $K_1 + T$ have treewidth two, and a connected graph containing a subdivision of K_4 has cyclomatic number $e - v + 1 \geq 3$, so connected graphs with at most $v(G)+1$ edges have no K_4-minor. The smallest graphs covered by Theorem 1.2 but by none of the results quoted above have six vertices: the 3-sun (a triangle with a further triangle attached along each of its edges), the strip of four triangles, and the triangular prism minus one of its three matching edges (Section 7). We have not found Theorem 1.2 stated in the literature.

The proof has two steps, both relying on ideas of [5]. The first (Lemma 3.1) is the averaging argument of [5, Claim 4.4], stated as a general embedding result: if a graph Γ on N vertices has at most $\lambda\, e(\Gamma)$ triangles, then $\overline{\Gamma}$ contains every graph without isolated vertices having at most $N/(624\lambda + 56)$ edges. The second (Lemma 4.1) uses the triangle-elimination process of [5, Section 4]: if H is a 2-tree on h vertices, the process eliminates every triangle of an H-free graph Γ, so Γ has at most $(h-3)\, e(\Gamma)$ triangles. In [5] these tools are used to find subdivisions of K_4; the point here is that for 2-trees they suffice on their own.

Consequences

Corollary 1.3. *Let G be a connected graph with $e(G) \leq v(G) + 2$.*

(a) If G contains neither K_4 nor K_4^*, then G is Ramsey size-linear.

(b) If G contains K_4, then G is not Ramsey size-linear.

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(c) *If G contains K_4^*, then G is Ramsey size-linear if and only if K_4^* is.*

Thus the threshold $v(G)+1$ of fact (i), which is sharp because of K_4, can be raised to $v(G)+2$ at the cost of a single unresolved graph, K_4^*.

For $k \geq 3$ let $W_k = K_1 + C_k$ be the wheel with k spokes, and for $k \geq 4$ let W_k^- denote W_k with one spoke removed; see Figure 1.

Proposition 1.4. *Let $k \geq 4$.*

(a) *W_k is not Ramsey size-linear, while W_k minus a rim edge is. Consequently, W_k is minimally non-Ramsey size-linear if and only if W_k^- is Ramsey size-linear.*

(b) *$W_4^- \cong K_4^*$.*

(c) *$m_2(W_k^-) = 2$. Hence an affirmative answer to Question 1.1 would imply that every wheel W_k with $k \geq 4$ is minimally non-Ramsey size-linear.*

(d) *For $k \in \{4,5\}$ every proper subgraph of W_k^- is Ramsey size-linear. Consequently each of the pairs $\{K_4^*, W_4\}$ and $\{W_5^-, W_5\}$ contains a minimally non-Ramsey size-linear graph. In particular, there are at least two minimally non-Ramsey size-linear graphs other than K_4: one of K_4^*, W_4 on five vertices and one of W_5^-, W_5 on six vertices.*

Proposition 1.4 does not depend on Theorem 1.2: it follows from facts (i)–(iii) and [5]. Erdős et al. already observed that K_4^* would be minimally non-Ramsey size-linear if it were not Ramsey size-linear [12]; the new point is the other member of each pair. Part (d) does not settle Wigderson's problem, since we cannot say which member of each pair is minimally non-Ramsey size-linear. It does show that *any* resolution of the question of Erdős et al. for K_4^*, in either direction, produces an explicit minimally non-Ramsey size-linear graph other than K_4: namely W_4 if K_4^* is Ramsey size-linear, and K_4^* itself otherwise. The same holds for W_5^- and W_5. Our computations (Section 7) show that, with the tools currently available, these are the only such pairs among the density-critical graphs on at most eight vertices.

Finally, in Section 6 we make explicit what the proof of [5, Theorem 4] actually produces, and in Section 7 we determine by an exhaustive computer search the minimal graphs on at most eight vertices that satisfy the hypothesis of Question 1.1 but whose Ramsey size-linearity does not follow from the results available. Apart from K_4^* there are 3, 10 and 35 of them on six, seven and eight vertices. The three on six vertices are W_5^-, the triangular prism and $K_{3,3}$, and all of them reduce to one of eleven cores (Proposition 7.2).

Organization. Section 2 collects preliminaries. Lemma 3.1 is proved in Section 3 and Lemma 4.1 in Section 4. Section 5 contains the proofs of Theorem 1.2, Corollary 1.3 and Proposition 1.4, Section 6 the consequences of the proof of [5], Section 7 the census of small cases, and Section 8 concluding remarks and open questions.

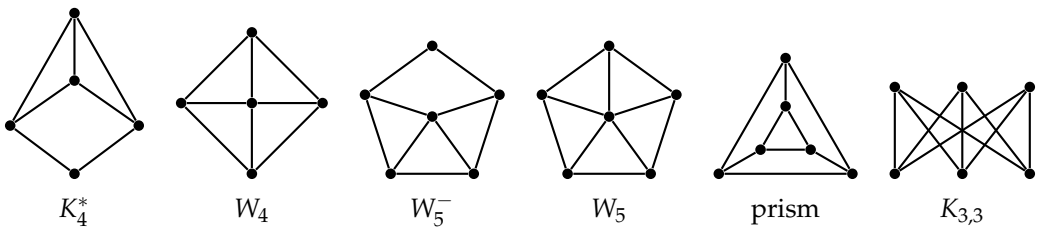

Figure 1. The graphs of Proposition 1.4(d), and the two remaining minimal unresolved graphs on six vertices. K_4^*, W_5^-, the prism and $K_{3,3}$ are the minimal unresolved graphs on at most six vertices (Section 7); W_4 and W_5 are not Ramsey size-linear.

