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The Fourier-Legendre Series of Bessel Functions of the First Kind

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The Fourier-Legendre Series of Bessel Functions of the First Kind

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Abstract: The Bessel function of the first kind $J_N(kx)$ is expanded in a Fourier-Legendre series, as is the modified Bessel functions of the first kind $I_N(kx)$. Known polynomial approximations for the range $-3 \leq x \leq 3$ (having five-digit accuracy) are compared with those arising from this Legendre series and substitute versions are given with twice the number of terms and range (with five-digit accuracy or having fifteen-digit accuracy over $-3 \leq x \leq 3$). It is shown that infinite series of like-powered contributors (involving ${}_2F_3$ hypergeometric functions) extracted from the Fourier-Legendre series may be summed, having values that are inverse powers of the first five primes $1/\left(2^i 3^j 5^k 7^l 11^m\right)$ for powers of order x^{24}.

Keywords: Bessel functions; Fourier-Legendre series; Laplace series; Generalized Hypergeometric Functions; polynomial approximations; Computational methods

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1. Introduction

The Strong Field Approximation (SFA) [1–6], unlike perturbation expansions that will not converge if an applied laser field is large, is an analytical approximation that is nonperturbative. Keating [7] applied it to the production of the positive antihydrogen ion and found that the reduction of the transition amplitudes to analytic form required the expansion of the resultant Bessel functions $J_N(kx)$ in a series of spherical harmonics. We have been unable to discover such a Laplace series [8] in the literature, nor does there seem to be an expansion of the Bessel function in a series of Legendre polynomials, to which the Laplace series reduces in the common case where the function is independent of the azimuthal angle. We recreate herein Keating's derivation, stripped of the specialized SFA terminology, and find that we can recast his series as a ${}_2F_3$ generalized hypergeometric function. We also extend the method to the modified Bessel function of the first kind $I_N(kx)$.

Since a Fourier-Legendre series is easily converted to a series in powers, we do so and compare the results to the known polynomial approximation[9] for the range $-3 \leq x \leq 3$ (having five-digit accuracy). We easily generate a substitute version with twice the number of terms that gives twice the range (with five-digit accuracy or having fifteen-digit accuracy over $-3 \leq x \leq 3$). Using 48-digit accuracy, we show that like-powered contributors extracted from the Fourier-Legendre series may be summed, having values that are inverse powers of the first five primes $1/\left(2^i 3^j 5^k 7^l 11^m\right)$ for powers of order x^{24}.

2. The Fourier-Legendre series of a Bessel function of the first kind

We begin with the assumption that the series

$$J_N(kx) = \sum_{L=0}^{\infty} a_{LN} P_L(x) \tag{1}$$

converges uniformly[10], where the coefficients are given by the orthogonality of the Legendre polynomials,

$$a_{LN} = \frac{2L+1}{2} \int_{-1}^{1} J_N(kx)\, P_L(x) dx \tag{2}$$

Following Keating's lead, we use Heine's integral representation of the Bessel function [11] for integer indices

$$J_N(kx) = \frac{i^{-N}}{\pi} \int_0^{\pi} e^{ikx\cos\theta} \cos(N\theta)\, d\theta \quad , \tag{3}$$

so that

$$a_{LN} = \frac{2L+1}{2} \int_{-1}^{1} \left[\frac{i^{-N}}{\pi} \int_0^{\pi} e^{ikx\cos\theta} \cos(N\theta)\, d\theta \right] P_L(x) dx \quad . \tag{4}$$

By switching the order of integration,

$$a_{LN} = \frac{2L+1}{2} \frac{i^{-N}}{\pi} \int_0^{\pi} \left[\int_{-1}^{1} e^{ikx\cos\theta} P_L(x) dx \right] \cos(N\theta)\, d\theta \quad , \tag{5}$$

we can use [12,13]

$$e^{ikx\cos\theta} = \sum_{l'} (2l'+1) i^{l'} j_{l'}(k\cos\theta) P_{l'}(x) \quad . \tag{6}$$

Then

$$\begin{aligned} a_{LN} &= \tfrac{2L+1}{2} \tfrac{i^{-N}}{\pi} \int_0^{\pi} \left[\int_{-1}^{1} \left(\sum_{l'=0}^{\infty} (2l'+1) i^{l'} j_{l'}(k\cos\theta) P_{l'}(x) \right) P_L(x) dx \right] \cos(N\theta)\, d\theta \\ &= \tfrac{2L+1}{2} \tfrac{i^{-N}}{\pi} \int_0^{\pi} \left[\left(\sum_{l'=0}^{\infty} (2l'+1) i^{l'} j_{l'}(k\cos\theta) \tfrac{2}{2l'+1} \delta_{l'L} \right) \right] \cos(N\theta)\, d\theta \\ &= (2L+1) \tfrac{i^{L-N}}{\pi} \int_0^{\pi} j_L(k\cos\theta) \cos(N\theta)\, d\theta \quad . \end{aligned} \tag{7}$$

