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Article

The Generalized w -Core Inverse in Banach Algebras with Involution

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Abstract

In this paper, we present the concept of the generalized w -core inverse within the context of a Banach $*$ -algebra. We delineate the characteristics of this new generalized inverse by using the generalized weighted core decomposition and elucidate its representations via the weighted g -Drazin inverse. Additionally, we delve into an examination of the generalized w -core orders. This work broadens the scope of the weighted core inverse and pseudo core inverse, previously defined for complex matrices and linear bounded operators, to a more general setting.

Keywords: core inverse; weighted core inverse; core-EP inverse; weighted g -Drazin inverse; Banach algebra

MSC: 15A10; 16U50; 46H05

1. Introduction

A Banach algebra $\mathcal{A}$ is called a Banach $*$ -algebra if there exists an involution $*$: $x \rightarrow x^*$ satisfying $(x + y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda}x^*$, $(xy)^* = y^*x^*$, $(x^*)^* = x$. Rakic et al. generalized the core inverse of a complex matrix to the case of an element in a ring (see [21]). An element a in a Banach $*$ -algebra $\mathcal{A}$ has core inverse if and only if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, xa^2 = a.$$

If such x exists, it is unique, and denote it by $a^\oplus$ (see [1,6]).

Zhu et al. introduced and studied w -core inverse for a ring element (see [30]). Let $a, w \in \mathcal{A}$. An element $a \in \mathcal{A}$ has w -core inverse if there exist $x \in \mathcal{A}$ such that

$$awx^2 = x, (awx)^* = awx, xaw = a.$$

If such x exists, it is unique, and denote it by $a_w^\oplus$. Let $\mathcal{A}_w^\oplus$ denote the set of all w -core invertible elements in $\mathcal{A}$. The w -core inverse was studied by many authors, e.g., [7,10–12,24,25,30,32]. As is well known,

$$a \in \mathcal{A}_w^\oplus \Leftrightarrow awx^2 = x, (awx)^* = awx, xaw = a, awxa = a, xawx = x$$

(see [30], [Theorem 2.13]).

Prasad et al. extended the concept of the core inverse and introduced the notion of core-EP inverse (i.e., pseudo core inverse) (see [8,19]). An element $a \in \mathcal{A}$ has core-EP inverse (i.e., pseudo core inverse) if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$

If such x exists, it is unique, and denote it by $a^\oplus$. The core-EP inverse has been investigated from many different views, e.g., [2,8,9,14–17,19,22,27,28].

The motivation of this paper is to introduce and examine a new type of generalized weighted inverse, which serves as a logical extension of the generalized inverses mentioned above. Let

$$\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = 0\}.$$

As is well known, $x \in \mathcal{A}^{qnil}$ if and only if $1 + \lambda x \in \mathcal{A}$ is invertible for any $\lambda \in \mathbb{C}$. Set $\mathcal{A}_w^{qnil} = \{x \in \mathcal{A} \mid xw \in \mathcal{A}^{qnil}\}$. In Section 2, we introduce generalized weighted core inverse in terms of a new kind of generalized weighted core decomposition. Many new properties of the w -core inverse and core-EP inverse are thereby obtained.

Definition 1.1. An element $a \in \mathcal{A}$ has generalized w -core decomposition if there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^\oplus, y \in \mathcal{A}_w^{qnil}.$$

Let $a, w \in \mathcal{A}$. We prove that $a \in \mathcal{A}$ has generalized w -core decomposition if and only if there exists unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Recall that $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}.$$

Such x is unique, if exists, and denote it by a^d . The g-Drazin inverse is of great importance in matrix and operator theory (see [3]). An element $a \in \mathcal{A}$ has generalized w -Drazin inverse x if there exists unique $x \in \mathcal{A}$ such that

$$awx = xwa, xwaawx = x \text{ and } a - awxwa \in \mathcal{A}^{qnil}.$$

We denote x by $a^{d,w}$ (see [13]). Evidently, $a^{d,w} = x$ if and only if $x = [(aw)^d]^2a = a[(wa)^d]^2 = (aw)^da(wa)^d$. In Section 3, we establish equivalences between the generalized w -core inverse and the weighted g-Drazin inverse for Banach algebra elements using involved images. We prove that $a \in \mathcal{A}_w^\oplus$ if and only if $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that $xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*$.

