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Moses Rahnama *

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Born's Rule from Record-Formation Constraints: A Conditional Uniqueness Theorem

Moses Rahnama

Mina Analytics, New York, United States; moses@minaanalytics.com

Abstract

We prove a conditional uniqueness theorem for projective measurements on pure states: given the L^2 Hilbert space structure of quantum mechanics, the Born rule $P = |\psi|^2$ is the only outcome-local probability assignment compatible with five operational postulates (outcome-locality, normalization, phase independence of the classical record, tensor product factorization, and continuity), with interference consistency used only as a post-hoc check. Physically, the theorem addresses a boundary problem: amplitudes support interference and cancellation before measurement, whereas a stabilized classical record cannot retain the full relative-phase distinction structure in accessible form. Irreversible record formation motivates this phase-insensitivity requirement, although no thermodynamic quantity enters the formal proof. Mathematically, phase independence reduces the map to a function of modulus, factorization and continuity force a power law, and consistency with L^2 normalization fixes the exponent to 2. Within this framework, the squared modulus is the unique classicalization map from phase-sensitive amplitudes to accessible record weights. The result is complementary in physical motivation to Gleason-type derivations but narrower in scope: it is confined to the pure-state projective setting and does not derive the general trace rule for mixed states and POVMs.

Keywords: born rule; quantum measurement; record formation; Landauer principle; thermodynamic constraints; quantum probability; Gleason's theorem; phase information loss; Hilbert space; decoherence; quantum foundations; information thermodynamics; Cauchy functional equation; tensor product factorization; quantum amplitudes

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1. Introduction

The Born rule, that the probability of obtaining outcome i from a quantum measurement is $P_i = |\alpha_i|^2$ where α_i is the amplitude, is one of the central postulates of quantum mechanics. Despite its empirical success, its foundational status remains debated:

- Is it a separate axiom, or can it be derived?
- Why specifically squared, rather than some other function?
- What is the physical mechanism that produces probabilities from amplitudes?

Previous derivations include:

- **Gleason's theorem** [1]: Shows that any non-contextual probability measure on a Hilbert space of dimension ≥ 3 must take the form $|\langle\phi|\psi\rangle|^2$
- **Deutsch-Wallace** [2]: Derives Born probabilities from rational decision theory in the Everett interpretation
- **Zurek's envariance** [3]: Derives the rule from environment-assisted invariance

We provide a complementary **conditional uniqueness theorem** for pure-state projective measurements: given the standard Hilbert space structure and five operational postulates motivated by record formation, the Born rule is the unique outcome-local probability assignment compatible with a phase-insensitive classical record.

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We work within standard complex Hilbert-space kinematics, equivalently the usual 2-norm geometry of quantum state space. This structure is not derived here. It is the conventional framework of quantum theory, independently supported by the observed second-order interference structure of experiments [21,22], by unitary representations of continuous symmetries and reversible dynamics [25,26], by the fact that the 2-norm is the unique L^p norm arising from an inner product (the Jordan-von Neumann characterization [24]), and by operational reconstruction programs that recover Hilbert-space state spaces from general physical principles [13,14]. Our claim is therefore conditional: within this standard kinematic setting, the additional constraints developed below restrict the admissible probability rule.

Mathematically, the proof follows a standard multiplicative-functional-equation route; the intended contribution is the explicit axiom set and its physical motivation via record formation, not a new proof technique. Physically, the theorem formalizes a boundary problem: amplitudes support interference and cancellation before measurement, whereas a stabilized classical record must carry nonnegative, exhaustive outcome weights. The postulates below encode the claim that record formation is the irreversible step at which the amplitude description is translated into a classical record description.

