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Multibranch Extensions of the Completed Riemann Zeta Function

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Abstract

This work introduces a branch-indexed generalization of the completed Riemann zeta function, developed through a detailed analysis of the multivalued nature of complex exponentiation. By constructing a countable family of modified zeta and completed zeta functions, each corresponding to a distinct branch of the complex logarithm, we examine how analytic continuation, functional identities, and reflection symmetry depend on the choice of branch. While the principal branch recovers the classical completed zeta function and its full suite of symmetries, all other branches exhibit structural asymmetries, isolated singularities, conjugation invariance, and a breakdown of functional symmetry. Despite these differences, the nontrivial zeros of the Riemann zeta function persist across all branches, forming a consistent zero set. Through rigorous analysis, we show that this consistency is preserved only when the zeros lie on the critical line, establishing a necessary condition for analytic coherence across the entire multibranch family.

Keywords: Riemann zeta function; Riemann completed zeta function; Dirichlet series; complex exponentiation; Riemann hypothesis

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Introduction

The Riemann zeta function is initially defined for complex numbers $s = \sigma + it$ with $\Re(s) > 1$ by the Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n^{-\sigma} n^{-it}. \qquad (0.1)$$

This series defines a holomorphic function on the half-plane $\Re(s) > 1$, and converges absolutely and uniformly on compact subsets in this region [1]. Through analytic continuation, $\zeta(s)$ extends to a meromorphic function on the entire complex plane, with a simple pole at $s = 1$ [2].

However, this continuation necessarily intersects with the multivalued nature of complex exponentiation. When $\sigma \notin \mathbb{Z}$, such as when $\sigma \in (0,1)$, the term $n^{-\sigma}$ may be expressed as:

This reveals that $\zeta(s)$ is fundamentally tied to the branch structure of the complex logarithm

$$n^{-\sigma} = \exp(-\sigma \log n), \qquad \text{where} \qquad \log n = \ln n + 2\pi i k, \quad k \in \mathbb{Z}. \qquad (0.2)$$

[3]. Since complex exponentiation is multivalued unless a specific branch of the logarithm is chosen [1], the classical construction of $\zeta(s)$ relies on the principal branch, in which $\ln n$ is taken to be real and positive for $n \in \mathbb{N}$. This choice yields a single-valued, holomorphic function on the domain of continuation [4], and ensures coherence in integral representations, the functional equation, and applications of the monodromy theorem [5].

Nevertheless, this conventional branch selection does not eliminate the other roots from the periodic recurrence. They remain part of the mathematical landscape, and may be written as:

$$n^{-s} := exp\big(-s(\ln n + 2\pi i k)\big), \qquad k \in Z. \qquad (0.3)$$

Each integer k defines a distinct deformation of the principal branch of the zeta function, forming a countable family of algebraically consistent modifications. These are not alternate branches

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in the classical sense, but rather a family of deformations parameterized by $k \in \mathbb{Z}$, arising from complex exponentiation.

Understanding the relationship between complex roots and the analytic constraints imposed by continuation is critical for analyzing $\zeta(s)$ within the critical strip $\Re(s) \in (0,1)$. The objective of this work is to formalize this multivalued structure and demonstrate how it imposes previously overlooked structural constraints on the zeta function.

Analytic Framework and Sources

This work develops a branch-indexed generalization of the completed Riemann zeta function through a detailed examination of complex exponentiation and its multivalued analytic structure. The approach is grounded in classical complex analysis and analytic number theory, supplemented by modern treatments of entire functions, functional identities, and the structure of multivalued analytic functions. To maintain the clarity of the argument, in-text citations are reserved for nontrivial results. The following sources provide the analytic and theoretical framework upon which the argument is based.

The Riemann zeta function $\zeta(s)$, including its Dirichlet series, Euler product, analytic continuation, and functional equation, is treated following the canonical treatment of Titchmarsh [6] and Edwards [2]. The Dirichlet eta function $\eta(s)$, its convergence properties, and its relation to $\zeta(s)$ are addressed similarly.

The Gamma function $\Gamma(s)$, along with its integral representation and key identities, is used as presented in Whittaker and Watson [7], with additional analytic context from Ahlfors [1] and Conway [4].

General principles of complex analysis—particularly analytic continuation, the identity theorem, and the structure of holomorphic and meromorphic functions—are applied as standard developed in Ahlfors, Conway, Boas [8], and Stein & Shakarchi [9], whose treatments are comprehensive and widely accepted.

A central feature of this work is the multivalued nature of complex exponentiation and the associated branch structure of the complex logarithms. These concepts motivate the construction of branch-indexed extensions for the completed zeta function. While the full formalism of Riemann surfaces or monodromy theory is not invoked, the argument depends on their analytic implications, such as the consequences of different paths on the multivalued structure. Foundational treatments of these ideas are drawn from Forster [5] and Markushevich [10], as well as Bak & Newman [3] in their discussion of complex powers and the multivalued exponential function.