2. Preliminaries

All graphs are finite and simple. For a graph Γ we write $v(\Gamma)$ and $e(\Gamma)$ for the numbers of vertices and edges, $N_\Gamma(x)$ and $d_\Gamma(x)$ for the neighbourhood and the degree of a vertex x, $T(\Gamma)$ for the number

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of triangles, and $\overline{\Gamma}$ for the complement. We say that Γ *contains* G if Γ has a (not necessarily induced) subgraph isomorphic to G, and that Γ is G-free otherwise. If $G \subseteq G'$ then $R(G,H) \le R(G',H)$; hence every subgraph of a Ramsey size-linear graph is Ramsey size-linear.

Treewidth.

A 2-*tree* is a graph obtained from a triangle by repeatedly adding a new vertex adjacent to both ends of an existing edge. A 2-tree on $k \ge 3$ vertices has $2k-3$ edges. We use the following standard facts; see [4,9,10].

(T1) A graph has treewidth at most two if and only if it has no K_4-minor.

(T2) Every graph of treewidth at most two with at least three vertices is a spanning subgraph of a 2-tree.

(T3) A graph has a K_4-minor if and only if it contains a subdivision of K_4.

Lemma 2.1 (Bradač, Gishboliner and Sudakov [5, Lemma 2.3])**.** *Let H be a graph with m edges, let $0 \le k \le v(H)$, and let A be a set of k vertices of largest degree in H. Let Γ be a graph on N vertices with average degree d, and let $\sigma' \colon A \to V(\Gamma)$ be an embedding of $H[A]$ into $\overline{\Gamma}$. If*

$$N \;\ge\; \frac{4m}{k+1}\,\max\Big\{\max_{a\in A} d_\Gamma(\sigma'(a)),\; 2d\Big\} + 2v(H),$$

then σ' extends to an embedding of H into $\overline{\Gamma}$.

Proof (included for completeness). Let D denote the maximum in the displayed inequality and let $W = \{u : d_\Gamma(u) \le 2d\}$. Fewer than $N/2$ vertices have degree larger than $2d$, so $|W| > N/2$. Embed the vertices of $V(H)\setminus A$ one at a time into W. When x is to be embedded, each of the k vertices of A has degree at least $d_H(x)$, so $(k+1)d_H(x) \le 2m$. The vertices that x must avoid are the images of the vertices already embedded (fewer than $v(H)$) and the Γ-neighbours of the images of its embedded H-neighbours; each such image lies in $\sigma'(A)$ or in W and so has Γ-degree at most D. Hence fewer than $2mD/(k+1) + v(H) \le N/2 < |W|$ vertices of W are excluded, and x can be embedded. □

Lemma 2.2 (Erdős, Faudree, Rousseau and Schelp [12, Lemma 2 and Corollary 3]; see also [5, Lemma 5.3])**.** *If G is connected and $G = G_1 \cup G_2$ with $|V(G_1)\cap V(G_2)| = 1$, then $R(G,H) \le R(G_1,H) + R(G_2,H)$ for every graph H without isolated vertices. Consequently, a connected graph is Ramsey size-linear if and only if all its blocks are.*

For disjoint unions one has $R(G_1 \cup G_2, H) \le R(G_1,H) + R(G_2,H) + v(G_1)$, so a graph is Ramsey size-linear if and only if all its blocks are, and it suffices to study 2-connected graphs.

3. Graphs with Few Triangles

The following lemma isolates the averaging argument of [5, Claim 4.4] as a general statement.

Lemma 3.1. *Let $\lambda \ge 0$ and let Γ be a graph on N vertices with $T(\Gamma) \le \lambda\, e(\Gamma)$. Let H be a graph with $m \ge 1$ edges and no isolated vertices. If $N \ge (624\lambda + 56)\, m$, then $\overline{\Gamma}$ contains H.*

Proof. If $e(\Gamma) = 0$ then $\overline{\Gamma}$ is complete and contains H, as $N \ge 2m \ge v(H)$. So let $e = e(\Gamma) > 0$ and let $d = 2e/N$ be the average degree. For a vertex x let

$$N'(x) = \{y \in N_\Gamma(x) : d_\Gamma(y) \le d_\Gamma(x)\}, \qquad d'(x) = |N'(x)|, \qquad t'(x) = e\big(\Gamma[N'(x)]\big).$$

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For every edge xy we have $y \in N'(x)$ or $x \in N'(y)$, so $\sum_x d'(x) \geq e$. Every edge of $\Gamma[N'(x)]$ forms a triangle with x, so $\sum_x t'(x) \leq 3T(\Gamma) \leq 3\lambda e$. Suppose first that $\lambda > 0$ and set

$$\Phi(x) = d'(x) - \frac{d_\Gamma(x) + 2d}{13} - \frac{t'(x)}{6\lambda}.$$

Since $\sum_x d_\Gamma(x) = Nd = 2e$, we get $\sum_x \Phi(x) \geq e - \frac{6e}{13} - \frac{e}{2} = \frac{e}{26} > 0$. Fix x with $\Phi(x) > 0$. Then $d'(x) > 0$ and

$$\text{(a) } d_\Gamma(x) < 13\, d'(x), \qquad \text{(b) } 2d < 13\, d'(x), \qquad \text{(c) } 2t'(x)/d'(x) < 12\lambda.$$

By (c) the graph $\Gamma[N'(x)]$ has average degree less than 12λ, so by the Caro–Wei (or Turán) bound it contains an independent set I with $|I| \geq d'(x)/(12\lambda + 1) \geq 1$. If $\lambda = 0$ then Γ is triangle-free, every $N'(x)$ is independent, and the same conclusion holds with $I = N'(x)$, using $\Phi(x) = d'(x) - (d_\Gamma(x) + 2d)/13$ and $\sum_x \Phi(x) \geq e - \frac{6}{13}e = \frac{7}{13}\, e > 0$.