Using the series expansion [14]

$$j_l(x) = \frac{1}{2}\sqrt{\pi} \left(\frac{x}{2}\right)^l \sum_{M=0}^{\infty} \frac{\left(-\frac{1}{4}\right)^M x^{2M}}{M!\Gamma\left(l+M+\frac{3}{2}\right)} \tag{8}$$

this becomes

$$a_{LN} = \frac{(2L+1)}{\sqrt{\pi}} i^{L-\mathrm{N}N} 2^{-L-1} \left(\sum_{M=0}^{\infty} \frac{\left(-\frac{1}{4}\right)^M k^{L+2M}}{M!\Gamma\left(L+M+\frac{3}{2}\right)} \right) \int_0^{\pi} \cos^{L+2M}(\theta) \cos(\mathrm{N}\theta)\, d\theta \quad . \tag{9}$$

Gröbner and Hofreiter[15] extended an integral over the interval $\left[0, \frac{\pi}{2}\right]$ that has three branches, to the interval $[0, \pi]$ with a prefactor $(1 + (-1)^{m+n})$ that renders the central one of the three possibilities nonzero only for even values for $m + n$.

$$\int_0^{\pi} cos^m\theta\, cos(n\theta) d\theta = (1 + (-1)^{m+n}) \frac{\pi}{2^{m+1}} \binom{m}{\frac{m-n}{2}} \quad [m \geq n > -1,\ m - n = 2K] \quad . \tag{10}$$

The other two branches, being for odd $m + n$ on $\left[0, \frac{\pi}{2}\right]$, are zero on $[0, \pi]$ when this prefactor is included. (Numerical integration also confirms that the contributions from the $\left[\frac{\pi}{2}, \pi\right]$ interval cancels the contributions from the $\left[0, \frac{\pi}{2}\right]$ on these branches.) The fifth edition of Gradshteyn and Ryzhik[16] (in which $m \longleftrightarrow n$) nevertheless included all three branches and this prefactor on the interval $[0, \pi]$. By their seventh edition, Gradshteyn and Ryzhik removed this integral entirely despite the correctness of the central branch on the interval $[0, \pi]$. Neither source noted the lower limit on m that we found: $[m \geq n > -1]$.

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The final form for the coefficient set of the Fourier-Legendre series for the Bessel function $J_N\left(kx\right)$ is then

$$\begin{aligned} a_{LN} &= \sqrt{\pi}(2L+1)2^{-L-1}i^{L-\mathrm{N}}\sum_{M=0}^{\infty}\frac{\left(\left(-\frac{1}{4}\right)^{M}k^{L+2M}\right)}{2^{L+2M+1}\left(M!\Gamma\left(L+M+\frac{3}{2}\right)\right)}\left(1+(-1)^{L+2M+\mathrm{N}}\right)\binom{L+2M}{\frac{1}{2}(L+2M-\mathrm{N})} \\ &= \frac{\sqrt{\pi}2^{-2L-2}(2L+1)k^L i^{L-N}}{\Gamma\left(\frac{1}{2}(2L+3)\right)}\left(1+(-1)^{L+N}\right)\binom{L}{\frac{L-N}{2}} \\ &\times \; {}_2F_3\left(\frac{L}{2}+\frac{1}{2},\frac{L}{2}+1;L+\frac{3}{2},\frac{L}{2}-\frac{N}{2}+1,\frac{L}{2}+\frac{N}{2}+1;-\frac{k^2}{4}\right) \end{aligned}, \tag{11}$$

where the final step is new with the present work.

The first 13 terms in the sum (1) are then, to 48-digit accuracy with $k=1$ (using E-7 as the programming shorthand for $\times 10^{-7}$),

$$\begin{aligned} J_0\left(x\right) \cong\; & 0.919730410089760239314421194080619970661964806513 P_0(x) \\ & -0.157942058625851887573713967144363701344798627015 P_2(x) \\ & +0.00343840094460110923299688787207291548407319575920 P_4(x) \\ & -0.0000291972184882872969366059098612566327416694039900 P_6(x) \\ & +1.31735695244778097765561656314327987844235800032724478896974 3\text{E-}7\; P_8(x) \\ & -3.68450084420820302717377109605886619909428401234772639 63\text{E-}10\; P_{10}(x) \\ & +7.01183003299384592820880332821144727852424580719192725747183780 83\text{E-}13\; P_{12}(x) \\ & -9.66596436985891226367199537275334645132271495693178359979990 35\text{E-}16\; P_{14}(x) \\ & +1.00963627682454644652534217092493625236608417531220118935645\text{E-}18\; P_{16}(x) \\ & -8.26665695592763785899197258417411650805067163797479423671369352 06\text{E-}22\; P_{18}(x) \\ & +5.44824486776275872589008283783943035835255048579364076292680 43\text{E-}25\; P_{20}(x) \\ & -2.95252718213735475167577460666339964484583374163248477360 21\text{E-}28\; P_{22}(x) \\ & +1.33885615885853446908089867009620010924155326611761105138108348 69\text{E-}31\; P_{24}(x). \end{aligned} \tag{12}$$