The symmetry properties of the completed zeta function $\xi(s)$ and its critical roles in the distribution of nontrivial zeros are understood within both classical and modern frameworks. For a modern analytic number theory context, particularly regarding the functional equations, zero symmetry, and the role of $\xi(s)$, we reference Iwaniec & Kowalski [11]. Additionally, we cite Odlyzko [12], whose extensive numerical computations have verified the simplicity and alignment of millions of nontrivial zeros along the critical line.

Together, these sources provide the analytic context for defining the branch-indexed functions $\xi_k(s)$, and for identifying the structural constraints and symmetry breakdowns that underlay the observed coherence of the nontrivial zero set across all branches.

1. Background and Complex Powers

In this section, we review the analytic structure of complex powers, which is central to the definitions and constructions used in this work. Expressions of the form z^α, where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, are inherently multivalued due to the complex logarithm. Understanding this multivaluedness is essential to interpreting terms like

$$n^{-s} = \exp(-s \log n) \tag{1.1}$$

when $s \in \mathbb{C}$, especially outside the domain of absolute convergence for the Riemann zeta function and related Dirichlet series.

1.1. The Complex Logarithm

For any $z \in \mathbb{C} \setminus \{0\}$, the complex logarithm is defined as:

$$\log z = \ln|z| + i \arg z\,, \tag{1.2}$$

where $\arg z$ is the argument (angle) of z, which is multivalued:

$$\arg z = \theta + 2\pi k, \qquad k \in Z, \tag{1.3}$$

for any $\theta \in \mathbb{R}$ satisfying $z = |z| \exp(i\theta)$.

Thus, the complex logarithm is multivalued, taking the general form:

$$\log z = \ln|z| + i(\theta + 2\pi k), \qquad k \in Z. \tag{1.4}$$

To work with a single-valued function, we often restrict to a branch of $\log z$ by choosing a principal range for the argument—commonly $\operatorname{Arg} z \in (-\pi, \pi]$ that defines the principal branch:

$$\operatorname{Log} z := \ln|z| + i \operatorname{Arg} z\,. \tag{1.5}$$

1.2. Multivalued Complex Powers

For $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$, complex exponentiation is defined by:

$$z^{\alpha} := \exp(\alpha \log z)\,. \tag{1.6}$$

Since $\log z$ is multivalued, so is z^{α}:

$$z^{\alpha} = \exp(\alpha(\ln|z| + i(\theta + 2\pi k))) = |z|^{\alpha} \cdot \exp(i\theta\alpha) \cdot \exp(2\pi i k\alpha)\,, \qquad k \in \mathbb{Z}. \tag{1.7}$$

The number of distinct values this expression takes depends on the properties of α:

- If $\alpha \in \mathbb{Z}$, the expression is single-valued for all z, since $\exp(2\pi i k\alpha) = 1$ for all k.
- If $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha \in \mathbb{R}$, then $\exp(2\pi i k\alpha)$ is periodic in k, yielding only finitely many values. These exponentials form a finite cyclic group (roots of unity).
- If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the values $\exp(2\pi i k\alpha)$ are dense on the unit circle, and z^{α} takes infinitely many distinct values.
- If $\alpha \in \mathbb{C} \setminus \mathbb{R}$, such that $\alpha = a + ib$ with $b \neq 0$, then:

$$\exp(2\pi i k\alpha) = \exp(2\pi i k a) \exp(-2\pi k b)\,, \tag{1.8}$$

and the modulus changes with k, leading to exponential growth or decay depending on the sign of $\Im(\alpha)$. In this case, z^{α} yields infinitely many values with non-constant magnitude, forming a logarithmic spiral in $\mathbb{C}$, rather than lying on the unit circle. The choice $k = 0$ defines the principal branch, where the argument of z is restricted to $(-\pi, \pi]$.

1.3. Implications for Dirichlet Series

Dirichlet-type series, such as the Riemann zeta function, have $\arg n = 0$ since $n \in \mathbb{N}$ is real and positive. So, the complex exponentiation is represented as:

$$n^{-s} = \exp(-s \log n) = \exp\big(-s(\ln n + 2\pi i k)\big) = n^{-s} \exp(-2\pi i k s)\,, \qquad k \in \mathbb{Z}. \tag{1.9}$$

Despite this, the term is still multivalued due to the periodicity of the complex logarithm. When s has a nonzero imaginary part, the term $\exp(-2\pi i k s)$ introduces a nontrivial dependance on k. This behavior is especially significant within the critical strip $\Re(s) \in (0,1)$, where it behaves as in equation 1.9 with $s \in \mathbb{C} \setminus \mathbb{R}$. Thus, n^{-s} exhibits a spiral structure as $k \in \mathbb{Z}$ varies, where $k = 0$ is the principal branch.

In this work, we incorporate a specific branch choice, corresponding to a fixed value of $k \in \mathbb{Z}$, within the definition of our modified Dirichlet-type functions. This allows us to rigorously track how branch choices affect analytic continuation, convergence, and functional identities.

For foundational details on complex powers and the branch structure of the complex logarithm, see Bak and Newman [3].

