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Posted Date: 6 June 2024

doi: 10.20944/preprints202406.0265.v1

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A Joint Limit Theorem for Epstein and Hurwitz Zeta-Functions

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Abstract: In the paper, we prove a joint limit theorem in terms of weak convergence of probability measures on $\mathbb{C}^2$ defined by means of the Epstein $\zeta(s;Q)$ and Hurwitz $\zeta(s,\alpha)$ zeta-functions. The limit measure in the theorem is explicitly given. For this, some restrictions on the matrix Q and parameter α are required. The theorem obtained extends and generalizes Bohr-Jessen's results characterising asymptotic behaviour of the Riemann zeta-function.

Keywords: dirichlet L-function; epstein zeta-function; hurwitz zeta-function; limit theorem; probability haar measure; weak convergence

MSC: 11M46; 11M06

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1. Introduction

Let $\mathbb{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, as usual, denote the sets of primes, positive integers, nonnegative integers, integers, real and complex numbers, respectively, $s=\sigma+it$ a complex variable, $n\in\mathbb{N}$, Q a positive defined $n\times n$ matrix, and $Q[\underline{x}]=\underline{x}^{\mathrm{T}}Q\underline{x}$ for $\underline{x}\in\mathbb{Z}^n$. In [1], Epstein considered a problem to find a zeta-function as general as possible and having the functional equation of the Riemann type. For $\sigma>\frac{n}{2}$, he defined the function

$$\zeta(s;Q)=\sum_{\underline{x}\in\mathbb{Z}^n\setminus\{\underline{0}\}}(Q[\underline{x}])^{-s}.$$

Now this function is called the Epstein zeta-function. It is analytically continuable to the whole complex plane, except for a simple pole at the point $s=\frac{n}{2}$ with residue

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\sqrt{\det Q}},$$

where $\Gamma(s)$ is the Euler gamma-function. Epstein also proved that $\zeta(s;Q)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(s;Q)=\sqrt{\det Q}\,\pi^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2}-s\right)\zeta\left(\frac{n}{2}-s;Q\right)$$

for all $s\in\mathbb{C}$.

It turned out that the Epstein zeta-function is an important number theoretical object having a series of practical applications, for example, in crystallography [2] and mathematical physics, more precisely, in quantum field theory and the Wheeler–DeWitt equation [3], [4].

Value distribution of $\zeta(s;Q)$, as other zeta-functions, is not simple, and was studied by many authors including Hecke [5], Selberg [6], Iwaniec [7], Bateman [8], Fomenko [9], Pańkowski and Nakamura [10]. In [11] and [12], the characterization of asymptotic behaviour of $\zeta(s;Q)$ was given in terms of probabilistic limit theorems. The latter approach for the Riemann zeta-function

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},\quad \sigma>1,$$

was proposed by Bohr in [13], and realized in [14], [15]. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and by meas$A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. For $A \in \mathcal{B}(\mathbb{C})$, define

$$P_{T,\sigma}^{Q}(A) = \frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma+it;Q) \in A\}.$$

Under restrictions that $Q[\underline{x}] \in \mathbb{Z}$ for all $\underline{x} \in \mathbb{Z}^n \setminus \{0\}$, and $n \geq 4$ is even, it was obtained [11] that $P_{T,\sigma}^{Q}$, for $\sigma > \frac{n-1}{2}$, converges weakly to an explicitly given probability measure P_{σ}^{Q} as $T \to \infty$. The discrete version of the latter theorem was given in [12].

The above restrictions on the matrix Q and [9], imply the decomposition

$$\zeta(s;Q) = \zeta(s;E_Q) + \zeta(s;F_Q) \tag{1}$$

with the zeta-function $\zeta(s;E_Q)$ of a certain Eisenstein series, and the zeta-function $\zeta(s;F_Q)$ of a certain cusp form.

Let χ be a Dirichlet character modulo q, and

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

the corresponding Dirichlet L-function having analytic continuation to the whole complex plane if χ is nonprincipal character, and except for a simple pole at the point $s = 1$ if χ is the principal character. Then (1), and [5], [7], lead to the representation

$$\zeta(s;Q) = \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s,\chi_k) L\left(s - \frac{n}{2} + 1, \hat{\chi}_l\right) + \sum_{m=1}^{\infty} \frac{b_Q(m)}{m^s}, \tag{2}$$

where χ_k and $\hat{\chi}_l$ are Dirichlet characters, $a_{kl} \in \mathbb{C}$, $k,l \in \mathbb{N}$, and the series with coefficients $b_Q(m)$ converges absolutely in the half-plane $\sigma > \frac{n-1}{2}$. Thus, the investigation of the function $\zeta(s;Q)$ reduces to that of Dirichlet L-functions which, for $\sigma > 1$, have the Euler product

$$L(s,\chi) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Our aim is to describe by probabilistic terms the joint asymptotic behaviour of the function $\zeta(s;Q)$ and a zeta-function having no Euler's product over primes. For this, the most suitable function is the classical Hurwitz zeta-function. Let $0 < \alpha \leq 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s,\alpha)$ was introduced in [16], and is defined, for $\sigma > 1$, by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}.$$

Moreover, $\zeta(s,\alpha)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1, $\zeta(s,1) = \zeta(s)$, and

$$\zeta\left(s, \frac{1}{2}\right) = \zeta(s)(2^s - 1).$$

Analytic properties of the function $\zeta(s,\alpha)$ depend on the arithmetic nature of the parameter α. Some probabilistic limit theorems for the function $\zeta(s,\alpha)$ can be found, for example, in [17].