When compared to the polynomial approximation given in Abramowitz and Stegun,[9]

$$J_0\left(x\right) \cong 1-2.25\left(\frac{x}{3}\right)^2+1.26562\left(\frac{x}{3}\right)^4-0.316387\left(\frac{x}{3}\right)^6+0.0444479\left(\frac{x}{3}\right)^8-0.0039444\left(\frac{x}{3}\right)^{10} +0.0002100\left(\frac{x}{3}\right)^{12}, \tag{13}$$

at the latter's limiting value of $x=3$ for five-digit accuracy, one finds 15-digit accuracy in the result, -0.260051954901933, much better than one would expect for doubling the number of powers in the approximation. In addition, the *range* of applicability of the new Legendre approximation truncated at x^{24}, if one is satisfied with five-digit accuracy, roughly doubles to $J_0\left(6.5\right) \cong 0.260095$.

3. Summing a set of infinite series

If the original polynomial approximation [9] has its powers of three folded into the coefficients,

$$J_0\left(x\right) \cong 1-0.25\,x^2+0.0156249\,x^4-0.000434001\,x^6+6.77456\text{E-}6\,x^8-6.6799\text{E-}8\,x^{10} +3.952\text{E-}10\,x^{12}\quad. \tag{14}$$

and notes that the third term is $1/64.0004$, one is led to wonder if each of these coefficients is made up of inverse powers of primes if one were to rederive a similar approximation using a process with greater accuracy.

We can easily expand the Legendre polynomials into their constituents terms and gather like powers in eq. (12) to give an updated polynomial approximation,

$$\begin{aligned} J_0\left(x\right) \cong\ & 1.000000000000000000000000000000000000000000000x^0 \\ - & 0.250000000000000000000000000000000000000000000x^2 \\ + & 0.015625000000000000000000000000000000000000000x^4 \\ - & 0.000434027777777777777777777777777777777777778x^6 \\ + & 6.781684027777777777777777777777777777777777709035743914\text{E-}6\ x^8 \\ - & 6.781684027777777777777777777777777777777776418460183472\text{E-}8\ x^{10} \\ + & 4.709502797067901234567901234567901234567901217102330430492\text{E-}10\ x^{12} \\ - & 2.402807549524439405391786344167296548248913366759383191168427975 41\text{E-}12\ x^{14} \\ + & 9.385966990329841427311665406903502141492077046170357821658 2006\text{E-}15\ x^{16} \\ - & 2.896903392077111551639402903365278385973981447743521706878477\text{E-}17\ x^{18} \\ + & 7.242258480192778879098507258412992706519050135221465476326\text{E-}20\ x^{20} \\ - & 1.496334396734045222954237036801210822096139838921547451367626 6228\text{E-}22\ x^{22} \\ + & 2.597802772107717400962216934172307551874835422820217913743326 9\text{E-}25\ x^{24} \quad . \\ = & 1x^0 - \frac{x^2}{2^2} + \frac{x^4}{2^6} - \frac{x^6}{2^8 3^2} + \frac{x^8}{2^{14} 3^2} - \frac{x^{10}}{2^{16} 3^2 5^2} + \frac{x^{12}}{2^{20} 3^4 5^2} - \frac{x^{14}}{2^{22} 3^4 5^2 7^2} + \frac{x^{16}}{2^{30} 3^4 5^2 7^2} \\ - & \frac{x^{18}}{2^{32} 3^8 5^2 7^2} + \frac{x^{20}}{2^{36} 3^8 5^4 7^2} - \frac{x^{22}}{2^{38} 3^8 5^4 7^2 11^2} + \frac{x^{24}}{2^{44} 3^{10} 5^4 7^2 11^2} \quad . \end{aligned} \tag{15}$$

We see that we have managed to rederive the first thirteen terms of the well-known series representation[17]

$$J_\nu\left(x\right) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)} \tag{16}$$

But the outcome worth the trouble of this investigation is that this process yields a set of infinite sums whose values are inverse powers of primes. Looking back at the coefficients in eq. (12) when multiplied by the constant terms in the Legendre polynomials that multiply them, whose first few are

$$\begin{aligned} \Big\{ P_0(x) = 1,\ P_2(x) = \frac{1}{2}\left(3x^2 - 1\right),\ P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \\ P_6(x) = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right) \Big\} \quad , \end{aligned} \tag{17}$$

there is no reason to suspect that

$$\begin{aligned} & 0.919730410089760239314421194080619970661964806513 \\ & -\frac{1}{2}\left(-0.157942058625851887573713967144363701344798627015\right) \\ & +\frac{3}{8}0.003438400944601109232996887872072915484073195759 20 \\ & -\frac{5}{16}\left(-0.000291972184882872969366059098612566327416694039900\right) \\ & +\cdots = 1 \end{aligned} \tag{18}$$

but one sees uniform convergence up through to accuracy of the calculation as one adds additional terms, as seen in Table 1.