Every $y \in I$ satisfies $d_\Gamma(y) \leq d_\Gamma(x) < 13\, d'(x) \leq 13(12\lambda + 1)|I|$, and also $2d < 13(12\lambda + 1)|I|$. If $|I| \geq v(H)$, then H embeds into $\overline{\Gamma}[I]$, which is complete. Otherwise put $k = |I|$, let A be a set of k vertices of largest degree in H, and let $\sigma' \colon A \to I$ be any bijection; it embeds $H[A]$ into $\overline{\Gamma}$ because I is independent in Γ. The maximum in Lemma 2.1 is at most $13(12\lambda + 1)k$, so

$$\frac{4m}{k+1} \cdot 13(12\lambda + 1)k + 2v(H) \;<\; 52(12\lambda + 1)m + 4m = (624\lambda + 56)\, m \;\leq N,$$

using $v(H) \leq 2m$. Lemma 2.1 extends σ' to an embedding of H into $\overline{\Gamma}$. □

Remark 3.2. *Lemma 3.1 yields the following sufficient condition: if there is λ_G such that every G-free graph Γ satisfies $T(\Gamma) \leq \lambda_G\, e(\Gamma)$, then $R(G, H) \leq (624\lambda_G + 56)\, e(H)$ and G is Ramsey size-linear. For $G = K_3$ it gives $R(K_3, H) \leq 56\, e(H)$, much weaker than the sharp bound $2e(H) + 1$ [14,18]; no attempt has been made to optimise the constants.*

4. Triangles in Graphs Without a Given 2-Tree

Lemma 4.1. *Let H be a 2-tree on $h \geq 3$ vertices. Then every H-free graph Γ satisfies $T(\Gamma) \leq (h - 3)\, e(\Gamma)$.*

Proof. The elimination process below is the one of [5, Section 4], and the embedding step follows [5, Proposition 1.1]. Let $\mathcal{T}$ be the set of triangles of Γ. As long as some edge of Γ lies in at least one and at most $h - 3$ triangles of $\mathcal{T}$, choose such an edge and delete from $\mathcal{T}$ all triangles containing it. After this step the chosen edge lies in no triangle of $\mathcal{T}$, and $\mathcal{T}$ only shrinks, so each edge is chosen at most once. Hence at most $(h - 3)\, e(\Gamma)$ triangles are deleted in total. Let $\mathcal{T}_0$ be the final set. By construction, every edge lying in some triangle of $\mathcal{T}_0$ lies in at least $h - 2$ triangles of $\mathcal{T}_0$.

We claim $\mathcal{T}_0 = \emptyset$, which gives $T(\Gamma) \leq (h - 3)\, e(\Gamma)$. Suppose not. Order $V(H)$ as $v_1, \ldots, v_h$ so that $v_1 v_2 v_3$ is a triangle and each v_i with $i \geq 4$ is adjacent to exactly two earlier vertices v_a, v_b, which are adjacent. Map v_1, v_2, v_3 onto the vertices of a triangle of $\mathcal{T}_0$. Inductively, assume $v_1, \ldots, v_{i-1}$ are embedded injectively so that every edge of $H[\{v_1, \ldots, v_{i-1}\}]$ is mapped to an edge lying in a triangle of $\mathcal{T}_0$. The image of $v_a v_b$ then lies in at least $h - 2$ triangles of $\mathcal{T}_0$, whose third vertices are distinct, while only $i - 3 \leq h - 3$ vertices other than the images of v_a, v_b are already used. Choose an unused third vertex z and map v_i to z; the two new edges belong to a triangle of $\mathcal{T}_0$, so the invariant is maintained. At the end we obtain a copy of H in Γ, a contradiction. □

5. Proofs of the Main Results

Proof of Theorem 1.2. If $v(G) \leq 2$ then $G \subseteq K_3$; otherwise, by (T1) and (T2), G is a spanning subgraph of a 2-tree G^* on h vertices. In both cases $G \subseteq G^*$ for a 2-tree G^* on h vertices. Let Γ be a graph on $N \geq (624h - 1816)\, m$ vertices that does not contain G. Then Γ is G^*-free, so $T(\Gamma) \leq (h - 3)\, e(\Gamma)$

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by Lemma 4.1, and Lemma 3.1 with $\lambda = h - 3$ shows that $\overline{\Gamma}$ contains H, because $624(h-3) + 56 = 624h - 1816$. □