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The statement of a joint limit theorem for the functions $\zeta(s;Q)$ and $\zeta(s,\alpha)$ requires some notations. Denote two tori

$$\Omega_1 = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\} \quad \text{and} \quad \Omega_2 = \prod_{m\in\mathbb{N}_0} \{s \in \mathbb{C} : |s| = 1\}.$$

With the product topology and pointwise multiplication, Ω_1 and Ω_2 are compact topological Abelian groups. Therefore,

$$\Omega = \Omega_1 \times \Omega_2$$

again is a compact topological group. Hence, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote elements of Ω by $\omega = (\omega_1, \omega_2)$, $\omega_1 = (\omega_1(p) : p \in \mathbb{P}) \in \Omega_1$ and $\omega_2 = (\omega_2(m) : m \in \mathbb{N}_0) \in \Omega_2$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define, for $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$, the $\mathbb{C}^2$-valued random element

$$\underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) = (\zeta(\sigma_1, \omega_1; Q), \zeta(\sigma_2, \omega_2, \alpha)),$$

where $\underline{\sigma} = (\sigma_1, \sigma_2)$,

$$\begin{aligned}\zeta(\sigma_1, \omega_1; Q) &= \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}\omega_1(k)\omega_1(l)}{k^{\sigma_1} l^{\sigma_1}} L(\sigma_1, \omega_1, \chi_k) L\Big(\sigma_1 - \frac{n}{2} + 1, \omega_1, \hat{\chi}_l\Big) \\ &+ \sum_{m=1}^{\infty} \frac{b_Q(m)\omega_1(m)}{m^{\sigma_1}},\end{aligned}$$

with

$$L(\sigma_1, \omega_1, \chi_k) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\chi_k(p)\omega_1(p)}{p^{\sigma_1}}\right)^{-1},$$

$$L\Big(\sigma_1 - \frac{n}{2} + 1, \omega_1, \hat{\chi}_l\Big) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\hat{\chi}_l(p)\omega_1(p)}{p^{\sigma_1 - \frac{n}{2} + 1}}\right)^{-1},$$

$$\omega_1(m) = \prod_{\substack{p^r | m \\ p^{r+1} \nmid m}} \omega_1^r(p), \quad m \in \mathbb{N},$$

and

$$\zeta(\sigma_2, \alpha, \omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^{\sigma_2}}, \quad m \in \mathbb{N}.$$

Let

$$L(\mathbb{P}, \alpha) = \{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0)\}.$$

Moreover, denote by $P_{\underline{\zeta},\underline{\sigma}}$ the distribution of the random element $\underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q)$, i.e.,

$$P_{\underline{\zeta},\underline{\sigma}}(A) = m_H\Big\{\omega \in \Omega : \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{C}^2).$$

The main the result of the paper is the following joint limit theorem of Bohr-Jessen type for the functions $\zeta(s;Q)$ and $\zeta(s,\alpha)$.

For brevity, we set

$$\underline{\zeta}(\underline{\sigma} + it, \alpha; Q) = (\zeta(\sigma_1 + it; Q), \zeta(\sigma_2 + it, \alpha)).$$

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Theorem 1. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and $\sigma_1 > \frac{n-1}{2}$, $\sigma_2 > \frac{1}{2}$. Then*

$$P_{T,\underline{\zeta},\underline{\sigma}}(A) = \frac{1}{T}\text{meas}\Big\{t \in [0,T] : \underline{\zeta}(\underline{\sigma}+it,\alpha;Q) \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{C}^2),$$

converges weakly to the measure $P_{\underline{\zeta},\underline{\sigma}}$ as $T \to \infty$.

For example, if the parameter α is transcendental, then the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$.

We divide the proof of Theorem 1 into several lemmas which are limit theorems in some spaces for certain auxiliary objects. The important place of the proof is the identification of the limit measure.

2. Limit Lemma on Ω

For $A \in \mathcal{B}(\Omega)$, set

$$P_{T,\Omega}(A) = \frac{1}{T}\text{meas}\Big\{t \in [0,T] : \Big(\Big(p^{-it}, p \in \mathbb{P}\Big), \Big((m+\alpha)^{-it}, m \in \mathbb{N}_0\Big)\Big) \in A\Big\}.$$

Lemma 1. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Then $P_{T,\Omega}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. The characters of the torus Ω are of the form

$$\prod_{p\in\mathbb{P}}{}^{*}\, \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}{}^{*}\, \omega_2^{l_m}(m),$$

where the star "$*$" shows that only a finite number of integers k_p and l_m are non-zeros. Therefore, the Fourier transform $F_{T,\Omega}(\underline{k},\underline{l})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, $\underline{l} = (l_m : l_m \in \mathbb{Z}, m \in \mathbb{N}_0)$, is given by

$$F_{T,\Omega}(\underline{k},\underline{l}) = \int_{\Omega} \left(\prod_{p\in\mathbb{P}}{}^{*}\, \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}{}^{*}\, \omega_2^{l_m}(m)\right) \mathrm{d}P_{T,\Omega}.$$

Thus, in view of the definition of $P_{T,\Omega}$,

$$\begin{aligned} F_{T,\Omega}(\underline{k},\underline{l}) &= \frac{1}{T}\int_0^T \left(\prod_{p\in\mathbb{P}}{}^{*}\, p^{-itk_p} \prod_{m\in\mathbb{N}_0}{}^{*}\, (m+\alpha)^{-itl_m}\right) \mathrm{d}t \\ &= \frac{1}{T}\int_0^T \exp\left\{-it\left(\sum_{p\in\mathbb{P}}{}^{*}\, k_p \log(p) + \sum_{m\in\mathbb{N}_0}{}^{*}\, l_m \log(m+\alpha)\right)\right\} \mathrm{d}t. \end{aligned} \tag{3}$$

We have to show that $F_{T,\Omega}(\underline{k},\underline{l})$ converges to the Fourier transform of the measure m_H as $T \to \infty$, i. e., to

$$F_{\Omega}(\underline{k},\underline{l}) = \begin{cases} 1 & \text{if } (\underline{k},\underline{l}) = (\underline{0},\underline{0}), \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where $\underline{0} = (0,\dots,0,\dots)$. Since the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$,

$$\mathcal{L}(\underline{k},\underline{l}) \stackrel{def}{=} \sum_{p\in\mathbb{P}}{}^{*}\, k_p \log(p) + \sum_{m\in\mathbb{N}_0}{}^{*}\, l_m \log(m+\alpha) \neq 0$$

for $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$. Therefore, in this case, equality (3) gives

$$F_{T,\Omega}(\underline{k}, \underline{l}) = \frac{1 - \exp\{-iT\mathcal{L}(\underline{k}, \underline{l})\}}{iT\mathcal{L}(\underline{k}, \underline{l})}.$$

Thus, for $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$,

$$\lim_{T\to\infty} F_{T,\Omega}(\underline{k}, \underline{l}) = 0.$$

Since, obviously, $F_{T,\Omega}(\underline{0}, \underline{0}) = 1$, this show that $F_{T,\Omega}(\underline{k}, \underline{l})$ converges to (4) as $T \to \infty$. The lemma is proved. □

Lemma 1 is a starting point for the proof of limit lemmas in $\mathbb{C}^2$ for certain objects given by absolutely convergent Dirichlet series.