Table 1. The constant term of the polynomial approximation as increasing numbers of terms are added from eq. (12), to 48 digit accuracy.

0.919730410089760239314421194080619970661964806513
0.998701439402686183101278177652801821334364120020
0.999990839756911599063652010604829164640891568430
0.999999963887689188843944699951660807338623340119
0.999999999909168357337955807722426205787682516277
0.999999999999841620300892054094280728854756176645
0.99999999999999979732605041136082528291421094660
0.999999999999999999801573588581204243621535893434
0.999999999999999999999846581786952424267000275450
0.999999999999999999999999903953851991585994488442
0.999999999999999999999999999503204200428971030 88
0.999999999999999999999999999999978412290228685090
0.999999999999999999999999999999999992008312509362
0.999999999999999999999999999999999999997449396440
0.999999999999999999999999999999999999999999290955
0.999999999999999999999999999999999999999999999827
1.00000000000000000000000000000000000000000000000

One may more formally concluded that

$$\sum_{L=0}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^L (2L+1) \binom{L}{\frac{L}{2}} \binom{2L}{L} \left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}} \left(-\frac{L}{2}\right)_{\frac{L}{2}} 2^{-3L-2} {}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1, L+\frac{3}{2}; -\frac{1}{4}\right)}{\frac{L}{2}! \left(\frac{1}{2}-L\right)_{\frac{L}{2}} \Gamma\left(\frac{1}{2}(2L+3)\right)} = 1 \tag{19}$$

where the superscript "(2)" on the sum indicates one is summing even values only (or one my retain the factor $(1+(-1)^L)$ in the sum), which is a result we have not seen in the literature. The Pochhammer symbols $\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}}$ and so on derive from a shift to[18]

$$P_n(x) = 2^{-n} \binom{2n}{n} x^n \, {}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}-n; \frac{1}{x^2}\right) \tag{20}$$

in the explicit sum. [19] One can see from the repeating digits that the third through sixth lines in eq. (15) have inverses that are powers of primes, and one can even see that the fifth term is $2^2 5^2$ times the sixth.

Subsequent terms are not at all obviously inverse powers of primes. Indeed, that eventuality requires a sufficient number of terms even at our 48-digit accuracy. It turns out that the inverse of the coefficient of the x^{16} term when including the 13 terms in the sum (15), that were sufficient for checking convergence of coefficients of the smallest powers, is **106542032486495**.61634840999175246241167161945 6197,whose integer part is in bold face. This is not a product of low-level primes. Adding one more term to (15) is sufficient to bring it to **106542032486400**.113376300998684305400416345779209, whose integer part is $2^{30} 3^4 5^2 7^2$. Adding another term gives **106542032486400**.000104784167278249059923631013442 and with every additional term added, the integer part remains the same while the fractional part diminishes by several decimal places. The coefficient of the x^{24} term required 40 terms in the sum (15) to establish convergence, and all of the coefficients in (15) include contributions from all 40 terms to establish a consistent floating-point set as well as the inverse prime version. One could likely use these as seed values for a 13-term optimization attempt that would be slightly more accurate than the present truncation approach, but simply adding more terms to a truncated series is much easier given the analytic form at our disposal in eq. (11).

The other 12 verified summed series given by the present approach (with $1 \leq h \leq 12$) are

$$\sum_{L=0}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^L 2^{-3L-2} \left((-1)^L+1\right)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L} {}_1F_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} \times \left[\frac{\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}-h}\left(-\frac{L}{2}\right)_{\frac{L}{2}-h}}{\left(\frac{1}{2}-L\right)_{\frac{L}{2}-h}}\right] = \frac{(-1)^h 2^{-2h}}{h!\Gamma(h+1)}, \tag{21}$$

from which one may verify the primes in the last two lines of eq. (15).

One may also use the relation[20]

$$(-a)_n = (-1)^n (a-n+1)_n \tag{22}$$

and the primary definition of the Pochhammer symbol[21]

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} /; \neg(-a \in \mathbb{Z} \land -a \geq 0 \land n \in \mathbb{Z} \land n \leq -a) \tag{23}$$

to rewrite the term in square brackets in infinite sum (21) to give

$$\begin{aligned} &\sum_{L=0}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^L 2^{-3L-2} \left((-1)^L+1\right)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L} {}_1F_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} \\ &\times \left[\frac{(-1)^{\frac{L}{2}-h}\left(h+\frac{1}{2}\right)_{\frac{L}{2}-h}(h+1)_{\frac{L}{2}-h}}{\left(h+\frac{L}{2}+\frac{1}{2}\right)_{\frac{L}{2}-h}}\right] \\ &= \sum_{L=0}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^L 2^{-3L-2} \left((-1)^L+1\right)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L} {}_1F_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} \\ &\times \left[\frac{(-1)^{\frac{L}{2}-h} 2^{2h-L}\Gamma(L+1)\Gamma\left(h+\frac{L}{2}+\frac{1}{2}\right)}{\Gamma(2h+1)\Gamma\left(L+\frac{1}{2}\right)}\right] = \frac{(-1)^h 2^{-2h}}{h!\Gamma(h+1)}. \end{aligned} \tag{24}$$