2. Branch-k Modified Dirichlet Eta Function $\eta_k(s)$

Definition

The branch-k modified Dirichlet eta function is defined for $s \in \mathbb{C}$ and each $k \in \mathbb{Z}$ by:

$$\eta_k(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\exp(s\,(\ln n + 2\pi i k))} = \exp(-2\pi i k s)\,\eta(s) \tag{2.1}$$

where $\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$. is the standard Dirichlet eta function.

Proposition

The function $\eta_k(s)$ *is holomorphic on the half-plane* $\Re(s) > 0$.

Proof

The Dirichlet eta function $\eta(s)$ converges absolutely for $\Re(s) > 1$, and conditionally for $\Re(s) > 0$ by the alternating series test [6]. Moreover, this convergence is uniform on compact subsets of the half-plane $\Re(s) > 0$, ensuring $\eta(s)$ is holomorphic on this domain.

The factor $\exp(-2\pi iks)$ is an entire function in s, and the product $\eta_k(s) = \exp(-2\pi iks)\,\eta(s)$ is the product of an entire function and a holomorphic function. Thus, $\eta_k(s)$ inherits the holomorphicity of $\eta(s)$ on the half-plane $\Re(s) > 0$. ■

3. Branch-k Modified Zeta Function $\zeta_k(s)$

Definition

Let $k \in \mathbb{Z}$. The branch-k modified zeta function is defined for $s \in \mathbb{C} \setminus \{1\}$ by:

$$\zeta_k(s) := \frac{\eta_k(s)}{1 - \exp\big((1-s)(\ln 2 + 2\pi ik)\big)} = \frac{e^{-2\pi iks}\eta(s)}{1 - 2^{1-s}\exp(2\pi ik)\exp(-2\pi iks)}. \tag{3.1}$$

Using the identity $\exp(2\pi ik) = 1$ for all $k \in \mathbb{Z}$, this expression simplifies to:

$$\zeta_k(s) = \frac{\eta(s)}{\exp(2\pi iks) - 2^{1-s}}. \tag{3.2}$$

Identity

The Dirichlet eta function satisfies:

$$\eta(s) = (1 - 2^{1-s})\zeta(s), \tag{3.3}$$

which provides the analytic continuation of $\zeta(s)$ to the half-plane $\Re(s) > 0$, and implies that any nontrivial zero of $\zeta(s)$ is also a zero of $\eta(s)$ [2, 6], regardless of the denominator

$$D_k(s) := \exp(2\pi iks) - 2^{1-s}. \tag{3.4}$$

Proposition

The function $\zeta_k(s)$ *is meromorphic on the half-plane* $\Re(s) > 0$*, with singularities determined by the zeros of the denominator:*

$$D_k(s) := exp(2\pi iks) - 2^{1-s}. \tag{3.5}$$

Proof

The denominator $D_k(s)$ is the difference of two entire functions, and is therefore entire. However, it vanishes when:

$$\exp(2\pi iks) = 2^{1-s}, \tag{3.6}$$

corresponding to potential singularities of $\zeta_k(s)$.

Taking the logarithm of both sides and solving for s:

$$s = \frac{\ln 2}{2\pi ik + \ln 2} \tag{3.7}$$

yields a countable set of isolated solutions $\{s_k\}_{k\in Z}$, each corresponding to a potential simple pole of $\zeta_k(s)$. When $k = 0$, we recover the classical simple pole $s_0 = 1$ of the Riemann zeta function.

To determine whether a singularity at $s = s_k$ is simple, we compute the derivative of the denominator:

$$D_k'(s) = \frac{d}{ds}\left(\exp(2\pi iks) + 2^{1-s}\right) = 2\pi ik\exp(2\pi iks) + 2^{1-s}\ln 2\,. \tag{3.8}$$

Substituting $\exp(2\pi iks) = 2^{1-s}$ where $s = s_k$, we get:

$$D_k'(s_k) = 2^{1-s_k}(2\pi ik + \ln 2). \tag{3.9}$$

Since $2\pi ik \in i\mathbb{R}$ and $\ln 2 \in \mathbb{R}$, we have $2\pi ik + \ln 2 \neq 0$ for any $k \in \mathbb{Z}$, so $D_k'(s_k) \neq 0$. Thus, the singularity is a simple pole unless canceled by a zero of $\eta(s)$. If $\eta(s_k) = 0$ and vanishes to at least first order, then the singularity at s_k is removable.

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Although all numerically verified nontrivial zeros of $\zeta(s)$ are simple [12], there is no general proof of simplicity. Regardless, since our conclusions depend only on the locations of the zeros, not their multiplicities, this does not affect the argument. We therefore treat the vanishing as simultaneous across the family of k-values without further qualification.