3. Absolutely Convergent Series

Fix $\beta > \frac{1}{2}$, and, for $N \in \mathbb{N}$, set

$$u_N(m) = \exp\left\{-\left(\frac{m}{N}\right)^{\beta}\right\}, \quad m \in \mathbb{N},$$

and

$$u_N(m, \alpha) = \exp\left\{-\left(\frac{m+\alpha}{N}\right)^{\beta}\right\}, \quad m \in \mathbb{N}_0.$$

Define

$$L_N\left(s - \frac{n}{2} + 1, \hat{\chi}_l\right) = \sum_{m=1}^{\infty} \frac{\hat{\chi}_l(m) u_N(m)}{m^{s-\frac{n}{2}+1}},$$

$$L_N\left(s - \frac{n}{2} + 1, \omega_1, \hat{\chi}_l\right) = \sum_{m=1}^{\infty} \frac{\hat{\chi}_l(m)\omega_1(m) u_N(m)}{m^{s-\frac{n}{2}+1}},$$

and

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{u_N(m, \alpha)}{(m+\alpha)^s},$$

$$\zeta(s, \omega_2, \alpha) = \sum_{m=0}^{\infty} \frac{\omega_2(m) u_N(m, \alpha)}{(m+\alpha)^s}.$$

Since $u_N(m)$ and $u_N(m, \alpha)$ decrease with respect to m exponentially, the above series are absolutely convergent for $\sigma > \sigma_0$ with arbitrary fixed finite σ_0. For $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$, let

$$\underline{\zeta}_N(\underline{\sigma}, \alpha; Q) = (\zeta_N(\sigma_1; Q), \zeta_N(\sigma_2, \alpha))$$

with

$$\zeta_N(\sigma_1; Q) = \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}}{k^{\sigma_1} l^{\sigma_1}} L(\sigma_1, \chi_k) L_N\left(\sigma_1 - \frac{n}{2} + 1, \hat{\chi}_l\right) + \sum_{m=1}^{\infty} \frac{b_Q(m)}{m^{\sigma_1}},$$

and

$$\underline{\zeta}_N(\underline{\sigma}, \omega, \alpha; Q) = (\zeta_N(\sigma_1, \omega_1; Q), \zeta_N(\sigma_2, \omega_2, \alpha))$$

with

$$\zeta_N(\sigma_1, \omega_1; Q) = \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}\omega_1(k)\omega_1(l)}{k^{\sigma_1} l^{\sigma_1}} L(\sigma_1, \omega_1, \chi_k) L_N\left(\sigma_1 - \frac{n}{2} + 1, \omega_1, \chi_k\right) + \sum_{m=1}^{\infty} \frac{b_Q(m)\omega_1(m)}{m^{\sigma_1}}.$$

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For $A \in \mathcal{B}(\mathbb{C}^2)$, define

$$P^{Q,\alpha}_{T,N,\underline{\sigma}}(A) = \frac{1}{T}\text{meas}\Big\{t \in [0,T] : \underline{\zeta}_N(\underline{\sigma}+it,\alpha;Q) \in A\Big\}$$

and

$$P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}(A) = \frac{1}{T}\text{meas}\Big\{t \in [0,T] : \underline{\zeta}_N(\underline{\sigma}+it,\omega,\alpha;Q) \in A\Big\}.$$

This section is devoted to weak convergence of $P^{Q,\alpha}_{T,N,\underline{\sigma}}$ and $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$ as $T \to \infty$. Let the mapping $v^{Q,\alpha}_{N,\underline{\sigma}} : \Omega \to \mathbb{C}^2$ be given by

$$v^{Q,\alpha}_{N,\underline{\sigma}}(\omega) = \underline{\zeta}_N(\underline{\sigma},\omega,\alpha;Q), \quad \sigma_1 > \frac{n-1}{2}, \quad \sigma_2 > \frac{1}{2},$$

and $V^{Q,\alpha}_{N,\underline{\sigma}} = m_H\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1}$, where, for $A \in \mathcal{B}(\mathbb{C}^2)$,

$$V^{Q,\alpha}_{N,\underline{\sigma}}(A) = m_H\left(\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1}A\right).$$

Since all Dirichlet series in the definition of $\underline{\zeta}_N(\underline{\sigma},\omega,\alpha;Q)$ are absolutely convergent in the considered region, the mapping $v^{Q,\alpha}_{N,\underline{\sigma}}$ is continuous, hence $(\mathcal{B}(\Omega),\mathcal{B}(\mathbb{C}^2))$-measurable. Therefore, the probability measure $V^{Q,\alpha}_{N,\underline{\sigma}}$ is defined correctly, see, for example, [18], Section 5.