4. Series arising from the $J_1(x)$ Fourier-Legendre series

The first 13 terms in the $J_1(x)$ Fourier-Legendre series (1) are

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$$
\begin{aligned}
J_1(x) = {} & 0.46359817059538106359411100393387024925807155027 2P_1(x) \\
& - 0.023865345658407397963072094164848661144756771804 9P_3(x) \\
& + 0.00031972435597200476385247576232560281161594051501 6P_5(x) \\
& - 1.9705191806665942502580629293911116487585393730843186079480334\text{E-6}\ P_7(x) \\
& + 6.9872474730978072187917594101570139463303280482950147787763\text{E-9}\ P_9(x) \\
& - 1.6105000560468750278070024429533270253824538358437085657441438624 64\text{E-11}\ P_{11}(x) \\
& + 2.6070865924416288429392481936199093177962128209450882960372588 34\text{E-14}\ P_{13}(x) \\
& - 3.1273114825407968821447136195674424403158051933811690518981 82\text{E-17}\ P_{15}(x) \\
& + 2.8914240817870507398273825966160635699016533236423299593 1\text{E-20}\ P_{17}(x) \\
& - 2.1236645347793691992144144557203173922053781801448888128016567 22\text{E-23}\ P_{19}(x) \\
& + 1.2690112017586735117145537075281857625899974494650324541452 76\text{E-26}\ P_{21}(x) \\
& - 6.2902019391359257635768713587386001405627426827888045791476098081 34\text{E-30}\ P_{23}(x) \\
& + 2.6281357969893254525738707742672131170306609821773481659610278 47\text{E-33}\ P_{25}(x).
\end{aligned} \tag{25}
$$

When compared to the polynomial approximation given in Abramowitz and Stegun [22],

$$
\begin{aligned}
J_1(x) \cong {} & \frac{x}{2} - 0.062499983x^3 + 0.0026041448x^5 - 0.00005424265x^7 + 6.7568816\text{E-7}x^9 \\
- {} & 5.3788\text{E-9}x^{11} + 2.087\text{E-11}x^{13} \quad ,
\end{aligned} \tag{26}
$$

at the latter's limiting value of $x = 3$ for five-digit accuracy, one finds 16-digit accuracy in the result, 0.3390589585259365, again much better than one would expect for doubling the number of powers in the approximation. In addition, the *range* of applicability of the new Legendre approximation, if one is satisfied with five-digit accuracy, roughly doubles to $J_1(6.5) \cong -0.153841$.

We can again expand the Legendre polynomials into their constituents terms and gather like powers in eq. (12) to give an updated polynomial approximation,

$$
\begin{aligned}
J_1(x) \cong {} & 0.49999999999999999999999999999999999996073036732708x \\
- {} & 0.062499999999999999999999999999999995064877689984265x^3 \\
+ {} & 0.0026041666666666666666666666666648306976626379121 84x^5 \\
- {} & 0.000054253472222222222222222221904840719303454131543 1x^7 \\
+ {} & 6.7816840277777777777774691877541020051259664182254220732966 6\text{E-7}\ x^9 \\
- {} & 5.6514033564814814814627959321470018420114848588327454860019\text{E-9}\ x^{11} \\
+ {} & 3.3639305693342151600733679654430218920551615491175665444193864291 42\text{E-11}\ x^{13} \\
- {} & 1.5017547184527725848458902100520196712921972004803304061942047588\text{E-13}\ x^{15} \\
+ {} & 5.2144261057349232956805346216422729234612607233549650639373514\text{E-16}\ x^{17} \\
- {} & 1.4484516955282091274063182645585361824309054991186853272349 1\text{E-18}\ x^{19} \\
+ {} & 3.2919352157321126612820014355943522224343714123520369874909542342 1\text{E-21}\ x^{21} \\
- {} & 6.2344608805335659655381607898522913641949830301704813933694043 2\text{E-24}\ x^{23} \\
+ {} & 9.9010539024527462540494936661697833309178308172225858987039 5\text{E-27}\ x^{25} \\
= {} & \frac{x}{2} - \frac{x^3}{2^4} + \frac{x^5}{2^7 3} - \frac{x^7}{2^{11} 3^2} + \frac{x^9}{2^{15} 3^2 5} - \frac{x^{11}}{2^{18} 3^3 5^2} + \frac{x^{13}}{2^{21} 3^4 5^2 7} + \frac{x^{15}}{2^{26} 3^4 5^2 7^2} + \frac{x^{17}}{2^{31} 3^6 5^2 7^2} \\
- {} & \frac{x^{19}}{2^{34} 3^8 5^3 7^2} + \frac{x^{21}}{2^{37} 3^8 5^4 7^2 11} - \frac{x^{23}}{2^{41} 3^9 5^4 7^2 11^2} + \frac{x^{24}}{2^{45} 3^{10} 5^4 7^2 11^2 13} \quad ,
\end{aligned} \tag{27}
$$