The aim of Section 4 is to characterize the generalized weighted core inverse of an element in a Banach $*$ -algebra in terms of other related generalized inverses, such as the weighted core inverse. It is shown that $a \in \mathcal{A}_w^\oplus$ if and only if $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}_w^\oplus$.

Finally, in Section 5, we introduce the generalized w -core order for elements in a Banach $*$ -algebra. Let $w \in \mathcal{A}$ and $a, b \in \mathcal{A}_w^\oplus$. We say that $a \leq_w^\oplus b$ provided that

$$awa_w^\oplus = bwa_w^\oplus, a_w^\oplus a = a_w^\oplus b.$$

The characterizations of the generalized weighted core order are present.

Throughout the paper, all Banach $*$ -algebras are complex with an identity. $\mathcal{A}^{d,w}$ and $\mathcal{A}_w^\oplus$ denote the sets of all weighted g-Drazin and w -core invertible elements in $\mathcal{A}$, respectively. Let $\mathbb{C}^{n \times n}$ be the Banach algebra of all $n \times n$ complex matrices with conjugate transpose $*$.

2. Generalized w -Core Decomposition

The objective of this section is to introduce the concept of the generalized w -core inverse within the framework of a Banach $*$ -algebra. We begin with

Theorem 2.1. Let $a, w \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}$ has generalized w -core decomposition.
- (2) There exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*y = ywz = 0, z \in \mathcal{A}_w^\oplus, y \in \mathcal{A}_w^{qnil}.$$

Set $x = z_w^\oplus$. Then

$$\begin{aligned} awx &= (z + y)wz_w^\oplus = zwz_w^\oplus, \\ (awx)^* &= awx, \\ awx^2 &= (awx)x = zwz_w^\oplus(z + y) = zwz_w^\oplus z = x. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} xawx &= z_w^\oplus(zwz_w^\oplus) = z_w^\oplus = x, \\ x(aw)^2x &= (xaw)(awx) = z_w^\oplus(z + y)wz_w^\oplus = z_w^\oplus zwz_w^\oplus \\ &= z_w^\oplus zwz_w^\oplus = zwz_w^\oplus = awx. \end{aligned}$$

Moreover, we have

$$awxa = (awx)a = zwz_w^\oplus(z + y) = zwz_w^\oplus z = z,$$

and so

$$a - awxa = a - z = y \in \mathcal{A}_w^{qnil}.$$

Then

$$\begin{aligned} \|(aw)^n - awx(aw)^n\| &= \|(a - awxa)w(aw)^{n-1}\| \\ &= \|yw(aw)^{n-1}\| = \|yawa(aw)^{n-2}\| \\ &= \|yw(z + y)w(aw)^{n-2}\| = \|(yw)^2(aw)^{n-2}\| \\ &= \dots = \|(yw)^n\|. \end{aligned}$$

Since $y \in \mathcal{A}_w^{qnil}$, we see that $\lim_{n \rightarrow \infty} \|(yw)^n\|^{\frac{1}{n}} = 0$. Therefore

$$\lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0,$$

as required.

(2) $\Rightarrow$ (1) By hypotheses, there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Then we check that

$$xawx = xaw(awx^2) = [x(aw)^2x]x = awx^2 = x.$$

Set $z = awxa$ and $y = a - awxa$. We verify that

$$\begin{aligned} ywz &= (a - awxa)wawxa = awawxa - awx(aw)^2xa \\ &= awawxa - aw(awx)a = 0, \\ z^*y &= (awxa)^*y = a^*(awx)y = a^*(awx)(a - awxa) \\ &= a^*aw(xa - xawxa) = 0. \end{aligned}$$

We claim that $z \in \mathcal{A}_w^\oplus$ and $z_w^\oplus = x$.

Claim 1. $x = zwx^2$. We verify that

$$zwx^2 = awx(awx^2) = awx^2 = x.$$

Claim 2. $(zwx)^* = zwx$. Clearly, we have $zwx = aw(xawx) = awx$, and then $(zwx)^* = (awx)^* = awx = zwx$.