Lemma 2. *Under hypotheses of Theorem 1, $P^{Q,\alpha}_{T,N,\underline{\sigma}}$ and $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$ both converge weakly to the same probability measure $V^{Q,\alpha}_{N,\underline{\sigma}}$ as $T \to \infty$.*

Proof. We apply the principle of preservation of weak convergence under continuous mappings, see Section 5 of [18]. By the definitions of $P^{Q,\alpha}_{T,N,\underline{\sigma}}$, $P_{T,\Omega}$ and $v^{Q,\alpha}_{N,\underline{\sigma}}$, we have

$$\begin{aligned} P^{Q,\alpha}_{T,N,\underline{\sigma}}(A) \quad &= \quad \frac{1}{T}\text{meas}\Big\{t \in [0,T] : \Big(\big(p^{-it}, p \in \mathbb{P}\big), \big((m+\alpha)^{-it}, m \in \mathbb{N}_0\big)\Big) \in (v^{Q,\alpha}_{N,\underline{\sigma}})^{-1}A\Big\} \\ & \quad\; P_{T,\Omega}\Big((v^{Q,\alpha}_{N,\underline{\sigma}})^{-1}A\Big) \end{aligned}$$

for every $A \in \mathcal{B}(\mathbb{C}^2)$. Thus, $P^{Q,\alpha}_{T,N,\underline{\sigma}} = P^{Q,\alpha}_{T,\Omega}\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1}$. This, continuity of $v^{Q,\alpha}_{N,\underline{\sigma}}$, Lemma 1 and Theorem 5.1 of [18] imply that $P^{Q,\alpha}_{T,N,\underline{\sigma}}$ converges to $V^{Q,\alpha}_{N,\underline{\sigma}}$ as $T \to \infty$.

It remains to show that $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$ also converges to $V^{Q,\alpha}_{N,\underline{\sigma}}$ as $T \to \infty$. Let $\hat{\omega} \in \Omega$, and the mapping $w^{Q,\alpha}_{N,\underline{\sigma}} : \Omega \to \mathbb{C}^2$ be given by

$$w^{Q,\alpha}_{N,\underline{\sigma}}(\omega) = \zeta_N(\underline{\sigma},\omega\hat{\omega},\alpha;Q).$$

Thus, we have that

$$w^{Q,\alpha}_{N,\underline{\sigma}}(\omega) = v^{Q,\alpha}_{N,\underline{\sigma}}(\omega)(a(\omega)), \tag{5}$$

where $a : \Omega \to \Omega$ is given by $a(\omega) = \omega\hat{\omega}$. By the same lines as in the case of $P^{Q,\alpha}_{T,N,\underline{\sigma}}$, we find that $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$ converges weakly to the measure $W^{Q,\alpha}_{N,\underline{\sigma}} = m_H\left(w^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1}$. However, by (5) and invariance of the Haar measure, we obtain

$$W^{Q,\alpha}_{N,\underline{\sigma}} = m_H\left(v^{Q,\alpha}_{N,\underline{\sigma}}(a)\right)^{-1} = \left(m_H a^{-1}\right)\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1} = m_H\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1} = V^{Q,\alpha}_{N,\underline{\sigma}}.$$

This completes the proof of the lemma. □

4. Approximation Lemmas

In this section, we approximate $\underline{\zeta}(\underline{\sigma}+it,\alpha;Q)$ by $\underline{\zeta}_N(\underline{\sigma}+it,\alpha;Q)$ and $\underline{\zeta}(\underline{\sigma}+it,\omega,\alpha;Q)$ by $\underline{\zeta}_N(\underline{\sigma}+it,\omega,\alpha;Q)$.

Let, for $\underline{z}_1=(z_{11},z_{12}), \underline{z}_2=(z_{21},z_{22})\in\mathbb{C}^2$,

$$\rho(\underline{z}_1,\underline{z}_2)=\Big(|z_{11}-z_{21}|^2+|z_{12}-z_{22}|^2\Big)^{1/2}.$$

Lemma 3. *For $\sigma_1>\frac{n-1}{2}$ and $\sigma_2>\frac{1}{2}$,*

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\rho\Big(\underline{\zeta}(\underline{\sigma}+it,\alpha;Q),\underline{\zeta}_N(\underline{\sigma}+it,\alpha;Q)\Big)\mathrm{d}t=0,$$

and, for almost all $\omega\in\Omega$,

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\rho\Big(\underline{\zeta}(\underline{\sigma}+it,\omega,\alpha;Q),\underline{\zeta}_N(\underline{\sigma}+it,\omega,\alpha;Q)\Big)\mathrm{d}t=0.$$

Proof. The first equality of the lemma is a corollary of the equalities

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T|\zeta(\sigma_1+it;Q)-\zeta_N(\sigma_1+it;Q)|\mathrm{d}t=0$$

and

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T|\zeta(\sigma_2+it,\alpha)-\zeta_N(\sigma_2+it,\alpha)|\mathrm{d}t=0. \tag{6}$$

The first of them was obtained in [11], Lemma 4. Its proof is based on the integral representation

$$L_N\Big(\sigma_1-\frac{n}{2}+1,\hat{\chi}_l\Big)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}L\Big(\sigma_1-\frac{n}{2}+1+z,\hat{\chi}_l\Big)l_N(z)\mathrm{d}z,$$

where

$$l_N(z)=\frac{1}{\beta}\Gamma\Big(\frac{z}{\beta}\Big)N^z,$$

and the mean square estimate for Dirichlet L functions in the half plane $\sigma>\frac{1}{2}$.

For the proof of (6), we use, for $\sigma_2>\frac{1}{2}$, the representation

$$\zeta_N(s,\alpha)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\zeta(s+z,\alpha)l_N(z)\mathrm{d}z. \tag{7}$$

Since $\sigma_2>\frac{1}{2}$, there exists $\epsilon>0$ such that $\frac{1}{2}+\epsilon<\sigma_2$. Let $\beta=\sigma_2$, and $\beta_1=\frac{1}{2}+\epsilon-\sigma_2$. The integrand in (7) has simple poles $z=0$ and $z=1-s$ in the strip $\beta_1<\mathrm{Re}z<\beta$. Therefore, by the residue theorem and (7),

$$\zeta_N(\sigma_2+it,\alpha)-\zeta(\sigma_2+it,\alpha)=\frac{1}{2\pi i}\int_{\beta_1-i\infty}^{\beta_1+i\infty}\zeta(\sigma_2+it+z,\alpha)l_N(z)\mathrm{d}z+l_N(1-\sigma_2-it).$$