2. From Amplitudes to Records

2.1. Quantum Amplitudes and Phase

Consider a quantum system in superposition:

$$|\psi\rangle = \sum_i \alpha_i |i\rangle, \quad \alpha_i = |\alpha_i| e^{i\phi_i} \tag{1}$$

Each amplitude α_i contains:

- **Magnitude** $|\alpha_i|$: The "weight" of pathway i
- **Phase** ϕ_i: Enables interference between pathways

The phase information is responsible for quantum interference:

$$\left| |\alpha_1| e^{i\phi_1} + |\alpha_2| e^{i\phi_2} \right|^2 = |\alpha_1|^2 + |\alpha_2|^2 + 2|\alpha_1||\alpha_2| \cos(\phi_1 - \phi_2) \tag{2}$$

2.2. Classical Records and Phase Information Loss

Before record formation, relative phase is physically active: branches with different phase relations can reinforce or cancel, and in a two-path interferometer nonzero pathway contributions can still yield a dark fringe. A classical measurement record, by contrast, is not a second copy of that full interference-capable description. In the record basis it does not exhibit interference between the recorded outcome branches. Therefore, creating such a record requires that relative-phase information is not retained in the record accessible to observers.

Once phase information is not accessible from the record, the map from the quantum state to the record is not reversible. The record is thus a classical ledger of accessible outcomes, not a faithful retention of the full phase-sensitive distinction structure present in the amplitudes. In the operational regime analyzed in Ref. [8], such irreversibility is accompanied by entropy export to a thermal bath during record formation under explicit applicability conditions. In the present paper that thermodynamic picture motivates the phase-independence axiom below; it is not used as a formal input to the proof [4,5,7].

2.3. The Central Question

What rule $f : \mathbb{C} \to \mathbb{R}^+$ converts interference-capable amplitudes into classical record weights while respecting both quantum kinematics and the operational constraints of classical record formation?

3. Operational Postulates

We impose five derivational constraints and one consistency check. The derivational constraints that enter the proof are:

Definition 1 (Outcome-Locality)**.** *For a measurement in a fixed basis, the probability assigned to outcome i depends only on its own amplitude α_i. We therefore assume a universal map $f : \mathbb{C} \to \mathbb{R}^+$ such that*

$$P_i = f(\alpha_i). \tag{3}$$

Definition 2 (Normalization)**.** *For any normalized quantum state $|\psi\rangle = \sum_i \alpha_i |i\rangle$ with $\sum_i |\alpha_i|^2 = 1$, the outcome-local probabilities must sum to unity. Here the state normalization uses the Hilbert space inner product and is independent of the probability rule:*

$$\sum_i f(\alpha_i) = 1 \tag{4}$$

Outcome-locality note. This postulate excludes globally renormalized rules of the form $p_i = g(|\alpha_i|)/\sum_j g(|\alpha_j|)$, which achieve normalization by construction for any power g. The physical motivation is twofold. First, in the record-formation picture each pointer branch couples to the bath and stabilizes independently; no branch has access to the global partition function $\sum_j g(|\alpha_j|)$ at the moment its record is written. Second, a globally renormalized rule makes the probability of a local outcome depend on the dimension N of the measurement basis (through the normalizing sum), so that adding an outcome with zero amplitude changes the probabilities of the others. This is not, by itself, a signalling protocol; rather, it violates null-extension invariance and makes outcome probabilities depend on how an event is embedded in a larger measurement context, a form of basis-context dependence the present framework excludes [19]. Outcome-locality is therefore a substantive physical assumption, not a convenience; the theorem's scope is limited to probability maps satisfying it.
Scope note. This work assumes the standard Hilbert space structure, including L^2 normalization and unitary dynamics. The result is conditional on those kinematic assumptions. Given that structure and the record formation constraints stated here, the Born rule is the unique probability assignment compatible with both.

Definition 3 (Phase Information Loss in Record Formation)**.** *The stabilized classical record, in the record (pointer) basis, does not retain relative-phase information between outcome branches. The probability assigned to each outcome must therefore be independent of phase:*

$$f(\alpha e^{i\phi}) = f(\alpha) \quad \forall \phi \in [0, 2\pi) \tag{5}$$

Formally, this corresponds to standard pointer-basis dephasing of the record subsystem in the channel description; in this paper it is used as a motivating operational idealization rather than a microscopic derivation.