Claim 3. $xzwx = z$. One checks that

$$xzwx = (xawx)awawxa = x(aw)^2xa = awxa = z.$$

Therefore $z \in \mathcal{A}_w^\oplus$. Moreover, we see that

$$\begin{aligned} \|(aw)^n - awx(aw)^n\| &= \|(a - awxa)w(aw)^{n-1}\| \\ &= \|yw(aw)^{n-1}\| = \|ytwaw(aw)^{n-2}\| \\ &= \|yw(z + y)w(aw)^{n-2}\| = \|(yw)^2(aw)^{n-2}\| \\ &= \dots = \|(yw)^n\|. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \|(yw)^n\|^{\frac{1}{n}} = 0,$$

and then $y \in \mathcal{A}_w^{qnil}$. This completes the proof. $\square$

Corollary 2.2. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}$ has generalized w -core decomposition.
- (2) There exist unique $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^\oplus, y \in \mathcal{A}_w^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) In view of Theorem 2.1, there exists $v \in \mathcal{A}$ such that

$$v = awv^2, (awv)^* = awv, v(aw)^2v = awv, \lim_{n \rightarrow \infty} \|(aw)^n - awv(aw)^n\|^{\frac{1}{n}} = 0.$$

Set $z = awva$ and $y = a - awva$. As in the proof of Theorem 2.1, we have

$$a = z + y, z^*y = ywz = 0, z \in \mathcal{A}_w^\oplus, y \in \mathcal{A}_w^{qnil}.$$

Suppose that there exist $b, c \in \mathcal{A}$ such that

$$a = b + c, b^*c = cwb = 0, b \in \mathcal{A}_w^\oplus, c \in \mathcal{A}_w^{qnil}.$$

Obviously, $awv = aw(awv^2) = (aw)^2v^2 = \dots = (aw)^n v^n$. Since $a = b + c$, we have $aw = bw + cw$. As $(cw)(bw) = (cwb)w = 0$, we have

$$(aw)^n = \sum_{i=0}^n (bw)^i (cw)^{n-i} = (cw)^n + \sum_{i=1}^n (bw)^i (cw)^{n-i}.$$

Hence,

$$[(aw)^n]^*c = [(cw)^n]^*c.$$

Since $cwb = 0$, we see that $(aw)^n b = (aw)^{n-1}(bw + cw)b = (aw)^{n-1}bwb = \dots = (bw)^n b$, and then $(aw)^n bw = (bw)^n bw = (bw)^{n+1}$. This implies that

$$(aw)^n bw[(bw)^\oplus]^{n+1} = (bw)^{n+1}[(bw)^\oplus] = (bw)(bw)^\oplus.$$

Hence

$$(aw)^n bw[(bw)^{\oplus}]^{n+1} bw = bw(bw)^{\oplus} bw = bw.$$

Therefore

$$\begin{aligned} b &= bw b_w^{\oplus} b = (bw)(b_w^{\oplus} b) \\ &= [(aw)^n bw[(bw)^{\oplus}]^{n+1} bw](b_w^{\oplus} b) \\ &= (aw)^n [bw[(bw)^{\oplus}]^{n+1} bw b_w^{\oplus} b] \\ &= (aw)^n [bw[(bw)^{\oplus}]^{n+1} b] \\ &= (aw)^n z, \end{aligned}$$

where $z = bw[(bw)^{\oplus}]^{n+1} b$. Accordingly,

$$\begin{aligned} b - z &= b - awva = b - awv(b + c) = b - awvb - awvc \\ &= b - (aw)^n v^n b - [(aw)^n v^n]^* c \\ &= b - (aw)^n v^n b - [(aw)^n v^n]^* c \\ &= b - (aw)v b - (v^n)^* ((aw)^n)^* c \\ &= (aw)^n z - (aw)v(aw)^n z - (v^n)^* ((cw)^n)^* c \\ &= [(aw)^n - (aw)v(aw)^n] z - (v^n)^* ((cw)^n)^* c \end{aligned}$$