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Hence,

$$\zeta_N(\sigma_2+it,\alpha)-\zeta(\sigma_2+it,\alpha) \ll \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\epsilon+it+i\tau,\alpha\right)\right|\left|l_N\left(\frac{1}{2}+\epsilon-\sigma_2+i\tau\right)\right|\mathrm{d}\tau + |l_N(1-\sigma_2-it)|$$

and

$$\begin{aligned}&\frac{1}{T}\int_0^T|\zeta(\sigma_2+it,\alpha)-\zeta_N(\sigma_2+it,\alpha)|\mathrm{d}t\\ &\ll \int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\epsilon+it+i\tau,\alpha\right)\right|\mathrm{d}t\right)\left|l_N\left(\frac{1}{2}+\epsilon-\sigma_2+i\tau,\right)\right|\mathrm{d}\tau\\ &+\frac{1}{T}\int_0^T|l_N(1-\sigma_2-it)|\mathrm{d}t \stackrel{def}{=} I_1(T,N)+I_2(T,N).\end{aligned} \tag{8}$$

It is well known, see, for example, [17], that, for $\frac{1}{2}<\sigma<1$,

$$\int_{-T}^{T}|\zeta(\sigma+it,\alpha)|^2\mathrm{d}t \ll_{\sigma,\alpha} T.$$

Therefore, for large T,

$$\begin{aligned}&\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\epsilon+it+i\tau,\alpha\right)\right|\mathrm{d}\tau \ll \left(\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\epsilon+it+i\tau,\alpha\right)\right|^2\mathrm{d}t\right)^{1/2}\\ &\le \left(\frac{1}{T}\int_{-|\tau|}^{T+|\tau|}\left|\zeta\left(\frac{1}{2}+\epsilon+it,\alpha\right)\right|^2\mathrm{d}t\right)^{1/2} \ll_{\epsilon,\alpha} \left(\frac{T+|\tau|}{T}\right)^{1/2}\\ &\ll_{\epsilon,\alpha}(1+|\tau|)^{1/2}.\end{aligned} \tag{9}$$

For the gamma-function, the estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c>0, \tag{10}$$

uniformly for σ in every finite interval is valid. Therefore,

$$l_N\left(\frac{1}{2}+\epsilon-\sigma_2+i\tau\right) \ll_{\sigma_2} N^{\frac{1}{2}+\epsilon-\sigma_2}\exp\left\{-\frac{c}{\sigma_2}|\tau|\right\}.$$

This together with (9) shows that

$$I_1(T,N) \ll_{\epsilon,\sigma_2,\alpha} N^{\frac{1}{2}+\epsilon-\sigma_2}\int_{-\infty}^{\infty}(1+|\tau|)^{1/2}\exp\left\{-\frac{c}{\sigma_2}|\tau|\right\}\mathrm{d}\tau \ll_{\epsilon,\sigma_2,\alpha} N^{\frac{1}{2}+\epsilon-\sigma_2}. \tag{11}$$

By (10) again,

$$l_N(1-\sigma_2-it) \ll_{\sigma_2} N^{1-\sigma_2}\exp\left\{-\frac{c}{\sigma_2}|t|\right\},$$

and thus,

$$I_2(T,N) \ll_{\sigma_2} N^{1-\sigma_2} \int\limits_0^\infty \exp\left\{-\frac{c}{\sigma_2}|t|\right\} \mathrm{d}t \ll_{\sigma_2} N^{1-\sigma_2} \frac{\log T}{T}.$$

Since $\frac{1}{2} + \epsilon - \sigma_2 < 0$, this, (11) and (8) prove (6).

The second equality of the lemma follows from the following two equalities

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T |\zeta(\sigma_1 + it, \omega_1; Q) - \zeta_N(\sigma_1 + it, \omega_1; Q)| \mathrm{d}t = 0$$

and

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T |\zeta(\sigma_2 + it, \alpha, \omega_2) - \zeta_N(\sigma_2 + it, \alpha, \omega_2)| \mathrm{d}t = 0$$

for almost all $\omega_1 \in \Omega_1$ and almost all $\omega_2 \in \Omega_2$, respectively.

The first of them has been obtained in [11], Lemma 7, while the second is proved similarly to equality (6) by using the representation, for $\sigma > \frac{1}{2}$,

$$\zeta_N(s, \alpha, \omega) = \frac{1}{2\pi i} \int\limits_{\beta - i\infty}^{\beta + i\infty} \zeta(s + z, \alpha, \omega) l_N(z) \mathrm{d}z$$

as well as the bound, for $\frac{1}{2} < \sigma < 1$ and almost all $\omega_2 \in \Omega_2$,

$$\int\limits_{-T}^T |\zeta(\sigma + it, \alpha, \omega_2)|^2 \mathrm{d}t \ll_{\sigma,\alpha} T,$$

see, for example, [17]. □

5. Tightness

Let $\{P\}$ be a family of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Remind that the family $\{P\}$ is called tight if, for every $\epsilon > 0$, there exists a compact set $K \subset \mathbb{X}$ such that

$$P(K) > 1 - \epsilon$$

for all $P \in \{P\}$. The family $\{P\}$ is relatively compact if every sequence $\{P_n\} \subset \{P\}$ contains a subsequence $\{P_n\}$ weakly convergent to a certain probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ as $n \to \infty$.

A property of relative compactness is useful for investigation of weak convergence of probability measures. By the classical Prokhorov theorem, see, for example, [18], every tight family $\{P\}$ is relatively compact as well. Therefore, often it is convenient to know the tightness of considered family. In our case, this concerns the measure $V_N^{Q,\alpha}$, $N \in \mathbb{N}$.

Lemma 4. *The family $\{V_N^{Q,\alpha} : N \in \mathbb{N}\}$ is tight.*

Proof. Consider the marginal measures of the measure $V_N^{Q,\alpha}$, i. e., for $A \in \mathcal{B}(\mathbb{C})$,

$$V_{N,\sigma_1}^Q(A) = V_{N,\underline{\sigma}}^{Q,\alpha}(A \times \mathbb{C})$$

and

$$V_{N,\sigma_2}^\alpha(A) = V_{N,\underline{\sigma}}^{Q,\alpha}(\mathbb{C} \times A).$$

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It is easily seen that the measure V^{Q}_{N,σ_1} appears in the process related to weak convergence of the measure $P^{Q}_{T,\underline{\sigma}}$ and the measure V^{α}_{N,σ_2} is used for study of

$$P^{\alpha}_{T,\sigma_2}(A) = \frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma_2 + it, \alpha) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

Thus, in [17], it was obtained the tightness of the family $\{V^{Q}_{N,\sigma_1} : n \in \mathbb{N}\}$, i. e., for every $\epsilon > 0$, there exists a compact set $K_1 \subset \mathbb{C}$ such that

$$V^{Q}_{N,\sigma_1}(K_1) > 1 - \frac{\epsilon}{2} \tag{12}$$

for all $N \in \mathbb{N}$. We will prove a similar inequality for V^{α}_{N,σ_2}.