Proof of Corollary 1.3. (b) follows from fact (ii), since $e(K_4) = 2 \cdot 4 - 2$. For (a) and (c), note first that if a connected graph G contains a connected subgraph S, then $e(G) - v(G) \geq e(S) - v(S)$: G arises from S by adding edges and by adding vertices each joined by at least one edge. If G has no K_4-minor, it is Ramsey size-linear by Theorem 1.2. Otherwise, by (T3), G contains a subdivision S of K_4, and $e(S) - v(S) = 2 \geq e(G) - v(G)$. Hence equality holds, which means that every edge of G outside S is a bridge: S is the only block of G with a cycle, and the other blocks are single edges. Now $S \neq K_4$ in case (a), and $S \ncong K_4^*$ unless G contains K_4^*; the only subdivisions of K_4 with fewer than six vertices are K_4 and K_4^*. So in case (a), S has at least six vertices and is Ramsey size-linear by [5]; single edges are Ramsey size-linear, and Lemma 2.2 shows that G is Ramsey size-linear. In case (c), the 2-connected graph K_4^* lies in a block of G, hence in S; since S is 2-connected and $e(S) - v(S) = e(K_4^*) - v(K_4^*)$, the argument above (applied to $K_4^* \subseteq S$) shows $S = K_4^*$. Thus G is obtained from K_4^* by attaching bridges, and it is Ramsey size-linear if K_4^* is, by Lemma 2.2; the converse holds because $K_4^* \subseteq G$. □

Proof of Proposition 1.4. Let c be the hub of W_k and $u_1u_2 \cdots u_ku_1$ its rim.

(a) W_k has $k+1$ vertices and $2k = 2(k+1) - 2$ edges, so it is not Ramsey size-linear by fact (ii). Deleting the rim edge u_ku_1 leaves the fan $K_1 + P_k$, which is Ramsey size-linear by fact (iii). Every proper subgraph of W_k is contained in $W_k - f$ for some edge f, and for $k \geq 4$ the automorphisms of W_k act transitively on the spokes and on the rim edges. Hence all proper subgraphs of W_k are Ramsey size-linear if and only if W_k^- is.

(b) Deleting the spoke cu_4 from W_4 leaves the complete graph on $\{c, u_1, u_2, u_3\}$ minus the edge u_1u_3, together with the path $u_1u_4u_3$; this is K_4 with the edge u_1u_3 subdivided by u_4.

(c) W_k^- has $k+1$ vertices and $2k - 1 = 2(k+1) - 3$ edges. Let S be a proper subset of its vertices with $|S| = j \geq 2$. If $c \notin S$, then S spans at most j edges of the rim cycle if $j = k$, and at most $j-1$ edges otherwise; in both cases at most $2j - 3$ edges when $j \geq 3$, and at most 1 edge when $j = 2$. If $c \in S$, then $S \setminus \{c\}$ is a proper subset of the rim, so it spans a linear forest with at most $j-2$ edges, and together with at most $j-1$ spokes S spans at most $2j-3$ edges. Thus $m_2(W_k^-) = 2$, and the second statement follows from (a).

(d) For $k = 4$: every proper subgraph of K_4^* is contained in $K_4^* - f$ for an edge f, which is connected (as K_4^* is 2-connected) and has $6 = v + 1$ edges, so it is Ramsey size-linear by fact (i). For $k = 5$, let $W_5^- = W_5 - cu_1$. If f is a rim edge, then deleting c from $W_5^- - f$ leaves a path, so $W_5^- - f$ is a subgraph of $K_1 + P_5$ and is Ramsey size-linear by fact (iii). If f is a spoke, then in $W_5^- - f$ the hub is adjacent to exactly three vertices of the rim C_5, so $W_5^- - f$ is a subdivision of K_4 with six vertices (branch vertices: c and its three neighbours), which is Ramsey size-linear by [5]. Hence every proper subgraph of W_k^- is Ramsey size-linear for $k \in \{4, 5\}$.

Now fix $k \in \{4, 5\}$. If W_k^- is not Ramsey size-linear, then it is minimally non-Ramsey size-linear, since all its proper subgraphs are Ramsey size-linear. If W_k^- is Ramsey size-linear, then W_k is minimally non-Ramsey size-linear by (a). In either case the pair $\{W_k^-, W_k\}$ contains a minimally non-Ramsey size-linear graph; the two pairs are disjoint and do not contain K_4, and $W_4^- \cong K_4^*$ by (b). □

6. More from the Proof of Bradač, Gishboliner and Sudakov

The proof of [5, Theorem 4] produces more than its statement asserts. We make this explicit, since it settles many small cases (Section 7). Throughout this section F is a graph with $m \geq 1$ edges and no isolated vertices, Γ is a graph on $N \geq Cm$ vertices whose complement contains no copy of F, and $C \gg C_1 \gg C_0$ are the constants of [5, Section 4]. Let G_+ be the set of edges lying in a triangle that survives the triangle-elimination process of Section 4 (run with threshold C_0), so that

$$\text{every edge of } G_+ \text{ lies in at least } C_0 \text{ triangles all of whose edges are in } G_+. \tag{1}$$

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Lemma 6.1. *Let $B \subseteq \Gamma$ with $E(B) \subseteq G_+$, and let K be obtained from B by repeatedly adding a new vertex joined to both ends of an existing edge. If $v(K) < C_0$, then Γ contains a copy of K extending the copy of B.*

Proof. Add the vertices of $K \setminus B$ in the order of the construction. When a vertex is to be joined to the ends of an edge e, the copy of e lies in G_+, so by (1) it lies in at least $C_0 > v(K)$ triangles with all edges in G_+; their apexes are distinct, so one of them is unused. The two new edges lie in G_+, so the invariant is maintained. □