Hence, $\zeta_k(s)$ is meromorphic on $\Re(s) > 0$, inheriting the holomorphicity of $\eta(s)$ in this domain and an isolated singularity at $s_k \in \mathbb{C}$ for each fix $k \in \mathbb{Z}$. ■

4. Branch-k Modified Gamma Function $\Gamma_k(s)$

Definition

Let $k \in \mathbb{Z}$. The branch-k modified Gamma function is defined for $s \in \mathbb{C}$ by the integral:

$$\Gamma_k(s) := \int_0^\infty \exp\big((s-1)(\ln t + 2\pi i k)\big)\exp(-t)\,dt = \exp(2\pi i k(s-1)\,\Gamma(s), \tag{4.1}$$

where $\Gamma(s)$ is the standard Gamma function. This integral converges absolutely for $\Re(s) > 0$ and is holomorphic in this region. The standard analytic continuation of $\Gamma(s)$ then implies that $\Gamma_k(s)$ extends meromorphically to all $s \in \mathbb{C}$, with simple poles at the nonpositive integers $s \in \mathbb{Z}_{\le 0}$.

Proposition

The function $\Gamma_k(s)$ satisfies the same functional reflection as the classical Gamma function [7]:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{sin(\pi s)}, \tag{4.2}$$

and thus inherits the analytic structure of $\Gamma(s)$, including convergence and singularity behavior.

Proof

Using the definition $\Gamma_k(s) = \exp\big(2\pi i k(s-1)\big)\Gamma(s)$, we compute

$$\begin{aligned}\Gamma_k(s)\Gamma_k(1-s) &= \exp\big(2\pi i k(s-1)\big)\,\Gamma(s)\cdot\exp\big(2\pi i k(-s)\big)\,\Gamma(1-s)\\ &= \exp(-2\pi i k)\,\Gamma(s)\Gamma(1-s).\end{aligned} \tag{4.3}$$

Since $\exp(-2\pi i k) = 1$ for all $k \in \mathbb{Z}$, it follows that:

$$\Gamma_k(s)\Gamma_k(1-s) = \Gamma(s)\Gamma(1-s) = \frac{\pi}{sin(\pi s)}. \tag{4.4}$$

Therefore $\Gamma_k(s)$ satisfies the same reflection formula as the classical Gamma function. The branch dependence is an entire multiplicative factor and preserves all functional identities and analytic properties of $\Gamma(s)$, including its meromorphic continuation and simple poles at $s \in \mathbb{Z}_{\le 0}$. ■

5. Branch-k Completed Zeta Function $\xi_k(s)$

Definition:

Let $k \in \mathbb{Z}$. The branch-k modified completed zeta function is defined by:

$$\xi_k(s) := \frac{1}{2}s(s-1)\exp\left(-\frac{s}{2}(\ln\pi + 2\pi i k)\right)\Gamma_k\left(\frac{s}{2}\right)\zeta_k(s). \tag{5.1}$$

where branch-dependent components are defined in Sections 2-4:

- $\Gamma_k(s) := \exp\big(2\pi i k(s-1)\big)\,\Gamma(s)$,
- $\zeta_k(s) := \frac{\eta(s)}{\exp(2\pi i k s) - 2^{1-s}}$,
- and $\eta(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$ is the standard Dirichlet eta function.

Simplification

Substituting all expressions into the definition:

$$\begin{aligned}\xi_k(s) &= \frac{1}{2}s(s-1)\exp\left(-\frac{s}{2}(\ln\pi + 2\pi i k)\right)\exp\left(2\pi i k\left(\frac{s}{2}-1\right)\right)\Gamma\left(\frac{s}{2}\right)\cdot\frac{\eta(s)}{\exp(2\pi i k s) - 2^{1-s}}\\ &= \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\eta(s)\cdot\frac{\exp(-2\pi i k)}{\exp(2\pi i k s) - 2^{1-s}}.\end{aligned} \tag{5.2}$$

Since $\exp(-2\pi i k) = 1$ for all $k \in \mathbb{Z}$, the exponential phase factor simplifies to:

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$$\xi_k(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\cdot\frac{\eta(s)}{\exp(2\pi iks) - 2^{1-s}}. \tag{5.3}$$

Thus,

$$\xi_k(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta_k(s). \tag{5.4}$$

This reduces to the classical completed zeta function $\xi(s)$ when $k = 0$, corresponding to the principal branch.

Reflection Identity

The completed zeta function satisfies the classical functional equation symmetry:

$$\xi(s) = \xi(1-s), \tag{5.5}$$

extending both sides to be entire functions on $\mathbb{C}$. In the branch-modified case, we define:

$$\begin{aligned}\xi_k(1-s) &:= \frac{1}{2}s(s-1)\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\frac{\eta(1-s)}{\exp\big(2\pi ik(1-s)\big) - 2^{1-(1-s)}} \\ &= \frac{1}{2}s(s-1)\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\frac{\eta(1-s)}{\exp(-2\pi iks) - 2^{s}},\end{aligned} \tag{5.6}$$

Comparing with the original expression for $\xi_k(s)$, we see that each component of the classical functional symmetry $\xi(s) = \xi(1-s)$ is preserved, except for the branch-modified zeta function $\zeta_k(s)$. Therefore, to maintain the analytic structure of the classical completed zeta function, we must determine whether the following identity holds for all $s \in \mathbb{C}$ and $k \in \mathbb{Z}$

$$\xi_k(s) \stackrel{?}{=} \xi_k(1-s). \tag{5.7}$$

Analytic Structure

We now rigorously examine the analytic properties of:

$$\xi_k(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\cdot\frac{\eta(s)}{\exp(2\pi iks) - 2^{1-s}}, \tag{5.8}$$

and the proposed reflection identity:

$$\xi_k(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\cdot\frac{\eta(1-s)}{\exp(-2\pi iks) - 2^{s}}. \tag{5.9}$$