The following is not used in the proof but is checked for consistency after the derivation:

Definition 4 (Interference Consistency (Operational))**.** *For two-path preparations with controllable relative phase, the predicted statistics must depend on the phase and exhibit a fringe pattern with nonzero visibility. When the relative phase is randomized (or the paths are fully decohered), the rule must reduce to a classical mixture with no phase dependence. This definition is operational and does not assume a specific functional form for the interference term.*

The remaining derivational constraints are:

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Definition 5 (Tensor Product Factorization)**.** *For independent systems A and B in product state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ with amplitudes α_i^A and α_j^B, the joint amplitude is $\alpha_{ij} = \alpha_i^A \alpha_j^B$. The joint probability factorizes:*

$$P(i,j) = f(\alpha_i^A \alpha_j^B) = f(\alpha_i^A)\, f(\alpha_j^B) \tag{6}$$

This factorization assumes the same outcome-local probability map f applies universally across such systems and that independent preparations yield product states whose records factorize. Whether weaker composition axioms suffice is an open question.

Definition 6 (Continuity)**.** *The function f is continuous in α. Small changes in amplitude produce small changes in probability. This excludes pathological (non-measurable) solutions to the multiplicative functional equation in Step 4; all physically meaningful probability rules satisfy this condition.*

Applicability conditions. The thermodynamic discussion below motivates the record-formation picture but does not enter the proof. It applies in the operational regime where Landauer-type bounds are valid. Norton [17] questions unconditional applications of Landauer's principle, whereas Ladyman and Robertson [18] defend it under specified conditions. Our application is conditional on C1 to C6 (summarized below; see Ref. [8] for experimental diagnostics and falsification criteria). Ref. [8] decomposes measurement into three stages: (1) reversible premeasurement (unitary correlation, no Landauer cost), (2) irreversible record formation (boundary event; Landauer cost paid here), and (3) optional reset (separate erasure cost). The Landauer bound applies to Stage 2 under six operational conditions: a thermal bath is present (C1), a classical register is created (C2), the apparatus is cyclic (C3), irreversibility arises from bath coupling (C4), no unaccounted work reservoir is present (C5), and $I(X;Y)$ is the actual mutual information (C6). The derivation should be read as conditional on those assumptions.

4. Derivation of the Born Rule

Theorem 1 (Born Rule from Record-Formation Constraints)**.** *For projective measurements on pure states expanded in the measurement basis, the unique function $f : \mathbb{C} \to \mathbb{R}^+$ satisfying Outcome-Locality, Normalization, Phase Information Loss, Tensor Product Factorization, and Continuity is $f(\alpha) = |\alpha|^2$. Interference Consistency is a post-hoc consistency check rather than a derivational constraint. Extension to the general trace rule $p(i) = \mathrm{tr}(\rho\, E_i)$ for mixed states and POVMs is not addressed here.*

Proof. Step 1: By Phase Information Loss, $f(\alpha) = f(|\alpha|)$. Thus f depends only on the modulus. Define $g : \mathbb{R}^+ \to \mathbb{R}^+$ by $g(r) = f(re^{i\phi})$ for any ϕ.

Step 2: By Normalization, when $\sum_i |\alpha_i|^2 = 1$:

$$\sum_i g(|\alpha_i|) = 1 \tag{7}$$

This must hold for all points on the unit sphere in amplitude space.

Step 3: Consider two specific cases on the unit sphere:

- $|\alpha_1|^2 = 1, |\alpha_2|^2 = 0$: Then $g(1) + g(0) = 1$
- $|\alpha_1|^2 = |\alpha_2|^2 = 1/2$: Then $2g(1/\sqrt{2}) = 1$

For certainty to map to certainty: $g(1) = 1$, equivalently by applying Normalization to the trivial one-outcome case. Hence $g(0) = 0$.