Write $B_1(t_1, t_2)$ for the graph with vertices $v, u_1, u_2, w, z_1, \dots, z_{t_1}, y_1, \dots, y_{t_2}$ and edges $vu_1, vu_2, vw, u_1u_2, u_1z_i, wz_i$ $(i \le t_1)$ and u_2y_j, wy_j $(j \le t_2)$. For a multigraph Q on a set A of size s, write $B_2(s, Q)$ for the graph with vertices v_1, v_2, A and one vertex x_f for each $f \in E(Q)$, and edges v_1a, v_2a $(a \in A)$ together with the two edges joining x_f to the ends of f. Finally, let c_1, c_2, c_3 be K_4 on $\{x, y, z, w\}$, the graph K_4^* obtained from it by subdividing xy once, and the graph H_3 obtained from it by subdividing xy twice. In c_1 the edge xy is called *dead*, in c_3 the edge of the subdivision path incident to y is called dead, and all other edges are *live*; in c_2 all edges are live.

Proposition 6.2. *With Γ as above and any fixed t_1, t_2, s and Q:*

(a) *Γ contains a copy of $B_1(t_1, t_2)$ all of whose edges lie in G_+;*
(b) *Γ contains a copy of $B_2(s, Q)$ all of whose edges lie in G_+;*
(c) *Γ contains a copy of c_1, of c_2 or of c_3 in which all live edges lie in G_+.*

Proof. Part (a) is what Case 1 of the proof of [5, Theorem 4] produces: a vertex v, an edge u_1u_2 of G_+ with $u_1, u_2 \in N_+(v)$, and $w \in N_+(v)$ such that u_1, w and u_2, w each have more than C_0 common neighbours in G_+; choosing $t_1 + t_2$ distinct such common neighbours gives the copy. Part (b) is Case 2: it produces v_1, v_2 and a set A' of common G_+-neighbours of v_1 and v_2 such that every pair in A' has more than C_0 common neighbours in G_+; there $|A'| \ge 3$, and the same averaging gives $|A'| \ge s$ once C_1 is large in terms of s and C_0. Choosing a distinct such common neighbour for each edge of Q gives the copy. Part (c) is the case analysis at the end of Case 3, whose possible outcomes are a copy of K_4 in which all edges except the one joining the two tips of a diamond lie in G_+, a copy of K_4^* in G_+, or a copy of H_3 in which all edges except one end edge of the subdivision path lie in G_+. These arguments use only [5, Claims 4.1 and 4.4], which hold for every Γ as above, and not the graph whose copy is being sought; hence they apply simultaneously. □

Corollary 6.3. *Every subgraph of a graph obtained from some $B_1(t_1, t_2)$ or some $B_2(s, Q)$ by 2-tree steps is Ramsey size-linear. So is every graph which, for each $i \in \{1, 2, 3\}$, is a subgraph of a graph obtained from c_i by 2-tree steps along live edges and along edges created by earlier steps.*

Proof. Combine Proposition 6.2 with Lemma 6.1 (applied to the live edges), taking C_0 larger than the order of the graph sought, and use that subgraphs of Ramsey size-linear graphs are Ramsey size-linear. □

Corollary 6.3 contains [5, Theorem 4], which is how that theorem is proved: a subdivision of K_4 on at least six vertices is a subdivision of H_1, H_2 or H_3, and a subdivided edge xy is a subgraph of a fan with hub y glued along xy by 2-tree steps.

7. The Remaining Small Cases

By Lemma 2.2 it suffices to consider 2-connected graphs. The next proposition describes the minimal graphs not covered by Theorem 1.2 and [5]. An *S-ear* of a subgraph S is a path of length at least one whose two (distinct) ends lie in S and whose inner vertices and edges do not belong to S.

Proposition 7.1. *Let G be a 2-connected graph with $m_2(G) \le 2$ which has a K_4-minor and is not a subdivision of K_4 with at least six vertices. Then G contains K_4^*, or G contains a subgraph $S \cup P$, where S is a subdivision of*

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K_4 with at least six vertices and P is an S-ear. Every such $S \cup P$ is 2-connected, has a K_4-minor and satisfies $e(S \cup P) = v(S \cup P) + 3$.

Proof. By (T3), G contains a subdivision S of K_4. We have $S \not\cong K_4$ because $m_2(K_4) > 2$. If $S \cong K_4^*$ we are done; otherwise $v(S) \geq 6$, and $S \neq G$ by assumption. If some edge of G outside S joins two vertices of S, it is an S-ear. Otherwise G has a vertex outside S; let C be a component of $G - V(S)$. Since G is 2-connected, C has neighbours in at least two vertices $x \neq y$ of S, and a path from x to y through C is an S-ear. The last sentence is clear, since adding an ear increases $e - v$ by one. □

Consequently, if only Theorem 1.2 and [5] are used, every minimal unresolved graph other than K_4^* has the form $S \cup P$ and has exactly $v + 3$ edges; Turán-type results can create exceptions, such as Q_3 below. By the path-extension result of [2], $S \cup P$ is Ramsey size-linear whenever P has length at least $d_S(x, y) + 3$, where x, y are its ends, so only short ears matter.

Computation.