Hence,

$$\|b - z\|^{\frac{1}{n}} \leq \|(aw)^n - (aw)v(aw)^n\|^{\frac{1}{n}} \|z\|^{\frac{1}{n}} + \|(v^n)^*\|^{\frac{1}{n}} \|((cw)^n)^*\|^{\frac{1}{n}} \|c\|^{\frac{1}{n}}.$$

Since $cw \in \mathcal{A}^{qnil}$,

then $1 - \bar{\lambda}cw \in \mathcal{A}^{-1}$; hence, $1 - \lambda(cw)^* \in \mathcal{A}^{-1}$. This implies that $(cw)^* \in \mathcal{A}^{qnil}$. Thus, we prove that $\lim_{n \rightarrow \infty} \|((cw)^n)^*\|^{\frac{1}{n}} = 0$. It follows that

$$\lim_{n \rightarrow \infty} \|b - z\|^{\frac{1}{n}} = 0.$$

Therefore $b = z$, and then $c = a - b = a - z = y$, as required.

(2) $\Rightarrow$ (1) This is trivial. $\square$

Theorem 2.3. Let $a, w \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}$ has generalized w -core decomposition.
- (2) There exists unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Proof. (2) $\Rightarrow$ (1) This is obvious by Theorem 2.1.

(1) $\Rightarrow$ (2) By hypothesis, there exists unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Assume that there exists $y \in \mathcal{A}$ such that

$$y = awy^2, (awy)^* = awy, y(aw)^2y = awy, \lim_{n \rightarrow \infty} \|(aw)^n - awy(aw)^n\|^{\frac{1}{n}} = 0.$$

Set $a_1 = awxa, a_2 = a - a_1$ and $b_1 = awya, b_2 = a - b_1$. As in the proof of Theorem 2.1, we prove that

$$\begin{aligned} a_1^* a_2 &= a_2 w a_1 = 0, a_1 \in \mathcal{A}_w^{\oplus}, a_2 \in \mathcal{A}_w^{qnil}, \\ b_1^* b_2 &= b_2 w b_1 = 0, b_1 \in \mathcal{A}_w^{\oplus}, b_2 \in \mathcal{A}_w^{qnil}. \end{aligned}$$

As in the proof of Corollary 2.2, we verify that $awxa = a_1 = b_1 = awya$. Therefore

$$x = (awxa)_w^\oplus = (a_1)_w^\oplus = (b_1)_w^\oplus = (awya)_w^\oplus = y.$$

Accordingly, $x = y$, the result follows. $\square$

We denote x in Theorem 2.3 by $a_w^\oplus$, and call it the generalized w -core inverse of a .

Corollary 2.4. Let $a = x + y$ be the generalized w -core decomposition of $a \in \mathcal{A}$. Then $a_w^\oplus = x_w^\oplus$.

Proof. Let $a = x + y$ be the generalized w -core decomposition of $a \in \mathcal{A}$. Similarly to the proof of Theorem 2.1, $x_w^\oplus$ is the generalized w -core inverse of a . So the theorem is true. $\square$

Theorem 2.5. Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}_w^\oplus$ if and only if there exists a projection $p \in \mathcal{A}$ such that

- (1) $(1 - p)a \in (1 - p)aw\mathcal{A}$ and $(1 - p)aw \in \mathcal{A}^\#$;
- (2) $aw + p \in \mathcal{A}$ is right invertible and $paw = pawp \in \mathcal{A}^{qnil}$.

Proof. (1) $\Rightarrow$ (2) Since $a \in \mathcal{A}_w^\oplus$, by using Theorem 2.3, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^\oplus, y \in \mathcal{A}_w^{qnil}.$$

Since $x \in \mathcal{A}_w^\oplus$, we have

$$x_w^\oplus = xw(x_w^\oplus)^2, (xwx_w^\oplus)^* = xwx_w^\oplus, x_w^\oplus xwx_w^\oplus = x_w^\oplus, x_w^\oplus xwx = x, x = xwx_w^\oplus x.$$

Let $p = 1 - xwx_w^\oplus$. Then $p^2 = p = p^*$ and $px = 0$. We directly check that

$$(xw + 1 - xwx_w^\oplus)(x_w^\oplus + 1 - xwx_w^\oplus) = 1 + x(1 - xwx_w^\oplus) \in \mathcal{A}^{-1}.$$