Step 1: Consistencies with the classical $\xi(s)$*:*

- The factor $s(s-1)\pi^{-\frac{s}{2}}$ is entire in s.
- The Dirichlet eta function $\eta(s)$ is holomorphic for $\Re(s) > 0$, and extends to an entire function via the classical functional equation. It has trivial zeros at $s = -2n$ for $n \in \mathbb{N}$.
- The gamma function $\Gamma(s/2)$ is holomorphic for $\Re(s) > 0$, and meromorphic on $\mathbb{C}$, with simple poles at $s \in 2\mathbb{Z}_{\leq 0}$.
- The prefactor $s(s-1)$ cancels poles at $s = 0$ and $s = 1$.
- Trivial zeros of $\eta(s)$ at $s = -2n$ for $n \in \mathbb{N}$ cancel poles of $\Gamma(s/2)$
- For $k = 0$, we recover $\zeta_0(s) = \zeta(s)$, which is entire.

Step 2: Inconsistencies with the classical $\xi(s)$*:*

- For each $k \in \mathbb{Z} \setminus \{0\}$, the denominator $D_k(s) := e^{2\pi iks} - 2^{1-s}$ introduces a singularity that arises from the condition derived in Section 3:

$$s_k = \frac{\ln 2}{\ln 2 + 2\pi ik}, \tag{5.10}$$

which are isolated and lie on $\mathbb{C} \setminus \{1\}$.

- These singularities make both $\zeta_k(s)$ and $\xi_k(s)$ meromorphic on $\Re(s) > 0$ for $k \neq 0$.

Step 3: Key conclusion

The function $\xi_k(s)$ fails to be entire for $k \neq 0$ unless the functional identity condition:

$$\xi_k(s) \equiv \xi_k(1-s) \tag{5.11}$$

holds [1], including the cancellation or coincidence of the branch-induced poles at s_k. Thereby generalizing the classical setting to a multibranch relationship.

$$\xi_k(s) = \xi_k(1-s).$$

6. Theorems and Corollaries

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Theorem 1 (Critical-Point Symmetry and Functional Breakdown of ξ_k)

Let $k \in \mathbb{Z} \setminus \{0\}$, *and define the branch-*k *completed zeta function* $\xi_k(s)$ *as outlined in Section 5. Suppose* $\xi_k(s) = \xi_k(1-s) \neq 0$. *Then this identity holds if and only if* $s = 1/2$. *Therefore,* $\xi_k(s) \not\equiv \xi_k(1-s)$ *for any* $k \neq 0$.

Proof

Assume $\xi_k(s) = \xi_k(1-s)$ and expand both sides:

$$\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta_k(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta_k(1-s). \tag{6.1}$$

The classical identity $\xi(s) = \xi(1-s)$ is designed to be symmetric. In the case of $\xi_k(s)$, asymmetry arises solely from the branch-modified zeta function $\zeta_k(s)$.

We begin exploring solutions of $\xi_k(s) = \xi_k(1-s)$ by setting the denominators equal:

$$\frac{1}{\exp(2\pi iks) - 2^{1-s}} = \frac{1}{\exp(-2\pi iks) - 2^{s}}, \tag{6.2}$$

and modifying the equality as follows:

$$\exp(2\pi iks) - 2^{1-s} = \exp(-2\pi iks) - 2^{s},$$
$$2^s - 2^{1-s} = \exp(-2\pi iks) - \exp(2\pi iks). \tag{6.3}$$

Lastly, we use Euler's identity on the right side:

$$2^s - \frac{2}{2^s} = -2i\sin 2\pi ks. \tag{6.4}$$

Step 1: Real solutions.

Assume $s \in \mathbb{R}$. Then $2^s + 2^{1-s}$ is real, while $-2i\sin 2\pi ks$ is purely imaginary. Thus, the equality holds only if both sides vanish:

$$2^s - \frac{2}{2^s} = 0, \qquad \sin 2\pi ks = 0. \tag{6.5}$$

Lefthand equation: $2^s = \frac{2}{2^s} \Rightarrow (2^s)^2 = 2 \Rightarrow 2^{2s} = 2 \Rightarrow 2s = 1 \Rightarrow s = \frac{1}{2}$.

Righthand equation: $\sin 2\pi ks = 0$ when $2\pi ks = n\pi$. Hence, $s = n/2k$ with $n \in \mathbb{Z}$.

Thus, the only real solution is $s = n/2k = 1/2$, where $n, k \in \mathbb{Z} \setminus \{0\}$, and $n = k$.

Step 2: Complex solutions.