Step 4: By Tensor Product Factorization:

$$g(r_1 \cdot r_2) = g(r_1) \cdot g(r_2) \tag{8}$$

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Combined with Continuity, this is Cauchy's multiplicative functional equation,[9] whose unique continuous solution on $\mathbb{R}^+$ is $g(r) = r^n$ for some $n \in \mathbb{R}$. Because g maps positive moduli to nonnegative probabilities and satisfies $g(1) = 1$, we require $n > 0$.

Step 5: The normalization constraint $\sum_i g(|\alpha_i|) = 1$ when $\sum_i |\alpha_i|^2 = 1$ implies

$$\sum_i |\alpha_i|^n = 1 \quad \text{whenever} \quad \sum_i |\alpha_i|^2 = 1. \tag{9}$$

In particular, for an equal-amplitude N-outcome superposition with $|\alpha_i| = 1/\sqrt{N}$, the constraint gives $N(1/\sqrt{N})^n = 1$, i.e. $N^{1-n/2} = 1$ for all N, hence $n = 2$.

Lemma (uniqueness of $n = 2$). The only power n for which the L^2 unit sphere is also an L^n unit sphere is $n = 2$.

Proof sketch. If $n > 2$, take $|\alpha_1| = |\alpha_2| = 1/\sqrt{2}$ and the rest zero. Then $\sum_i |\alpha_i|^2 = 1$ but $\sum_i |\alpha_i|^n = 2^{1-n/2} < 1$, a contradiction. If $n < 2$, take $|\alpha_1| = \sqrt{1-\epsilon^2}$, $|\alpha_2| = \epsilon$ (exactly on the L^2 sphere: $|\alpha_1|^2 + |\alpha_2|^2 = 1$). Then $\sum_i |\alpha_i|^n = (1-\epsilon^2)^{n/2} + \epsilon^n$. For small $\epsilon > 0$, Bernoulli's inequality gives $(1-\epsilon^2)^{n/2} > 1 - \frac{n}{2}\epsilon^2$, so

$$\sum_i |\alpha_i|^n > 1 + \epsilon^2\left(\epsilon^{n-2} - \frac{n}{2}\right). \tag{10}$$

Since $n - 2 < 0$, we can choose ϵ small enough that $\epsilon^{n-2} > \frac{n}{2}$, giving $\sum_i |\alpha_i|^n > 1$, again a contradiction. Hence $n = 2$.

Step 6 (consistency check): The operational interference requirement is met by $f(\alpha) = |\alpha|^2$, which yields the standard phase-dependent fringe visibility in two-path experiments and reduces to a classical mixture under phase randomization. Other power rules generally distort the phase dependence and visibility.

Therefore $f(\alpha) = |\alpha|^2$. $\square$

Remark on scope. Our axioms do not conjure $n = 2$ from nothing: the 2-norm, which quantum mechanics requires independently for unitarity, already constrains the space of possibilities. The contribution is showing that $n = 2$ is the *unique* rule compatible with the five derivational constraints simultaneously; the Born rule is forced by consistency, not postulated. On an L^p sphere with $p \neq 2$, the same argument would yield $n = p$. This norm-relativity acquires physical content because the 2-norm is independently selected: it is the only L^p norm arising from an inner product [24], and dedicated multi-path experiments bound third-order interference to less than 10^{-4} of the pairwise term [23], consistent with the second-order-only structure that Sorkin [21] showed is equivalent to quadratic probabilities.

5. Why Squaring? Interpreting the Map to Record Weights

The proof establishes uniqueness, but the deeper question is what physical role "squaring" plays at the boundary from amplitudes to records. We therefore give two interpretations:

5.1. Information-Theoretic Interpretation

Each amplitude $\alpha_i = |\alpha_i|e^{i\phi_i}$ carries a magnitude and a phase. Only the *relative* phases $\phi_i - \phi_j$ between branches are physically observable (global phase is unobservable). Classical probability is a positive real number; the map from $\mathbb{C}$ to $\mathbb{R}^+$ must therefore be insensitive to phase.