We enumerated all 2-connected graphs G with at most eight vertices and $m_2(G) \leq 2$. For $v(G) \leq 7$ we used the atlas of all graphs on at most seven vertices, as implemented in NetworkX. For $v(G) = 8$ we used that such a graph has minimum degree 2 or 3 (as $e(G) \leq 2v(G) - 3$), so it is obtained from a graph on seven vertices with $m_2 \leq 2$ by adding a vertex of degree 2 or 3; isomorphism classes were separated by Weisfeiler–Leman hashes followed by exact isomorphism tests. An independent enumeration, adding a vertex with an arbitrary neighbourhood to every graph on seven vertices, gave the same number, 1224. Each graph was classified by the following sufficient conditions for Ramsey size-linearity, all taken from the literature except (C3): (C1) $e(G) \leq v(G) + 1$ [12, Theorem 4]; (C2) $G - v$ is a forest for some vertex v, that is, G is a subgraph of some $K_1 + T$ [12, Theorem 3]; (C3) treewidth at most two (Theorem 1.2); (C4) G is a subdivision of K_4 with at least six vertices [5]; (C5) G is bipartite and all vertices of one colour class have degree at most two, whence its Turán number is $O(n^{3/2})$ [1,13]; (C6) G is a subgraph of $K_{3,3} - e$ or of $Q_3 - e$, whose Turán numbers are stated to be $O(n^{3/2})$ in [12, p. 395] (for (C5) and (C6) we use [12, Theorem 5]); (C7) G arises from a graph whose blocks satisfy one of the previous conditions by adding a path of length at least $d(x, y) + 3$ between two of its vertices x, y [2, Corollary 2]; (C8) G is a subgraph of a graph satisfying one of (C1), (C2) or (C4)–(C7); (C9) G is a subgraph of a 2-tree extension of some $B_1(t_1, t_2)$ or $B_2(s, Q)$ (Corollary 6.3); (C10) G lies in the extension class of each of c_1, c_2, c_3 (Corollary 6.3). Treewidth was tested by series–parallel reduction. A graph is *unresolved* if none of (C1)–(C8) applies, and *minimal unresolved* if moreover all its proper subgraphs are resolved. Conditions (C9) and (C10) were tested by a reduction: a graph lies in the extension class of a base B if and only if it can be reduced to a subgraph of B by repeatedly deleting a vertex of degree at most two and, when its degree is two, adding the edge between its two neighbours (an edge which must be live). Table 1 separates the contributions of (C3) and of (C9)–(C10).

Table 1. 2-connected graphs with $m_2 \leq 2$ on $v \leq 8$ vertices. "Earlier results" means (C1), (C2) and (C4)–(C8); the next two columns count the graphs resolved in addition by Theorem 1.2 and then by Corollary 6.3.

v	2-conn., $m_2 \leq 2$	earlier results	+ Theorem 1.2	+ Corollary 6.3	unresolved	minimal unres.
3	1	1	0	0	0	0
4	2	2	0	0	0	0
5	6	5	0	0	1	1
6	26	15	3	0	8	3
7	144	39	18	14	73	10
8	1224	115	127	139	843	35

On five vertices Theorem 1.2 adds nothing: Erdős et al. had already settled every graph on at most five vertices except K_4^* [12, Section 3]. On six vertices it settles exactly three graphs not covered before, namely the 3-sun, the strip of four triangles and the prism minus a matching edge.

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After all ten conditions, 49 minimal unresolved graphs remain: K_4^*; on six vertices W_5^-, the triangular prism and $K_{3,3}$; and 10 and 35 graphs on seven and eight vertices, listed in Appendix A. They are governed by few cores. Call a graph a *core* if it has minimum degree at least three, and reduce a graph by repeatedly deleting a vertex of degree at most two, adding the edge between its two neighbours when the degree is two; each step undoes a 2-tree step, possibly after deleting edges.

Proposition 7.2. *Each of the* 49 *minimal unresolved graphs on at most eight vertices reduces to exactly one of the following eleven cores (listed with the number of graphs reducing to it):*

- *six cores which are* not *Ramsey size-linear, having* $2v-2$ *edges:* K_4 (5 *graphs), the wheel* W_4 (22 *graphs), and four graphs on six vertices with ten edges* (1 *graph each);*
- *five cores with* $m_2 = 2$ *and minimum degree three: the triangular prism* (10 *graphs),* $K_{3,3}$ (5)*, the cube* Q_3*, a graph on seven vertices with eleven edges, and a cubic graph on eight vertices* (1 *graph each).*

In particular K_4^* and W_5^- are the smallest members of the first group: they are obtained from the non-Ramsey size-linear graphs K_4 and W_4 by subdividing one edge. The second group contains the graphs $K_{3,3}$ and Q_3 of [12, Question 2]. No graph of either group lies in one of the extension classes of Corollary 6.3, so settling any of them requires a new argument.

Density-critical graphs.

Call G *density-critical* if $e(G) = 2v(G) - 2$ and every proper subset S of $V(G)$ with $|S| \geq 2$ spans at most $2|S| - 3$ edges. By fact (ii) such a graph is not Ramsey size-linear, and every minimal non-Ramsey size-linear graph G with $m_2(G) > 2$ is density-critical. A density-critical graph has minimum degree exactly three. There are 1, 1, 4, 15 and 109 of them on $4, \dots, 8$ vertices. For each, we determined the unresolved 2-connected proper subgraphs (every proper subgraph lies in some $G - f$, and every 2-connected subgraph of $G - f$ lies in a block of $G - f$). The argument of Proposition 1.4(d) applies to G exactly when all of them are minimal unresolved; this happens only for K_4 (with no unresolved subgraphs), for W_4 (with K_4^*) and for W_5 (with W_5^-).

8. Concluding Remarks

A programme.