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again the first thirteen terms of the well-known series representation[17]. The bridge from the Legendre series to the above gives us another set of infinite series (of which we explicate $0 \leq h \leq 12$, above),

$$\begin{aligned}
&\sum_{L=1}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^{L-1}\left((-1)^{L+1}+1\right)(2L+1)2^{-3L-2}\binom{L}{\frac{L-1}{2}}\binom{2L}{L}(-1)^{-h+\frac{L}{2}-\frac{1}{2}}\,{}_1F_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};-\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(-h+\frac{L}{2}-\frac{1}{2}\right)!} \\
&\quad\times\left[\frac{\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L-1}{2}-h}\left(-\frac{L}{2}\right)_{\frac{L-1}{2}-h}}{\left(\frac{1}{2}-L\right)_{\frac{L-1}{2}-h}}\right] \\
&\quad=\sum_{L=1}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^{L-1}\left((-1)^{L+1}+1\right)(2L+1)2^{-3L-2}\binom{L}{\frac{L-1}{2}}\binom{2L}{L}\,{}_1F_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};-\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(-h+\frac{L}{2}-\frac{1}{2}\right)!} \\
&\quad\times\left[(-1)^{-h+\frac{L}{2}-\frac{1}{2}}\frac{(h+1)_{-h+\frac{L}{2}-\frac{1}{2}}\left(h+\frac{3}{2}\right)_{-h+\frac{L}{2}-\frac{1}{2}}}{\left(h+\frac{L}{2}+1\right)_{-h+\frac{L}{2}-\frac{1}{2}}}\right] \\
&\quad=\sum_{L=1}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^{L-1}\left((-1)^{L+1}+1\right)(2L+1)2^{-3L-2}\binom{L}{\frac{L-1}{2}}\binom{2L}{L}\,{}_1F_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};-\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(-h+\frac{L}{2}-\frac{1}{2}\right)!} \\
&\quad\times\left[(-1)^{-h+\frac{L}{2}-\frac{1}{2}}\frac{2^{2h-L+1}\Gamma(L+1)\Gamma\left(h+\frac{L}{2}+1\right)}{\Gamma(2h+2)\Gamma\left(L+\frac{1}{2}\right)}\right]=\frac{(-1)^h 2^{-2h-1}}{h!\Gamma(h+2)} \quad . \qquad (28)
\end{aligned}$$

5. Series arising from the $I_n(x)$ Fourier-Legendre series

Because the modified Bessel functions of the first kind $I_N(kx)$ are related to the ordinary Bessel functions by the relation [23]

$$I_n(z) = i^{-n} J_n(iz) \quad , \qquad (29)$$

we merely need to multiply by i^{-n} and set $k = i$ in eq. (11) to obtain the $I_0(x)$ Fourier-Legendre series, the first 13 terms of which are

$$\begin{aligned}
I_0(x) \cong\; & 1.086521097023589815837941923492506345739351342949 P_0(x) \\
&+ 0.1758046819215242662605951354261250016059765417743 P_2(x) \\
&+ 0.003709009244052882533923838165527032588349792239 48 P_4(x) \\
&+ 0.000030951052709924321986137446087776016407689073 9973(x) \\
&+ 1.3812597347197735383200523052245059142459821160852513603676 98\text{E-7}\; P_8(x) \\
&+ 3.834312601086373005317788906125573404760824243263591291 25\text{E-10}\; P_{10}(x) \\
&+ 7.257172450096213936720667660411977911684715455492292944521042 5096\text{E-13}\;\; P_{12}(x) \\
&+ 9.962746978836018020128433111635975115045384492620974776154548 8\text{E-16}\; P_{14}(x) \\
&+ 1.037251346110052630963705477046735526583079222769985819861 19\text{E-18}\; P_{16}(x) \\
&+ 8.470496863240475339343499321604115510305543113181010728012323846\text{E-22}\; P_{18}(x) \\
&+ 5.570541399858852219278260483523686660360142281793012955676544 6\text{E-25}\; P_{20}(x) \\
&+ 3.013347383234528850224689041823201292250019444317848356273 6\text{E-28}\; P_{22}(x) \\
&+ 1.364338005353527272638479093175249413053044529736612748510141592 57\text{E-31}\; P_{24}(x). \qquad (30.)
\end{aligned}$$