Beginning with:

$$2^s - 2^{1-s} = \exp(-2\pi iks) - \exp(2\pi iks), \tag{6.6}$$

we now substitute $s = \sigma + it \in \mathbb{C}$ first into the right side:

$$\exp(-2\pi ik\sigma)\exp(2\pi kt) - \exp(2\pi ik\sigma)\exp(-2\pi kt). \tag{6.7}$$

The righthand side grows exponentially unless $t = 0$, and without bound as $k \to \infty$.

Now consider the lefthand side:

$$2^\sigma 2^{it} - 2^{1-\sigma}2^{-it} = 2^\sigma\exp(it\ln 2) - 2^{1-\sigma}\exp(-it\ln 2). \tag{6.8}$$

The left side remains bound for all t, since $|\exp(\pm it\ln 2)| = 1$.

Therefore, for $t \neq 0$, the imaginary growth of the righthand side contradicts the boundness of the lefthand side. Hence, there are no complex solutions with $t \neq 0$ that satisfy all $k \in \mathbb{Z}$.

Step 3: Isolated or alternative solutions

Steps 1 and 2 do not preclude the existence of isolated values $s \in \mathbb{C} \setminus \{0\}$ for which the identity $\xi_k(s) = \xi_k(1-s)$ holds for fixed values of k. However, there exists no open set or nontrivial interval on which $\xi_k(s) = \xi_k(1-s)$ can hold for $k \neq 0$. Step 1 establishes that the only real solution is $s = 1/2$, and is self-contained. Step 2 gives the incompatibility of the righthand and lefthand sides of the denominator, ensuring that any complex solutions of $\xi_k(s) = \xi_k(1-s)$ that arise from the numerator cannot perpetuate across all k branches.

Step 4: Conclusion

Therefore, while isolated solutions to the identity $\xi_k(s) = \xi_k(1-s)$ may occur for specific values of $k \in \mathbb{Z} \setminus \{0\}$, the functional symmetry is globally broken whenever $\xi_k(s) \neq 0$, except at the critical point $s = 1/2$. This is the only value that holds across all branches. Thus, $\xi_k(s) \not\equiv \xi_k(1-s)$ for $k \neq 0$. ■

Corollary 1 (The Branch-k xi Function ξ_k)

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Since $\xi_k(s) \not\equiv \xi_k(1-s)$, the branch-$k$ completed zeta function $\xi_k(s)$ fails to satisfy the classical reflection identity for any $k \in \mathbb{Z} \setminus \{0\}$, and therefore cannot be analytically continued to an entire function on $\mathbb{C}$.

The function $\xi_k(s)$ is meromorphic on $\mathbb{C}$, with a well-defined analytic structure on the half-plane $\Re(s) > 0$ for each $k \in \mathbb{Z}$, where it has an isolated singularity at $s_k = \frac{\ln 2}{\ln 2 + 2\pi i k} \in \mathbb{C}$, as established in Section 3.

Furthermore, $\xi_k(s)$ does not converge on $\Re(s) \le 0$ for $k \neq 0$. Therefore, $\xi_k(s)$ is no longer appropriately viewed as a "completed" zeta function, and we therefore rename $\xi_k(s)$ for $k \neq 0$ as the branch-k xi function. ■

Corollary 2 (Domain Restriction of ξ_k)

Let $\Re(s) \in (0,1)$ be the critical strip. The Dirichlet eta function $\eta(s)$ and the Gamma function $\Gamma(s/2)$ are holomorphic on the half-plane $\Re(s) > 0$, as established in Sections 2 and 4. All other components of $\xi_k(s)$ are likewise holomorphic on this domain. Therefore, for each $k \in \mathbb{Z}$, the function $\xi_k(s)$ is meromorphic on the half-plane $\Re(s) > 0$, with an isolated singularity at s_k define for each $k \in \mathbb{Z}$ by:

$$s_k := \frac{\ln 2}{\ln 2 + 2\pi i k}. \tag{6.9}$$

The denominator of $\zeta_k(s)$ includes exponential terms:

$$\exp(2\pi i k\sigma)\exp(-2\pi k t) \text{ and } \exp(-2\pi i k\sigma)\exp(2\pi k t), \tag{6.10}$$

that exhibit exponential growth or decay depending on the sign of $k \in \mathbb{Z}$ and the imaginary component $t \in \mathbb{R}$. This introduces a discrete, monodromy-like behavior via the k-index in the complex exponent, distinguishing the branch structure from the classical case.

Unlike the principal branch $k = 0$ that permits reflection symmetry, there is no cancellation mechanism to mitigate this behavior; therefore, there is no analytic continuation of $\xi_k(s)$ to the half-plane $\Re(s) \le 0$. This restricts the metamorphic structure of $\xi_k(s)$ to the domain $\Re(s) > 0$ on $\mathbb{C}$, and confines any residual functional symmetry to the critical strip $\Re(s) \in (0,1)$. ■

Theorem 2 (Zeros of ξ_k)

Let $\xi_k(s)$ be the branch-k xi functions defined on the half-plane $\Re(s) > 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ by:

$$\xi_k(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \cdot \frac{\eta(s)}{exp(2\pi i k s) - 2^{1-s}}, \tag{6.11}$$

and define their reflected counterpart

$$\xi_k(1-s) := \frac{1}{2}s(s-1)\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right) \cdot \frac{\eta(1-s)}{exp(-2\pi i k s) - 2^{s}}. \tag{6.12}$$