Repeating the proofs of Lemmas 1 and 2 leads to weak convergence of

$$P^{\alpha}_{T,N,\sigma_2}(A) = \frac{1}{T}\text{meas}\{t \in [0,T] : \zeta_N(\sigma_2 + it, \alpha) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

to V^{α}_{N,σ_2} as $T \to \infty$. Introduce a random variable θ_T defined on a certain probability space $(\Xi, \mathcal{A}, \mu)$ and uniformly distributed in $[0,T]$. Define

$$\xi^{\alpha}_{T,N,\sigma_2} = \xi^{\alpha}_{T,N,\sigma_2}(\sigma) = \zeta_N(\sigma_2 + i\theta_T, \alpha),$$

and denote by $\xrightarrow{D}$ the convergence in distribution. Then, the above remark can be written as

$$\xi^{\alpha}_{T,N,\sigma_2} \xrightarrow[T\to\infty]{D} \xi^{\alpha}_{N,\sigma_2}, \tag{13}$$

where $\xi^{\alpha}_{N,\sigma_2}$ is a random variable with distribution V^{α}_{N,σ_2}. Since the series for $\zeta_N(s,\alpha)$ is absolutely convergent, we have

$$\begin{aligned} \sup_{N\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta_N(\sigma_2 + it, \alpha)|^2 \mathrm{d}t &= \sup_{N\in\mathbb{N}} \sum_{m=1}^{\infty} \frac{v_N^2(m,\alpha)}{(m+\alpha)^{2\sigma_2}} \le \sum_{m=1}^{\infty} \frac{1}{(m+\alpha)^{2\sigma_2}} \\ &\le C_{\alpha,\sigma_2} < \infty. \end{aligned}$$

Then, in view of (13),

$$\begin{aligned} \sup_{N\in\mathbb{N}} \mu\left\{\left|\xi^{\alpha}_{N,\sigma_2}\right| \ge \sqrt{C_{\alpha,\sigma_2}\left(\frac{\epsilon}{2}\right)^{-1}}\right\} &= \sup_{N\in\mathbb{N}} \limsup_{T\to\infty} \mu\left\{\left|\xi^{\alpha}_{T,N,\sigma_2}\right| \ge \sqrt{C_{\alpha,\sigma_2}\left(\frac{\epsilon}{2}\right)^{-1}}\right\} \\ &\le \sup_{N\in\mathbb{N}} \frac{1}{C_{\alpha,\sigma_2}}\frac{\epsilon}{2} \limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta_N(\sigma_2 + it, \alpha)|^2 \mathrm{d}t \\ &\le \frac{\epsilon}{2}. \end{aligned} \tag{14}$$

Let $K_2 = \left\{z \in \mathbb{C} : |z| \le \sqrt{C_{\alpha,\sigma_2}\left(\frac{\epsilon}{2}\right)^{-1}}\right\}$. Then K_2 is a compact set in $\mathbb{C}$, and, by (14),

$$V^{\alpha}_{N,\sigma_2}(K_1) > 1 - \frac{\epsilon}{2} \tag{15}$$

for all $N \in \mathbb{N}$.

Now, define $K = K_1 \times K_2$. Then K is a compact set in $\mathbb{C}^2$. Moreover, taking into account (12) and (15) gives

$$V^{Q,\alpha}_{N,\underline{\sigma}}(\mathbb{C}^2 \setminus K)) \le V^{Q}_{N,\sigma_1}(\mathbb{C} \setminus K_1) + V^{\alpha}_{N,\sigma_2}(\mathbb{C} \setminus K_2) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

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for all $N \in \mathbb{N}$. Thus, $V^{Q,\alpha}_{N,\underline{\sigma}}(K) \geq 1 - \epsilon$ for all $N \in \mathbb{N}$, and the proof is completed. □

6. Limit Theorems

Now we are ready to prove weak convergence for $P_{T,\underline{\zeta},\underline{\sigma}}$ and

$$P^{\Omega}_{T,\underline{\zeta},\underline{\sigma}}(A) = \frac{1}{T}\text{meas}\Big\{t \in [0,T] : \underline{\zeta}(\underline{\sigma} + it, \omega, \alpha; Q) \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{C}^2).$$

Proposition 1. *Suppose that the set $L(\mathbb{P};\alpha)$ is linearly independent over $\mathbb{Q}$, and $\sigma_1 > \frac{n-1}{2}$, $\sigma_2 > \frac{1}{2}$. Then $P_{T,\underline{\zeta},\underline{\sigma}}$ and $P^{\Omega}_{T,\underline{\zeta},\underline{\sigma}}$, for almost all $\omega \in \Omega$, both converge to the same probability measure $P_{\underline{\sigma}}$ as $T \to \infty$.*

Proof. Let θ_T be the same random variable as in Section 5. Introduce the $\mathbb{C}^2$-valued random elements

$$\underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}} = \underline{\zeta}_N(\underline{\sigma} + i\theta_T, \alpha; Q)$$

and

$$\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}} = \underline{\zeta}(\underline{\sigma} + i\theta_T, \alpha; Q).$$

Moreover, let $\underline{\xi}^{Q,\alpha}_{N,\underline{\sigma}}$ be $\mathbb{C}^2$-valued random element having the distribution $V^{Q,\alpha}_{N,\underline{\sigma}}$. Then the assertion of Lemma 2 for $P^{Q,\alpha}_{T,N,\underline{\sigma}}$ can be written as

$$\underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}} \xrightarrow[T\to\infty]{D} \underline{\xi}^{Q,\alpha}_{N,\underline{\sigma}}. \tag{16}$$