When compared to the polynomial approximation given in Abramowitz and Stegun,[24]

$$\begin{aligned} I_0(x) &\cong 1 + 3.5156229\left(\frac{x}{3.75}\right)^2 + 3.0899424\left(\frac{x}{3.75}\right)^4 + 1.2067492\left(\frac{x}{3.75}\right)^6 \\ &+ 0..2659732\left(\frac{x}{3.75}\right)^8 + 0.0360768\left(\frac{x}{3.75}\right)^{10} + 0.0045813\left(\frac{x}{3.75}\right)^{12} \\ &= 1 + 0.25\,x^2 + 0.0156252\,x^4 + 0.00043394\,x^6 + 6.801234\text{E-}6\,x^8 + 6.56017\text{E-}8\ \ x^{10} \\ &+ 5.9240\text{E-}10\,x^{12} \end{aligned} \tag{31}$$

at the latter's limiting value of $x = 3.75$ for seven-digit accuracy, one finds 15-digit accuracy in the result, 9.118945860844, about what one would expect for doubling the number of powers in the approximation. In addition, the *range* of applicability of the new Legendre approximation, if one is satisfied with six-digit accuracy, roughly doubles to $I_0(7.5) \cong 268.1613$. It is only in the latter form of the polynomial approximation, in comparison with the J_0 version (14) that one would suspect that the two are related by reversing all of the negative signs in the J_0 version. That the correspondence is not exact for the higher-power terms likely is a result of the optimization scheme in the two cases having slightly different ranges of validity.

We may, however, use our truncation method to verify that the two series are identical apart from the sign reversals of the terms containing i^{-2m} multiplied by x^{2m} for m odd. The full series [25] indeed fulfills this correspondence:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)} \quad . \tag{32}$$

We thereby have another set of summed series given by our bridge from the $I_0(x)$ Fourier-Legendre series for $h \geq 0$

$$\begin{aligned} &\sum_{L=0}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^L k^L 2^{-3L-2}\left((-1)^L+1\right)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L}\,{}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};-\frac{k^2}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} \\ &\times \left[\frac{\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}-h}\left(-\frac{L}{2}\right)_{\frac{L}{2}-h}}{\left(\frac{1}{2}-L\right)_{\frac{L}{2}-h}}\right]_{k\to i} \\ &= \sum_{L=0}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^{2L} 2^{-3L-2}\left((-1)^L+1\right)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L}\,{}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} \\ &\times \left[\frac{\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}-h}\left(-\frac{L}{2}\right)_{\frac{L}{2}-h}}{\left(\frac{1}{2}-L\right)_{\frac{L}{2}-h}}\right] \\ &= \sum_{L=0}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^{2L} 2^{-3L-2}\left((-1)^L+1\right)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L}\,{}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} \\ &\times \left[\frac{2^{2h-L}\Gamma(L+1)\Gamma\left(h+\frac{L}{2}+\frac{1}{2}\right)}{\Gamma(2h+1)\Gamma\left(L+\frac{1}{2}\right)}\right] = \frac{2^{-2h}}{h!\Gamma(h+1)} \quad . \end{aligned} \tag{33}$$

Indeed, since $J_\nu(z)$ is well defined for complex z, we may develop a summed series using this method for any complex value of k. For instance, for $k \to \frac{1+i}{\sqrt{2}}$ one has

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$$\sum_{L=0}^{\infty} {}^{(2)} \frac{\sqrt{\pi} i^L k^L 2^{-3L-2} \left((-1)^L + 1\right) (2L+1) \binom{L}{\frac{L}{2}} \binom{2L}{L} {}_1F_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{k^2}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!}$$

$$\times \left[\frac{\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}-h}\left(-\frac{L}{2}\right)_{\frac{L}{2}-h}}{\left(\frac{1}{2}-L\right)_{\frac{L}{2}-h}}\right]_{k\to\frac{1+i}{\sqrt{2}}} = \frac{(-1)^h (1+i)^{2h} 2^{-3h}}{h!\Gamma(h+1)}$$

$$= \left\{1, -\frac{i}{2^2}, -\frac{1}{2^6}, \frac{i}{2^8 3^2}, \frac{1}{2^{14} 3^2}, -\frac{i}{2^{16} 3^2 5^2}, -\frac{1}{2^{20} 3^4 5^2}, \cdots\right\} \text{ for } [h = 0, 1, 2, \cdots] \,. \quad (34)$$