The squared modulus $\alpha \mapsto |\alpha|^2 = \alpha \cdot \alpha^*$ is a natural phase-insensitive positive map:

$$|\alpha e^{i\phi}|^2 = (\alpha e^{i\phi})(\alpha e^{i\phi})^* = \alpha\alpha^* e^{i\phi} e^{-i\phi} = |\alpha|^2 \tag{11}$$

The phase cancels through multiplication with the complex conjugate. The theorem in Sec. IV shows this is not merely natural but forced: no other outcome-local map satisfies all five derivational postulates.

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5.2. Thermodynamic Interpretation

Under the operational conditions where Landauer bounds apply, record formation can be viewed as an irreversible transition from a coherent quantum description to a classical record description. In this paper that thermodynamic picture motivates the phase-independence axiom; it is not part of the formal proof. A full microscopic derivation linking loss of relative phase access to a specific heat cost remains open (cf. the conclusion and Ref. [8]).

The relative phase between branches carries information that enables interference. When a classical record is created, that relative-phase information is not retained in the accessible record basis. For a controlled two-phase preparation with $\Delta\phi \in \{0, \pi\}$ and a detector placed at a point of maximal contrast, the relative phase acts as a classical binary label: constructive and destructive interference correspond to two distinguishable settings.

In that specific illustrative setting, if the record-formation channel is modeled as irreversibly discarding access to that classical phase label under the applicability conditions inherited from Paper A, the associated heat scale is of order $k_B T \ln 2$. More generally, any quantity such as I_{phase} requires an explicit phase prior and measurement context, so we do not use it as a general derivational input.

In this interpretive picture, the Born rule probability $P = |\alpha|^2$ is the phase-insensitive residue after the relative-phase degree of freedom has been excluded from the accessible classical record.

6. What the Record Loses: Information Content of Phase

Having established $f(\alpha) = |\alpha|^2$ in Sec. IV, we now use this result to analyze the information content of phase and its connection to the Landauer bound. This subsection is illustrative: it uses standard two-path interference statistics implied by the Born rule and does not introduce an additional axiom in the derivation.

6.1. Mutual Information Analysis

Consider two distinct experimental configurations for a two-path interferometer:

- X: The relative phase $\Delta\phi = \phi_1 - \phi_2$, uniformly distributed on $[0, 2\pi)$
- Y_{int}: The detector outcome in the interference basis (which output port clicks, with paths recombined)
- Y_{rec}: The path-basis record (which-path detector, no recombination)

In the interference configuration, the detection probability is phase-dependent:

$$P(Y_{\text{int}} = 1 | \Delta\phi) = \frac{1}{2}(1 + \cos\Delta\phi) \tag{12}$$

The mutual information between phase and interference outcome is:

$$I(X; Y_{\text{int}}) = H(Y_{\text{int}}) - H(Y_{\text{int}}|X) \tag{13}$$

For a continuous uniform phase, $I(X; Y_{\text{int}})$ depends on the detector setting and is less than 1 bit. In a two-phase preparation with $\Delta\phi \in \{0, \pi\}$ and a detector at maximal contrast, $H(Y_{\text{int}}|X) = 0$ and $I(X; Y_{\text{int}}) = 1$ bit.

In the path-basis (record) configuration, the outcome probability is phase-independent:

$$P(Y_{\text{rec}}) = |\alpha_1|^2 \quad \text{(independent of phase)} \tag{14}$$

Thus $I(X; Y_{\text{rec}}) = 0$: the classical record carries no information about the relative phase.

In the two-phase preparation, the transition from interference detection to path-basis recording corresponds to the loss of 1 bit of phase information. Connecting that loss to a Landauer cost requires the additional operational assumptions of Paper A. (For a continuous uniform phase the mutual

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information is less than 1 bit; for an equiprobable K-valued discrete phase alphabet, the associated entropy is $\log_2 K$ bits.)