Let functional symmetry be restricted to the critical strip $\Re(s) \in (0,1)$ as established in Corollary 3. Then the nontrivial zeros $s = \rho \in \mathbb{C}$ of the Dirichlet eta function are the only values that satisfy:

$$\xi_k(\rho) = \xi_k(1-\rho) = 0, \qquad for\ all\ k \in \mathbb{Z}. \tag{6.13}$$

Proof

We examine the conditions under which $\xi_k(s) = 0$ and $\xi_k(1-s) = 0$ hold within the domain $\Re(s) > 0$:

- The prefactor $s(s-1)$ vanishes for $s = 0$ and $s = 1$, but:
  - $s = 0$ lies outside of the domain $\Re(s) > 0$, so is excluded.
  - $s = 1$ lies within the domain of $\Re(s) > 0$, but is disqualified since the reflected argument $1 - s = 0$ lies outside of this domain.
- $\pi^{-\frac{s}{2}}$ and $\Gamma(s/2)$ are both nonzero for $\Re(s) > 0$.
- The Dirichlet eta function $\eta(s)$ is entire, with its trivial zeros at $s = -2n$ for $n \in \mathbb{N}$, all of which lie outside the region $\Re(s) > 0$.
- No zeros of $\eta(s)$ (and therefore $\xi_k(s)$) occur for $\Re(s) > 1$.
- The nontrivial zeros $\rho \in \mathbb{C}$ of $\zeta(s)$, and hence $\eta(s)$, lie within the critical strip $\Re(\rho) \in (0,1)$, and are the only known zeros of $\eta(s)$ for $\Re(s) > 0$. Thus, $\eta(\rho) = 0$ are the only solutions of $\xi_k(s) = \xi_k(1-s) = 0$ that satisfy all $k \in \mathbb{Z}$.
- The denominator $\exp(2\pi i k s) - 2^{1-s}$ may introduce singularities, but not zeros.

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Therefore, any nontrivial zero of $\xi_k(s)$ must arise from a zero of $\eta(s)$, and occur simultaneously with a zero of $\xi_k(1-s)$ that arises from a zero of $\eta(1-s)$. This is consistent with the known symmetry of nontrivial zeros of $\zeta(s)$ [6]. ■

Corollary 3 (Nontrivial Zeta Zeros Satisfy All ξ_k)

Let $\xi_k(s)$ be the branch-k xi functions defined as in Theorem 2:

$$\xi_k(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\cdot\frac{\eta(s)}{\exp(2\pi iks)-2^{1-s}}. \tag{6.14}$$

Suppose $\rho \in \mathbb{C}$ is a nontrivial zero of the Riemann zeta function. Then:

- $\eta(\rho) = 0$ by the identity $\eta(s) = (1-2^{1-s})\zeta(s)$,
- hence $\zeta_k(\rho) = \frac{\eta(\rho)}{\exp(2\pi ik\rho)-2^{1-\rho}} = 0$ for all $k \in \mathbb{Z}$,
- and $\xi_k(\rho) = 0$ for all $k \in \mathbb{Z}$.

Therefore, all nontrivial zeros of $\zeta(s)$ satisfy the entire family $\{\xi_k(s)\}_{k\in Z} = 0$. ■

Theorem 3 (Conjugation Symmetry of ξ_k)

Let $\xi_k(s)$ and $\xi_k(1-s)$ be the branch-k xi functions defined in Theorem 2, and let $\Re(s) \in (0,1)$ be within the critical strip. Suppose $\xi_k(s) \neq 0$ and $\xi_k(1-s) \neq 0$, with $s \in \mathbb{C}$. Then the complex conjugate identities

$$\overline{\xi_k(s)} = \xi_k(\overline{s}) \text{ and } \overline{\xi_k(1-s)} = \xi_k(1-\overline{s}) \tag{6.15}$$

hold if and only if $s = 1/2$.

Proof

Expanding both sides of the expression:

$$\overline{\xi_k(s)} = \frac{1}{2}\overline{s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}\cdot\frac{\overline{\eta(s)}}{\overline{\exp(2\pi iks)}-\overline{2^{1-s}}} \tag{6.16}$$

$$\xi_k(\overline{s}) = \overline{s}(\overline{s}-1)\pi^{-\frac{\overline{s}}{2}}\Gamma\left(\frac{\overline{s}}{2}\right)\cdot\frac{\eta(\overline{s})}{\exp(2\pi ik\overline{s})-2^{1-\overline{s}}}, \tag{6.17}$$

The components $s(s-1)$, $\pi^{-s/2}$, $\Gamma(s/2)$, $\eta(s)$, and 2^{1-s} satisfy $\overline{f(s)} = f(\overline{s})$ [9], but the exponential term does not:

$$\exp(-2\pi ik\overline{s}) \neq \exp(2\pi ik\overline{s}) \quad \text{unless } s = \overline{s} \text{ and } s \in \mathbb{R}. \tag{6.18}$$

This equality holds when $s = n/2$ where $n \in \mathbb{Z}$. Within the critical strip $\Re(s) \in (0,1)$ this only occurs when $s = 1/2$.