Let $q = [x_w^\oplus + 1 - xwx_w^\oplus][1 + x(1 - xwx_w^\oplus)]^{-1}$. Then $(xw + p)q = 1$. This implies that $xw + p \in \mathcal{A}$ is right invertible. Moreover, we have

$$\begin{aligned} 1 + ywq &= 1 + [ywxw(x_w^\oplus)^2 + yw - (ywx)wx_w^\oplus][1 + x(1 - xwx_w^\oplus)]^{-1} \\ &= 1 + yw[1 + x(1 - xwx_w^\oplus)] = 1 + yw \in \mathcal{A}^{-1}. \end{aligned}$$

By using Jacobson's lemma, $1 + qyw \in \mathcal{A}^{-1}$. Therefore we check that

$$\begin{aligned} pa &= p(x + y) = py = (1 - xwx_w^\oplus)y = y \in \mathcal{A}^{qnil}, \\ paw(1 - p) &= ywxwx_w^\oplus = 0, \\ aw + p &= xw + yw + p = (xw + p)[1 + qyw] \in \mathcal{A} \text{ is right invertible.} \end{aligned}$$

Since $(1 - p)a = xwx_w^\oplus(x + y) = xwx_w^\oplus x = x \in \mathcal{A}_w^\oplus$, it follows by [30, Theorem 2.10] that $(1 - p)aw \in \mathcal{A}^\#$ and $(1 - p)a \in (1 - p)aw\mathcal{A}$, as required.

(2) $\Rightarrow$ (1) By hypothesis, there exists a projection $p \in \mathcal{A}$ such that $(1 - p)a \in \mathcal{A}_w^\oplus$;

$$aw + p \in \mathcal{A} \text{ right invertible, } paw(1 - p) = 0, pa \in \mathcal{A}_w^{qnil}.$$

Set $x = (1 - p)a$ and $y = pa$. Then

$$\begin{aligned} x^*y &= [a^*(1 - p)^*]pa = 0, \\ ywx &= paw(1 - p)a = 0, \\ y &= pa \in \mathcal{A}_w^{qnil}. \end{aligned}$$

Write $(aw + p)q = 1$ for some $q \in \mathcal{A}$. Then $(1 - p)awq = (1 - p)(aw + p)q = 1 - p$, and so $(1 - p)awq(1 - p)a = (1 - p)a$ and $[(1 - p)awq]^* = (1 - p)^* = 1 - p = (1 - p)awq$. Hence, $(1 - p)a \in \mathcal{A}^{(1,3)}$.

By hypothesis, $(1 - p)a \in (1 - p)aw\mathcal{A}$ and $(1 - p)aw \in \mathcal{A}^\#$. In light of [30, Lemma 2.8], $w \in \mathcal{A}^{|| (1-p)a}$. According to [30, Theorem 2.6], $(1 - p)a \in \mathcal{A}_w^\oplus$. That is, $x \in \mathcal{A}_w^\oplus$. Therefore $a \in \mathcal{A}_w^\oplus$. $\square$

Corollary 2.6. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^\oplus$ if and only if there exists a projection $p \in \mathcal{A}$ such that*

- (1) $(1 - p)a \in \mathcal{A}^\#$;
- (2) $a + p \in \mathcal{A}$ is right invertible and $pa = pap \in \mathcal{A}^{qnil}$.

Proof. This is obvious by choosing $w = 1$ in Theorem 2.5. $\square$

3. Characterizations by Weighted g-Drazin Inverses

The goal of this section is to elucidate the generalized w -core inverse through the lens of the image associated with the weighted g-Drazin inverse. Evidently, $a_w^\oplus = (aw)^\oplus$. But $aw \in \mathcal{A}^\oplus$ do not imply $a \in \mathcal{A}_w^\oplus$. For instance, letting $a = \begin{pmatrix} -i & 1 \\ 0 & i \end{pmatrix}, w = \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix}$. Then $aw \in \mathbb{C}^{2 \times 2}$ has the core-inverse $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, while a has not w -core inverse. Contract to this observation, we now derive the following result which enable us to investigate the generalized w -core inverse by using the weighted g-Drazin inverse.