7. Connection to Prior Derivations

7.1. Gleason's Theorem

Gleason's theorem [1] proves that in Hilbert spaces of dimension ≥ 3, any probability measure satisfying non-contextuality must be given by the Born rule.

Our derivation is **complementary mainly in physical motivation and axiom choice**:

- **Gleason**: Shows the rule is *mathematically necessary* given non-contextuality (dimension ≥ 3)
- **This work**: Within the class of outcome-local maps for pure-state projective measurements, shows that the rule is uniquely fixed by record-basis phase independence, factorization, continuity, and L^2 normalization

Our axiom set (factorization (composition) + phase independence (symmetry) + continuity) is structurally closer to the "composition + symmetry" family of derivations (Hardy [13], Masanes and Müller [14]) than to Gleason's measure-theoretic non-contextuality. Phase independence in the record basis replaces Gleason's non-contextuality; this is a different assumption, but the two are structurally related because both constrain how probabilities may depend on measurement context. Phase independence is a statement about insensitivity to relative phase in the accessible record basis, whereas non-contextuality is a statement about independence from the embedding measurement context. Generalized Gleason-type results by Busch [15] and Caves *et al.* [16] extend the Born rule to POVMs and to qubits without the $d \geq 3$ restriction; our theorem does not cover that generality.

Stacey [19] cautions that many Born-rule derivations merely relocate Gleason's assumptions. Our axiom set differs structurally (phase independence replaces non-contextuality, no $d \geq 3$ restriction), though the concern about assumption-shuffling is legitimate. Our theorem applies to qubits because outcome-locality and tensor-product factorization are assumed directly rather than extracted from Gleason's lattice geometry; this is a difference in route and motivation, not a replacement for the more general Gleason-type literature.

7.2. Masanes, Galley, and Müller

Masanes, Galley, and Müller [10] derive the Born rule and measurement postulates from the remaining quantum postulates plus subsystem consistency (a composition axiom related to our Tensor Product Factorization postulate) without thermodynamic input, using a stronger set of operational postulates. Galley and Masanes [11] classify alternatives to the Born rule in terms of informational properties, providing a different perspective on the uniqueness question.

7.3. Thermodynamic Approaches via Divergence Minimization

Other recent work has explored thermodynamic routes to the Born rule through KL-divergence minimization or structural-persistence arguments.[12] Our approach differs in using an explicit axiom set rather than an optimization principle, and in providing a direct proof via Cauchy's functional equation that only $n = 2$ survives.

7.4. Envariance

Zurek's envariance derivation [3] uses environment-assisted invariance: probabilities must be invariant under swaps that leave the environment unchanged.

Our framework is *consistent with* envariance and provides a physical mechanism that supports it:

- The environment is the thermal bath receiving the Landauer heat
- Swaps that leave the bath state unchanged cannot change probabilities
- Phase information is not retained in the record accessible to the observer

Envariance is consistent with entropy flow to the bath during record formation.

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8. Why Not Linear? Why Not Quartic?

8.1. Linear Rule

Consider the linear rule $P_i = |\alpha_i|$. It could be normalized with an L^1 norm, but this breaks the Hilbert-space structure required for unitary evolution and a well-defined inner product. Quantum mechanics uses the L^2 norm precisely because it preserves the inner product and probability under unitary dynamics. With L^2 normalization, $\sum_i |\alpha_i| \neq 1$ generically, and linear probabilities fail to reproduce observed interference.

8.2. Quartic Rule

Consider the quartic rule $P_i = |\alpha_i|^4$. It satisfies phase independence but fails normalization on the L^2 unit sphere: for $|\alpha_1|^2 = |\alpha_2|^2 = 1/2$, we get $\sum_i |\alpha_i|^4 = 1/2 \neq 1$.