By Proposition 7.2, settling Question 1.1 for all graphs on at most eight vertices amounts to settling the families attached to eleven cores. For the six cores that are not Ramsey size-linear, the question is whether subdividing edges of such a core, just enough to bring m_2 down to 2, always yields Ramsey size-linear graphs; K_4^* and W_5^- are the smallest instances. For the five cores with $m_2 = 2$ the question concerns the cores themselves, and for $K_{3,3}$ it requires $R(K_{3,3}, K_n) = O(n^2)$. Testing each of the 49 graphs against the three configurations of Proposition 6.2(c) separately, with the dead edges declared live or not, shows how far the method is from each of them. For 45 graphs (all families except that of K_4) none of the three extension classes applies even when every edge is live, so a new structure is needed. For K_4^* and three other graphs with core K_4 the obstruction is the configuration c_3, even with all its edges live. For one graph on eight vertices, H_3 with a triangle glued along each end edge of its subdivision path (graph6 `GJyGcc`), the only obstruction is the dead edge of c_3; in the notation of [5, Case 3], it is present in Γ unless the set U of tips induces a triangle-free graph all of whose edges have codegree at most seven.

Short cycles in links.

For a vertex a let $L_a = G_+[N_+(a)]$ be its link in G_+; by (1) it has minimum degree at least C_0. If L_a contains a cycle of length $k \in \{3, 4, 5\}$, then a together with this cycle spans a wheel $W_k = K_1 + C_k$ (with $W_3 = K_4$) all of whose edges lie in G_+, and Lemma 6.1 applies to it. Testing the 49 minimal unresolved graphs against the extension classes of W_3, W_4, W_5 gives the following.

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- If some link contains a cycle of length at most five, whichever length it is, then Γ contains `GJyGcc`. Equivalently, if Γ does not contain `GJyGcc`, then every link has girth at least six; this can also be checked by hand (a C_3 in L_a gives a K_4 in G_+; a C_4 gives two tips of a diamond in G_+ with a third common neighbour; a C_5 gives the required path between the tips).
- If some link contains a C_4 or a C_5, then Γ contains W_5^-, 19 further graphs with core W_4, and `GJyGcc`.
- If some link contains a C_3 or a C_4, then Γ contains K_4^*, `GJyGcc` and two further graphs with core K_4.

In the quotients of $\tilde{A}_2$-buildings of Remark 8.3 every link is the incidence graph of a projective plane and has girth six, so none of these hypotheses follows from (1) alone; a proof would have to use [5, Claim 4.1]. For K_4^* the third item is little more than a reformulation, since every neighbourhood in a K_4^*-free graph is C_4-free; for W_5^- and `GJyGcc` it is a genuine reduction.

Question 8.1. *Let Γ be as in Section 6. Must some link L_a contain a cycle of length at most five? Must some link contain a cycle of length four or five?*

A positive answer to the second part would show that W_5^- is Ramsey size-linear and hence, by Proposition 1.4, that W_5 is an explicit minimally non-Ramsey size-linear graph.

The graph W_5^-.

By Proposition 1.4, deciding whether W_5^- is Ramsey size-linear would, in either direction, produce an explicit minimally non-Ramsey size-linear graph other than K_4: W_5 if the answer is positive, and W_5^- itself (together with a negative answer to Question 1.1) otherwise. The same is true for K_4^*, but K_4^* has resisted several attempts [2,5,12], and W_5^-, whose rim is longer, may be more accessible. Natural intermediate steps, in analogy with [5], are the following.

Question 8.2. *Is $R(W_5^-, F) = O(e(F))$ for every bipartite graph F without isolated vertices? Is $R(W_5^-, K_n) = O(n^2)$?*

The general bounds of [5] give $R(W_5^-, K_n) = O(n^3)$.

Remark 8.3 (Limits of the method; sketch)**.** *The only property of G used in the proof of Theorem 1.2 is that G-free graphs have $O(e)$ triangles. This fails for K_4 (consider complete tripartite graphs) and, less obviously, for W_5^-. Let Δ be the Bruhat–Tits building of $\mathrm{PGL}_3(\mathbb{F}_q((t)))$. Its vertex links are point–line incidence graphs of the projective plane $\mathrm{PG}(2,q)$ [17], so every edge of its 1-skeleton lies in exactly $q+1$ triangles and every neighbourhood induces a bipartite graph of girth six. Suppose Δ contained W_5^-, with hub c, rim path $u_1u_2u_3u_4$ in the neighbourhood of c, and a further common neighbour $w \neq c$ of u_1 and u_4. By girth six, u_1 and u_4 are at distance three in the link of c, i.e., opposite. In the CAT(0) metric of Δ [6, Ch. II.10A], where all triangles are equilateral with side one, the path u_1cu_4 is then a geodesic of length 2, and u_1wu_4 would be a second geodesic of the same length between the same points, contradicting the uniqueness of geodesics. Cocompact lattices in $\mathrm{PGL}_3(\mathbb{F}_q((t)))$ exist (see e.g., [8]) and are residually finite, being finitely generated linear groups; hence Δ has finite quotients in which every ball of radius two is isomorphic to a ball of Δ. Their 1-skeleta are finite W_5^--free graphs with $T/e = (q+1)/3$. Thus proving that W_5^- is Ramsey size-linear requires more than triangle counting.*

Question 8.4. *For which graphs G is there a constant λ_G such that every G-free graph Γ satisfies $T(\Gamma) \leq \lambda_G\, e(\Gamma)$? By Lemma 4.1 this holds when G has treewidth at most two. Does it fail for every graph with a K_4-minor?*

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Constants.