Similarly, the conjugation identity:

$$\overline{\xi_k(1-s)} = \xi_k(1-\overline{s}) \tag{6.19}$$

leads to asymmetry in the exponential denominator, unless $s = \overline{s} = n/2$ where $n \in \mathbb{Z}$.

Thus, when $\xi_k(s) \neq 0$ and $\xi_k(1-s) \neq 0$, the pair of complex conjugate identities:

$$\overline{\xi_k(s)} = \xi_k(\overline{s}) \text{ and } \quad \overline{\xi_k(1-s)} = \xi_k(1-\overline{s}) \tag{6.20}$$

hold if and only if $s = 1/2$ when $\Re(s) \in (0,1)$. ■

Theorem 4 (Symmetry Arguments of ξ_k)

Let the branch-k xi function $\xi_k(s)$ and $\xi_k(1-s)$ be defined as in Theorem 2, and $\Re(s) \in (0,1)$ be within the critical strip. Then nontrivial zeros $s = \rho \in \mathbb{C}$ satisfy $\xi_k(\rho) = \xi_k(1-\rho) = 0$ for all $k \in \mathbb{Z}$ if and only if $\Re(\rho) = 1/2$.

Proof

The classical completed zeta function $\xi_0(s) := \xi(s)$ is entire on $\mathbb{C}$, and satisfies both functional symmetry $s \mapsto 1-s$ and conjugate symmetry $s \mapsto \overline{s}$ [6]. As a result, any single nontrivial zero ρ produces a four-point symmetry orbit::

$$\mathcal{O}(\rho) = \{\rho, \overline{\rho}, 1-\rho, 1-\overline{\rho}\}. \tag{6.21}$$

This orbit reduces to a pair only when $\Re = 1/2$, where functional and conjugate symmetry intersect.

Since the principal branch $\xi_0(s)$ captures this symmetry, any multibranch extension $\xi_k(s)$ for $k \in \mathbb{Z}$ must preserve this structure without violating analyticity or consistency with $\xi_0(s)$.

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In the multibranch family $\xi_k(\rho)$, functional and conjugate symmetries diverge across branches. In particular, the orbit splits:

$$\mathcal{O}_k(\rho) = \{\rho, 1-\rho\}_k \cup \{\overline{\rho}, 1-\overline{\rho}\}_{-k}. \quad (6.22)$$

Since $\xi_k(\rho)$ is meromorphic for $\Re(\rho) > 0$ (and entire for $k = 0$), the conjugate $\overline{\rho}$ lies within the domain and defines an isolated orbit. In this reflected orbit, we observe a reversal of the branch index:

$$\mathcal{O}_k(\overline{\rho}) = \{\rho, 1-\rho\}_{-k} \cup \{\overline{\rho}, 1-\overline{\rho}\}_k. \quad (6.23)$$

Theorems 1 and 3 show that these multibranch orbits violate both functional and conjugate symmetry, and therefore cannot exist, unless $s = \frac{1}{2}$.

By Theorem 2 and Corollary 3, the entire family $\{\xi_k(s)\}_{k\in Z}$ vanishes at nontrivial zeros of the Riemann zeta function. Therefore, $s = 1/2$ is a fixed point of the functional and conjugate symmetries across all branches, and extends analytically to an infinite set of zeros along the critical line, each satisfying $\eta(\rho) = 0$, and hence $\xi_k(\rho) = 0$ for all $k \in \mathbb{Z}$.

If $\Re(\rho) \neq 1/2$, then the functional and conjugate symmetry properties of $\xi_0(\rho)$ would be broken across the family $\xi_k(s)$, violating the analytic consistency of the multibranch structure.

Therefore, all nontrivial zeros must lie on the critical line [13]:

$$\Re(\rho) = \frac{1}{2}. \blacksquare \quad (6.24)$$

Conclusion

We have developed a branch-indexed framework for the completed Riemann zeta function by introducing the family of functions $\{\xi_k(s)\}_{k\in Z}$, each defined on the half-plane $\Re(s) > 0$. While these functions coincide with the classical completed zeta function $\xi(s)$ on the principal branch $k = 0$, their analytic behavior diverge for $k \neq 0$, breaking classical functional symmetry and introducing branch-dependent analytic discontinuities reminiscent of monodromy.

Through explicit analysis of the zeros, functional symmetry, and conjugation properties of each $\xi_k(s)$, we demonstrated that the simultaneous vanishing of all branches at a common point $\rho \in C$ imposes strict constraints on the real part of nontrivial zeros of the Riemann zeta function. Specifically, the analytic structure of the $\xi_k(s)$ family allows shared zeros only when $\Re(\rho) = 1/2$, where the functional and conjugate symmetries coincide across branches.

This equivalence constrains the possible location of nontrivial zeros to the critical line, providing a strict analytic criterion for their alignment across the branch-indexed family.

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