Lemma 4 together with Prokhorov's theorem implies that the family $\{V^{Q,\alpha}_{N,\underline{\sigma}} : N \in \mathbb{N}\}$ is relatively compact. Hence, we have a sequence $\{V^{Q,\alpha}_{N_r,\underline{\sigma}}\} \subset \{V^{Q,\alpha}_{N,\underline{\sigma}}\}$ and a probability measure $V^{Q,\alpha}_{\underline{\sigma}}$ on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ such that

$$\underline{\xi}^{Q,\alpha}_{N_r,\underline{\sigma}} \xrightarrow[r\to\infty]{D} V^{Q,\alpha}_{\underline{\sigma}}. \tag{17}$$

Now, it is a time for application of Lemma 3. Thus, using Lemma 3, we obtain that, for every $\epsilon > 0$,

$$\begin{aligned}
&\lim_{r\to\infty}\limsup_{T\to\infty} \mu\Big\{\rho\Big(\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}, \underline{\xi}^{Q,\alpha}_{T,N_r,\underline{\sigma}}\Big) \geq \epsilon\Big\} \\
= &\lim_{r\to\infty}\sup_{T\to\infty} \frac{1}{T}\text{meas}\{t \in [0,T] : \rho(\underline{\zeta}(\underline{\sigma} + it, \alpha; Q), \underline{\zeta}_{N_r}(\underline{\sigma} + it, \alpha; Q)) \geq \epsilon\} \\
\leq &\lim_{r\to\infty}\sup_{T\to\infty} \frac{1}{\epsilon T}\int_0^T \rho(\underline{\zeta}(\underline{\sigma} + it, \alpha; Q), \underline{\zeta}_{N_r}(\underline{\sigma} + it, \alpha; Q))\mathrm{d}t = 0.
\end{aligned}$$

This equality, and relations (16) and (17) show that Theorem 4 from [18] can be applied for the random elements $\underline{\xi}^{Q,\alpha}_{T,N_r,\underline{\sigma}}$, $\underline{\xi}^{Q,\alpha}_{N_r,\underline{\sigma}}$ and $\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}$. Thus, we have

$$\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}} \xrightarrow[T\to\infty]{D} V^{Q,\alpha}_{\underline{\sigma}}, \tag{18}$$

in other words, $P_{T,\underline{\zeta},\underline{\sigma}}$ converges weakly to the measure $V^{Q,\alpha}_{\underline{\sigma}}$ as $T \to \infty$.

It remains to prove that $P^{\Omega}_{T,\underline{\zeta},\underline{\sigma}}$, as $T \to \infty$, converges weakly to the measure $V^{Q,\alpha}_{\underline{\sigma}}$ as well. Relation (18) shows that the limit measure $V^{Q,\alpha}_{\underline{\sigma}}$ does not depend on the sequence $\{V^{Q,\alpha}_{N_r,\underline{\sigma}}\}$. Since the family $\{V^{Q,\alpha}_{N,\underline{\sigma}}\}$ is relatively compact, the latter remark implies the relation

$$\underline{\xi}^{Q,\alpha}_{N,\underline{\sigma}} \xrightarrow[N\to\infty]{D} V^{Q,\alpha}_{\underline{\sigma}}. \tag{19}$$

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Define the random elements
$$\underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}}(\omega) = \underline{\zeta}_N(\underline{\sigma} + i\theta_T, \omega, \alpha; Q)$$
and
$$\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}(\omega) = \underline{\zeta}(\underline{\sigma} + i\theta_T, \omega, \alpha; Q).$$
By Lemma 2, for $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$, the relation
$$\underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}}(\omega) \xrightarrow[T\to\infty]{D} \underline{\xi}^{Q,\alpha}_{N,\underline{\sigma}} \tag{20}$$
holds. Moreover, Lemma 3, for every $\epsilon > 0$ and almost all $\omega \in \Omega$, implies
$$\begin{aligned}&\lim_{N\to\infty}\limsup_{T\to\infty} \mu\left\{\rho\left(\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}(\omega), \underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}}(\omega)\right) \geq \epsilon\right\} \\ \leq\ &\lim_{N\to\infty}\limsup_{T\to\infty} \frac{1}{\epsilon T}\int_0^T \rho\left(\underline{\zeta}(\underline{\sigma} + it, \omega, \alpha; Q), \underline{\zeta}_N(\underline{\sigma} + it, \omega, \alpha; Q)\right) \mathrm{d}t = 0.\end{aligned}$$
This, (19), (20) and Theorem 4.2 of [18] yield, for almost all $\omega \in \Omega$, the relation
$$\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}(\omega) \xrightarrow[T\to\infty]{D} V^{Q,\alpha}_{\underline{\sigma}},$$
i. e., that $P^{\Omega}_{T,\underline{\zeta},\underline{\sigma}}$, as $T \to \infty$, converges weakly to $V^{Q,\alpha}_{\underline{\sigma}}$. The proposition is proved. □

7. Proof of Theorem

Let $t \in \mathbb{R}$ and $e_t = \big((p^{-it} : p \in \mathbb{P}), ((m+\alpha)^{-it}, m \in \mathbb{N}_0)\big)$. Obviously, e_t is an element of Ω. Using e_t, define a transformation $g_t : \Omega \to \Omega$ by
$$g_t(\omega) = e_t\omega, \quad \omega \in \Omega.$$
In virtue of the invariance of the Haar measure m_H, g_t is a measurable measure preserving transformation on Ω. Then $\mathcal{G}_t = \{g_t : t \in \mathbb{R}\}$ is the one-parameter group of transformations on Ω. A set $A \in \mathcal{B}(\Omega)$ is invariant with respect to $\mathcal{G}_t$ if, for every $t \in \mathbb{R}$ the sets $A_t = g_t(A)$ and A can differ one from another at most by a set of m_H-measure zero. All invariant sets form a σ-subfield of $\mathcal{B}(\Omega)$. We say that the group $\mathcal{G}_t$ is ergodic if its σ-field of invariant sets consists only of sets having m_H-measure 1 or 0.