The constant $624h$ in Theorem 1.2 is far from optimal; for $G = K_3$ the truth is $2e(H) + 1$. It would be interesting to determine the order of magnitude of the best constant for 2-trees on h vertices.

Supplementary Materials: The following supporting information can be downloaded at the website of this paper posted on Preprints.org. **File S1** (`supplementary_material_S1.zip`) contains: the Python scripts of the computer search of Section 7 and of the tests of Section 8 (`rsl_census.py`, `bgs_extract.py`, `case3.py`, `census_final.py`, `cores.py`, `families_final.py`, `diagnostico.py`, `wheels.py` and `export_results.py`); the file `census_results.csv`, which lists all 1403 graphs of the census with the conditions (C1)–(C10) that they satisfy; the expected output of every script; and a file `README.txt` with instructions.

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Data Availability Statement: All data of this paper are contained in or produced by Supplementary File S1 (see Supplementary materials). The scripts require Python 3 and NetworkX, and reproduce Table 1, Proposition 7.2, the analysis of density-critical graphs, the tests of Section 8 and the lists of Appendix A.

Conflicts of Interest: The author declares no conflicts of interest.

Use of artificial intelligence tools: This work was carried out with the assistance of the large language model Claude (Anthropic). The tool was used for literature search, for proposing and checking proofs, for writing and running the Python code of the computer search in Section 7 and of the tests in Section 8, and for drafting the text of the manuscript. The author independently checked the proofs of Lemmas 3.1 and 4.1 and is responsible for the content of the manuscript.

Appendix A. Minimal Unresolved Graphs on Seven and Eight Vertices

The graphs are given in graph6 format, as produced by NetworkX.

7 vertices (10 graphs).

```
Fhf_g   FheoW   FhEKw   FhEMg   FhdQW   FMjDO
FhEJW   FMohg   FlBHo   FhUk_
```

8 vertices (35 graphs).

```
G[EoIS  GxUA?k  GxUA?[  GeoJ@g  Gewa?k  GxSICK
GxSQCK  GEtBDG  GjCHSg  GjCHSK  G`DbKW  GhogKc
GFwcAK  GxOYCc  GxOYCS  GxSAKW  GhT@Kg  Ggogl_
GxOWSK  GHt@Kg  GHFEKW  GhFKBC  GhFK@c  GxU?Gs
GHhGm_  GFw_MC  GhEKcc  GhdWbC  GhUgMC  GhUgLC
GhdYCS  GJyGcc  GJyGcK  GMs`KK  GhELQg
```

References

1. N. Alon, M. Krivelevich and B. Sudakov, Turán numbers of bipartite graphs and related Ramsey-type questions, *Combin. Probab. Comput.* 12 (2003), 477–494.
2. P. N. Balister, R. H. Schelp and M. Simonovits, A note on Ramsey size-linear graphs, *J. Graph Theory* 39 (2002), 1–5.
3. T. F. Bloom, Erdős Problems, https://www.erdosproblems.com, Problems #79 and #566–#570. Accessed September 2026.
4. H. L. Bodlaender, A partial k-arboretum of graphs with bounded treewidth, *Theoret. Comput. Sci.* 209 (1998), 1–45.
5. D. Bradač, L. Gishboliner and B. Sudakov, On Ramsey size-linear graphs and related questions, *SIAM J. Discrete Math.* 38 (2024), 225–242.
6. M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.
7. S. Cambie, A. Freschi, P. Morawski, K. Petrova and A. Pokrovskiy, Ramsey number of a cycle versus a graph of a given size, arXiv:2601.10238, 2026.
8. D. I. Cartwright and T. Steger, A family of $\tilde{A}_n$-groups, *Israel J. Math.* 103 (1998), 125–140.

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9. R. Diestel, *Graph Theory*, 5th edition, Springer, Berlin, 2017.
10. R. J. Duffin, Topology of series-parallel networks, *J. Math. Anal. Appl.* 10 (1965), 303–318.
11. P. Erdős, Some of my favourite problems in number theory, combinatorics, and geometry, *Resenhas* 2 (1995), 165–186.
12. P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Ramsey size linear graphs, *Combin. Probab. Comput.* 2 (1993), 389–399.
13. Z. Füredi, On a Turán type problem of Erdős, *Combinatorica* 11 (1991), 75–79.
14. W. Goddard and D. J. Kleitman, An upper bound for the Ramsey numbers $r(K_3, G)$, *Discrete Math.* 125 (1994), 177–182.
15. E. K. Hng, M. Ji and A. Lamaison, Ramsey size linear and generalization, arXiv:2603.25453, 2026.
16. O. Janzer, Disproof of a conjecture of Erdős and Simonovits on the Turán number of graphs with minimum degree 3, arXiv:2109.06110, 2021.
17. M. Ronan, *Lectures on Buildings*, Academic Press, Boston, 1989.
18. A. F. Sidorenko, The Ramsey number of an N-edge graph versus triangle is at most $2N + 1$, *J. Combin. Theory Ser. B* 58 (1993), 185–196.
19. J. Spencer, Asymptotic lower bounds for Ramsey functions, *Discrete Math.* 20 (1977/78), 69–76.
20. Y. Wigderson, Infinitely many minimally non-Ramsey size-linear graphs, *European J. Combin.* 128 (2025), 104175.

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