Lemma 3.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}$ has generalized w -core decomposition.
- (2) $aw \in \mathcal{A}^\oplus$.
- (3) $a \in \mathcal{A}^{d,w}$ and there exists unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

In this case, $a_w^\oplus = x = (aw)^\oplus$.

Proof. (1) $\Rightarrow$ (2) Since $a \in \mathcal{A}_w^\oplus$, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^\oplus, y \in \mathcal{A}_w^{qnil}.$$

Hence, $aw = xw + yw$. Since $x \in \mathcal{A}_w^\oplus$, we have

$$xw(x_w^\oplus)^2 = x_w^\oplus, (xwx_w^\oplus)^* = xwx_w^\oplus, xwx_w^\oplus x = x.$$

Then

$$xw(x_w^\oplus)^2 = x_w^\oplus, (xwx_w^\oplus)^* = xwx_w^\oplus, x(wx_w^\oplus)^2 = wx_w^\oplus.$$

This shows that $xw \in \mathcal{A}^\oplus$. Obviously, $yw \in \mathcal{A}^{qnil}$. Moreover, we check that

$$(xw)^*(yw) = w^*(x^*y)w = 0, (yw)(xw) = (ywx)w = 0.$$

In light of [4, Corollary 2.2], $aw \in \mathcal{A}^\oplus$. Moreover, we have $a_w^\oplus = x_w^\oplus = (xw)^\oplus = (aw)^\oplus$.

(2) $\Rightarrow$ (1) Let $x = (aw)^\oplus$. Then $aw \in \mathcal{A}^d$. In view of [4, Theorem 2.5], we have

$$x = awx^2, (awx)^* = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

We easily check that

$$\begin{aligned} \|awx - x(aw)^2x\|^{\frac{1}{n}} &= \|(aw)^n x^{n+1} - x(aw)^{n+1} x^{n+1}\|^{\frac{1}{n}} \\ &\leq \|(aw)^n - x(aw)^{n+1}\|^{\frac{1}{n}} \|x^{n+1}\|^{\frac{1}{n}}. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \|awx - x(aw)^2x\|^{\frac{1}{n}} = 0.$$

Therefore $x(aw)^2x = awx$, and then $a \in \mathcal{A}_w^\oplus$.

(2) $\Rightarrow$ (3) In view of [4, Theorem 2.5], $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Moreover, we have $(aw)(aw)^d x = (aw)^\oplus$. Since $x = awx^2$, by induction, we have $x = (aw)^n x^{n+1}$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned} \|x - (aw)(aw)^d x\|^{\frac{1}{n}} &= \|x - (aw)(aw)^d x\|^{\frac{1}{n}} \\ &= \|[1 - (aw)(aw)^d](aw)^n x^{n+1}\|^{\frac{1}{n}} \\ &= \|(aw)^n - (aw)^d(aw)^{n+1}\|^{\frac{1}{n}} \|x\|^{1+\frac{1}{n}}. \end{aligned}$$

As $\lim_{n \rightarrow \infty} \|(aw)^n - (aw)^d(aw)^{n+1}\|^{\frac{1}{n}} = 0$, we see that

$$\lim_{n \rightarrow \infty} \|x - (aw)(aw)^d x\|^{\frac{1}{n}} = 0,$$

and therefore $x = (aw)(aw)^d x = (aw)^\oplus$, as required.

(3) $\Rightarrow$ (2) Since $a \in \mathcal{A}^{d,w}$, we have $aw \in \mathcal{A}^d$. Therefore $aw \in \mathcal{A}^\oplus$ by Lemma 3.1. $\square$

We are ready to prove:

Theorem 3.2. Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}_w^\oplus$ if and only if

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

In this case, $a_w^\oplus = x$.