Lemma 5. *Suppose that the set $L(\mathbb{P}, \alpha)$ is linearly independent over $\mathbb{Q}$. Then the group $\mathcal{G}_t$ is ergodic.*

Proof. We fix an invariant set A of the group $\mathcal{G}_t$, and consider its indicator function I_A. We will prove that, for almost all $\omega \in \Omega$, $I_A(\omega) = 1$ or $I_A(\omega) = 0$. For this, we will use the Fourier transform method.

By the proof of Lemma 1, we know that characters χ of Ω are of the form
$$\chi(\omega) = \prod_{p\in\mathbb{P}}^{*} \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}^{*} \omega_2^{l_m}(m),$$
where the star "$*$" indicates that only a finite number of integers k_p and l_m are non-zeros. Hence, if χ is a non-trivial character,
$$\chi(g_t) = \prod_{p\in\mathbb{P}}^{*} p^{-itk_p} \prod_{m\in\mathbb{N}_0}^{*} (m+\alpha)^{-itl_m}$$

$$= \exp\left\{-it\left(\sum_{p\in\mathbb{P}}^{*} k_p \log(p) + \sum_{m\in\mathbb{N}_0}^{*} l_m \log(m+\alpha)\right)\right\}.$$

Since χ is a non-principal character, i. e., $\chi(\omega) \not\equiv 1$. The linear independence of the set $L(\mathbb{P},\alpha)$ shows that

$$\sum_{p\in\mathbb{P}}^{*} k_p \log(p) + \sum_{m\in\mathbb{N}_0}^{*} l_m \log(m+\alpha) \neq 0$$

for $k_p \not\equiv 0$ and $l_m \not\equiv 0$. These remarks implies the existence of $t_0 \neq 0$ such that

$$\chi(g_{t_0}) \neq 1. \tag{21}$$

Moreover, by the invariance of A, for almost all $\omega \in \Omega$,

$$I_A(g_{t_0}) = I_A(\omega). \tag{22}$$

Let $\hat{h}$ denotes the Fourier transform of h. Then, by (22), invariant of m_H and multiplicativity of characters

$$\hat{I}_A(\chi) = \int_{\Omega} I_A(\omega)\chi(\omega)\mathrm{d}m_H = \chi(g_{t_0}) \int_{\Omega} I_A(\omega)\chi(\omega)\mathrm{d}m_H = \chi(g_{t_0})\hat{I}_A(\chi).$$

Thus, (21) gives

$$\hat{I}_A(\chi) = 0. \tag{23}$$

Now, suppose that $\chi(\omega) \equiv 1$, and $\hat{I}_A(\chi) = a$. Then

$$\hat{a}(\chi) = \int_{\Omega} a(\chi)\chi(\omega)\mathrm{d}m_H = a\int_{\Omega} \chi(\omega)\mathrm{d}m_H = \begin{cases} a & \text{if } \chi(\omega) \equiv 1, \\ 0 & \text{otherwise}, \end{cases}$$

by orthogonality of characters. This, and (23) gives

$$\hat{I}_A(\chi) = \hat{a}(\chi).$$

The latter equality shows that $I_A(\omega) = a$ for almost all $\omega \in \Omega$. In other words, $a = 1$ or $a = 0$ for almost all $\omega \in \Omega$. Thus, $I_A(\omega) = 1$ or $I_A(\omega) = 0$ for almost all $\omega \in \Omega$. Therefore, $m_H(A) = 1$ or $m_H(A) = 0$, and the proof is completed. □

For convenience, we remind the classical Birkhoff-Khintchine ergodic theorem, see, for example, [19].

Lemma 6. *Suppose that a random process $\xi(t,\hat{\omega})$ is ergodic with finite expectation $\mathbb{E}|\xi(t,\hat{\omega})|$, and sample paths integrable almost surely in the Riemann sense over every finite interval. Then, for almost all ω,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \xi(t,\hat{\omega})\mathrm{d}t = \mathbb{E}\xi(0,\hat{\omega}).$$

Proof of Theorem 1. In virtue of Proposition 1, it suffices to identify the limit measure $P_{\underline{\sigma}}$ in it, i. e., to show that $P_{\underline{\sigma}} = P_{\underline{\zeta},\underline{\sigma}}$.

Let $A \in \mathcal{B}(\mathbb{C}^2)$ be a continuity set of the measure $P_{\underline{\sigma}}$. Then, by Proposition 1, for almost all $\omega \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{t \in [0,T] : \underline{\zeta}(\underline{\sigma}+it,\omega,\alpha;Q) \in A\right\} = P_{\underline{\sigma}}(A). \tag{24}$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the random variable

$$\xi = \xi(\omega) = \begin{cases} 1 & \text{if } \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A, \\ 0 & \text{otherwise,} \end{cases}$$

Obviously,

$$\mathbb{E}\xi = \int_{\Omega} \xi \mathrm{d}m_H = m_H\Big\{\omega \in \Omega : \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A\Big\}. \tag{25}$$

By Lemma 5, the random process $\xi(g_t(\omega))$ is ergodic. Therefore, an application of Lemma 6 yields

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi(g_t(\omega))\mathrm{d}t = \mathbb{E}\xi \tag{26}$$

for almost all $\omega \in \Omega$. On the other hand, from the definitions of ξ and $\mathcal{G}_t$, we have

$$\frac{1}{T} \int_0^T \xi(g_t(\omega))\mathrm{d}t = \frac{1}{T}\mathrm{meas}\Big\{t \in [0, T] : \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A\Big\}.$$

Therefore, equalities (25) and (26), for almost all $\omega \in \Omega$, lead to

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\Big\{t \in [0, T] : \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A\Big\} = P_{\underline{\zeta},\underline{\sigma}}(A).$$

This together with (24) shows that

$$P_{\underline{\sigma}}(A) = P_{\underline{\zeta},\underline{\sigma}}(A). \tag{27}$$

Since A is an arbitrary continuity set of $P_{\underline{\sigma}}$, equality (27) is valid for all $A \in \mathcal{B}(\mathbb{C}^2)$. This proves the theorem. □

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