8.3. General Power Rule

For the general power rule $P_i = |\alpha_i|^n$, only $n = 2$ satisfies all constraints simultaneously within the present outcome-local framework. A full assessment of non-Born alternatives requires specifying the surrounding measurement theory, not just a bare power-law substitution. No other power satisfies all constraints simultaneously.

9. Experimental Probes and Consistency Checks

The following are not predictions specific to this framework, but probes and consistency checks showing that the record-formation picture is compatible with known physics. Calorimetric protocols are developed in Ref. [8].

9.1. Heat-Information Correlation

For measurements creating different amounts of classical information, the heat dissipated should scale with the Shannon entropy of the outcome distribution, with a Landauer prefactor under the applicability conditions:

$$\langle Q \rangle \gtrsim k_B T \ln 2 \, I(X;Y) \tag{15}$$

where $I(X;Y) = H(Y) - H(Y|X)$ is the classical mutual information between the prepared state X and the measurement record Y. For ideal projective measurements with known prior, $I(X;Y) = H(P)$ (the Shannon entropy of the outcome distribution); the general bound uses mutual information. [6,8] Calorimetric protocols and sensitivity estimates are provided in Ref. [8].

9.2. Decoherence Interpolation

For weak measurements with partial phase information loss, the unnormalized detection rate (intensity) interpolates between quantum and classical:

$$I_{\text{eff}} = (1-\epsilon) \cdot |\psi_1 + \psi_2|^2 + \epsilon \cdot (|\psi_1|^2 + |\psi_2|^2) \tag{16}$$

where ϵ can be operationally defined from fringe visibility, e.g. $\epsilon = 1 - V$ with $V = (I_{\max} - I_{\min})/(I_{\max} + I_{\min})$. Note that I_{eff} is not a probability: it can exceed unity for constructive interference. Probabilities are obtained by normalizing I_{eff} over all outcomes.

10. Conclusion

We have proved a conditional uniqueness theorem for projective measurements on pure states: given the L^2 Hilbert space structure of quantum mechanics, the Born rule is the unique outcome-local probability assignment compatible with Outcome-Locality, Normalization, record-basis Phase Information Loss, Tensor Product Factorization, and Continuity. No thermodynamic quantity enters the formal proof. In physical terms, the theorem concerns the transition from an interference-capable amplitude description to a classical record description. Relative phase is dynamically relevant before

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measurement, but a stabilized record cannot retain that full distinction structure in accessible form. Rather, the record-formation picture motivates the phase-independence axiom. It also identifies why the map to classical outcome weights should be irreversible. The squared modulus $|\alpha|^2$ is the unique such rule that:

1. Maps phase-sensitive amplitudes to phase-insensitive record weights
2. Preserves interference before measurement (quantum consistency)
3. Satisfies normalization and factorization for classical outcomes

The Born rule is not an arbitrary postulate; within the class of outcome-local probability maps, no other assignment is consistent with both quantum kinematics and the operational constraints stated here. Put differently, it is the only classicalization map from phase-sensitive amplitudes to accessible record weights that satisfies all five postulates. "Why squared?" has a conditional answer: because the L^2 norm, together with outcome-local probability assignment and record-basis phase independence, admits no other consistent rule.

A natural next step is to derive phase independence from an explicit CPTP model of the record-formation channel, which would upgrade the thermodynamic content from motivation to derivation. Generalization to mixed states and POVMs likewise remains open in this framework. More broadly, connecting the axiom set to the recent thermodynamic stage decompositions of measurement [20] may clarify the relationship between record formation and the Born rule at a deeper level.

Supplementary code (`simulation.py`) verifies all numerical claims: algebraic identities, uniqueness-lemma sweeps, alternative-rule falsification, information-theoretic quantities, and cross-consistency with Ref. [8]. These checks confirm internal mathematical consistency; they do not validate the physical premises.

Supplementary Materials: The following supporting information can be downloaded at the website of this paper posted on Preprints.org.

Data Availability Statement: The supplementary simulation code and generated consistency reports are available in the project repository

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