Proof. $\Rightarrow$ Choose $x = a_w^\oplus$. Then $aw \in \mathcal{A}^\oplus$ and $x = (aw)^\oplus$. In view of [4, Theorem 3.3], $aw \in \mathcal{A}^d$ and

$$x(aw)x = x, x\mathcal{A} = (aw)^d\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^d)^*.$$

Since $a^{d,w} = [(aw)^d]^2 a = a[(wa)^d]^2 = (aw)^d a(wa)^d$, we easily check that $(aw)^d = [(aw)^d]^2 aw = a^{d,w}a$, and then

$$(aw)^d\mathcal{A} = a^{d,w}\mathcal{A}.$$

On the other hand, we have $(a^{d,w})^* = [(aw)^d aw]^* [(aw)^d]^* = [(aw)^d]^* = [((aw)^d)^2 aw]^* = w^*(a^{d,w})^*$. Thus, $\mathcal{A}[(aw)^d]^* = \mathcal{A}(a^{d,w})^*$. Therefore

$$x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*$$

.

$\Leftarrow$ By hypothesis, There exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

As the argument above, we have

$$(aw)^d\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}[(aw)^d]^* = \mathcal{A}(a^{d,w})^*.$$

Therefore we have

$$xawx = x, x\mathcal{A} = (aw)^d\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^d)^*.$$

In light of [4, Theorem 3.3], $aw \in \mathcal{A}^\oplus$. According to Lemma 3.1, $a_w^\oplus = x$. $\square$

An element $a \in \mathcal{A}$ has pseudo w -core decomposition if there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^\oplus, y \in \mathcal{A}_w^{nil}.$$

The preceding x is unique if it exists, and we denote its w -core inverse by the pseudo w -core inverse of a , i.e., $a_w^\oplus = x_w^\oplus$. Evidently, $a_w^\oplus = z$ if and only if there exists $n \in \mathbb{N}$ such that

$$z = awz^2, (awz)^* = awz, z(aw)^2z = awz \text{ and } (aw)^n = awz(aw)^n.$$

In this case, $a_w^\oplus = z$.

Corollary 3.3. *Let $a, w \in \mathcal{A}$. Then a has pseudo w -core inverse if and only if*

- (1) $a \in \mathcal{A}_w^\oplus$.
- (2) aw has Drazin inverse.

Proof. $\Rightarrow$ Obviously, $a \in \mathcal{A}_w^\oplus$ and $aw \in \mathcal{A}^\oplus$. In view of [8, Theorem 2.3], aw has Drazin inverse, as desired.

$\Rightarrow$ In view of Theorem 3.2, there exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

Since aw has Drazin inverse, we have $a^{d,w} = [(aw)^d]^2a = [(aw)^D]^2a = a^{D,w}$. Hence,

$$x(aw)x = x, x\mathcal{A} = (aw)^D\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^D)^*.$$

Analogously to Theorem 3.2, we prove that $aw \in \mathcal{A}^\oplus$. Similarly to Lemma 3.1, a has pseudo w -core inverse. $\square$

Let $\mathcal{R}(X)$ represent the range space of a complex matrix X and X^* be the conjugate transpose of X . We improve [2, Theorem 3.5] and provide a new characterizations of pseudo W -core inverse for any complex matrix.

Corollary 3.4. *Let $A, X, W \in \mathbb{C}^{n \times n}$. Then the following are equivalent:*

- (1) X is the pseudo W -core inverse of A .
- (2) $XAWX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(AW)^D$.

Proof. Obviously, $AW \in \mathbb{C}^{n \times n}$ has Drazin inverse. Since $A^{D,W} = [(AW)^D]^2A$ and $(AW)^D = [(AW)^D]^2AW = A^{D,W}W$, we see that $\mathcal{R}(A^{D,W}) = \mathcal{R}(AW)^D$. Therefore we obtain the result by Theorem 3.2 and Corollary 3.3. $\square$

If a and x satisfy the equations $a = axa$ and $(ax)^* = ax$, then x is called $(1,3)$ -inverse of a and is denoted by $a^{(1,3)}$. We use $\mathcal{A}^{(1,3)}$ to stand for sets of all $(1,3)$ -invertible elements in $\mathcal{A}$. We now derive

Theorem 3.5. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}_w^\oplus$.
- (2) $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}^{(1,3)}$.
- (3) $a \in \mathcal{A}^{d,w}$ and there exists a projection $q \in \mathcal{A}$ such that $a^{d,w}\mathcal{A} = q\mathcal{A}$.