Although eq. (1) allows one to easily compute the Fourier-Legendre series for any $J_n(x)$ or $I_n(x)$, to enable readers to find these series for higher indices by recursion [26] we give the first 13 terms in the $I_1(x)$ Fourier-Legendre series to complete the required set:

$$\begin{aligned}
I_1(x) = {} & 0.5386343421852555592809081051666335756049569052 16\, P_1(x) \\
+ {} & 0.02618069164825977449795296407260333183639698663 42\, P_3(x) \\
+ {} & 0.00034198519125508062362100943615073441391897864 8008\, P_5(x) \\
+ {} & 2.0776519716996569638602670707248637741802282324973820998109474 \text{ E-6 } P_7(x) \\
+ {} & 7.2990015186624314149055763249328768942564072680398941990154 \text{ E-9 } P_9(x) \\
+ {} & 1.6714434829548537391625277672032146307860407068348851091689302762 84 \text{ E-11 } P_{11}(x) \\
+ {} & 2.6927445514592352327349366664527043330876567797721216993944 3154 \text{ E-14 } P_{13}(x) \\
+ {} & 3.2181067547711624558537598385452819550922951740079191127736 09 \text{ E-17 } P_{15}(x) \\
+ {} & 2.9666246467734030748241964359375417408463038623776610366 96 \text{ E-20 } P_{17}(x) \\
+ {} & 2.1736868837209010315684360476557479238624896516567372397932 24472 \text{ E-23 } P_{19}(x) \\
+ {} & 1.2963262657895548755467112949983444063453578351506262182718 28 \text{ E-26 } P_{21}(x) \\
+ {} & 6.4148551021514157335965882965788329904477400966783833642 7 \text{ E-30 } P_{23}(x) \\
+ {} & 2.6763899251428757868630742853871958302121521857760824566097 6892 \text{ E-33 } P_{25}(x). \quad (35)
\end{aligned}$$

Because of the above-noted correspondence between the power-series versions of $J_0(x)$ and $I_0(x)$ (reversing all of the negative signs in the former to achieve the latter), there is no need to display the $I_1(x)$ power-series version either since the same correspondence applies. We simply give the last set of infinite series,

$$\sum_{L=1}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^{L-2} k^L 2^{-3L-2}\left((-1)^{L+1}+1\right)(2L+1)\binom{L}{\frac{L-1}{2}}\binom{2L}{L}{}_1F_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};-\frac{k^2}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L-1}{2}-h\right)!}$$
$$\times\left[\frac{\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L-1}{2}-h}\left(-\frac{L}{2}\right)_{\frac{L-1}{2}-h}}{\left(\frac{1}{2}-L\right)_{\frac{L-1}{2}-h}}\right]_{k\to i}$$
$$=\sum_{L=1}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^{2L-2} 2^{-3L-2}\left((-1)^{L+1}+1\right)(2L+1)\binom{L}{\frac{L-1}{2}}\binom{2L}{L}{}_1F_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L-1}{2}-h\right)!}$$
$$\times\left[\frac{(-1)^{\frac{L-1}{2}-h}(h+1)_{\frac{L-1}{2}-h}\left(h+\frac{3}{2}\right)_{\frac{L-1}{2}-h}}{\left(h+\frac{L}{2}+1\right)_{\frac{L-1}{2}-h}}\right]$$
$$=\sum_{L=1}^{\infty}{}^{(2)} \frac{\sqrt{\pi}(-1)^{L-1} 2^{-3L-2}\left((-1)^{L+1}+1\right)(2L+1)\binom{L}{\frac{L-1}{2}}\binom{2L}{L}{}_1F_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L-1}{2}-h\right)!}$$
$$\times\left[(-1)^{\frac{L-1}{2}-h}\frac{2^{2h-L+1}\Gamma(L+1)\Gamma\left(h+\frac{L}{2}+1\right)}{\Gamma(2h+2)\Gamma\left(L+\frac{1}{2}\right)}\right]=\frac{2^{-2h-1}}{h!\Gamma(h+2)}. \tag{36}$$

6. Conclusions

We have found the Fourier-Legendre series of modified Bessel functions of the first kind $I_N(kx)$ based on that found by Keating [7] for the Bessel functions of the first kind $J_N(kx)$, and show that Keating's coefficients, comprised of infinite-series, can be reduced to ${}_2F_3$ functions. For $N=0$ and 1 we give numerical values for those coefficients up through x^{24} with 48-digit accuracy, and find that the resultant approximation gives 10 more decimal places of accuracy that the polynomial approximations given in Abramowitz and Stegun [9,22,24] or a roughly double their $-3 \leq x \leq 3$ range at their five-digit accuracy.

Each of these infinite Fourier-Legendre series may be decomposed into an infinite sum of infinite series, by gathering like powers from the Legendre polynomials in each of the terms in the Fourier-Legendre series. We show that each of these infinite sub-series converges to values that are inverse powers of the first five primes $1/\left(2^i 3^j 5^k 7^l 11^m\right)$ for powers up to x^{24}. That these values recapitulate the coefficients of the known power series expansions of Bessel functions[17,25] is no surprise since we expect such an expansion to be unique. But given the relative paucity of infinite series whose values are known (e.g., two dozen pages in Gradshteyn and Ryzhik compared to their 900 pages of known integrals), having even one such to add to the total has the potential to be of use to future researchers. We add an infinite set of infinite series of ${}_2F_3$ functions whose values are now known.

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