In this case, $a_w^\oplus = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = a^{d,w}wq$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. Let $x = a_w^\oplus$. By virtue of Theorem 2.1, there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Let $z = (wa)^2wx$. Then

$$\begin{aligned} a^{d,w}z &= [(aw)^d]^2a[(wa)^2wx] \\ &= (aw)^2(aw)^dx \\ &= (aw)^2(aw)^d(aw)^\oplus \\ &= (aw)(aw)^\oplus. \end{aligned}$$

Therefore $(a^{d,w}z)^* = [(aw)(aw)^\oplus]^* = (aw)(aw)^\oplus = a^{d,w}z$. Moreover, we verify that

$$\begin{aligned} a^{d,w}za^{d,w} &= (aw)(aw)^\oplus a^{d,w} \\ &= (aw)(aw)^\oplus [(aw)^d]^2a \\ &= (aw)(aw)^\oplus (aw)^2[(aw)^d]^4a \\ &= (aw)^2[(aw)^d]^4a \\ &= [(aw)^d]^2a \\ &= a^{d,w}. \end{aligned}$$

Accordingly, $a^{d,w} \in \mathcal{A}^{(1,3)}$, as required.

(2) $\Rightarrow$ (1) Let $x = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}$. Then we check that

$$\begin{aligned} xawx &= a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}awa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ &= a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}(aw)a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ &= a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a^{d,w}(wa)wa^{d,w}(a^{d,w})^{(1,3)} \\ &= a^{d,w}wa^{d,w}w(awa^{d,w})(a^{d,w})^{(1,3)} \\ &= a^{d,w}w[a^{d,w}wa]wa^{d,w}(a^{d,w})^{(1,3)} \\ &= [a^{d,w}wawa^{d,w}]wa^{d,w}(a^{d,w})^{(1,3)} \\ &= [a^{d,w}wawa^{d,w}]wa^{d,w}(a^{d,w})^{(1,3)} \\ &= a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ &= x. \end{aligned}$$

Clearly, $x\mathcal{A} \subseteq a^{d,w}\mathcal{A}$. Also we see that

$$a^{d,w} = (a^{d,w}w)^2a = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a^{d,w}wa = xa^{d,w}wa;$$

hence, $a^{d,w}\mathcal{A} \subseteq x\mathcal{A}$. Thus $x\mathcal{A} = a^{d,w}\mathcal{A}$.

We easily verify that

$$\begin{aligned} x &= a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}] \\ &= a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}]^* = a^{d,w}w[(a^{d,w})^{(1,3)}]^*[a^{d,w}]^*; \end{aligned}$$

and then, $\mathcal{A}x \subseteq \mathcal{A}(a^{d,w})^*$. Also we check that

$$\begin{aligned} [a^{d,w}]^* &= [a^{d,w}(a^{d,w})^{(1,3)}a^{d,w}]^* = [(a^{d,w}(a^{d,w})^{(1,3)})^*a^{d,w}]^* \\ &= [a^{d,w}]^*a^{d,w}(a^{d,w})^{(1,3)} \\ &= [a^{d,w}]^*[a^{d,w}wa^{d,w}a^{d,w}](a^{d,w})^{(1,3)} \\ &= [a^{d,w}]^*[a^{d,w}wa]wa^{d,w}(a^{d,w})^{(1,3)} \\ &= [a^{d,w}]^*[awa^{d,w}]wa^{d,w}(a^{d,w})^{(1,3)} \\ &= [a^{d,w}]^*aw[a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}] \\ &= [a^{d,w}]^*awx, \end{aligned}$$

and then $\mathcal{A}(a^{d,w})^* \subseteq \mathcal{A}x$. Hence $\mathcal{A}x = \mathcal{A}(a^{d,w})^*$. Accordingly, $a \in \mathcal{A}_w^\oplus$ by Theorem 3.2.

(2) $\Rightarrow$ (3) By hypothesis, $a^{d,w} \in \mathcal{A}^{(1,3)}$, and so $a^{d,w} = a^{d,w}(a^{d,w})^{(1,3)}a^{d,w}$ and $[a^{d,w}(a^{d,w})^{(1,3)}]^* = a^{d,w}(a^{d,w})^{(1,3)}$. Let $q = a^{d,w}(a^{d,w})^{(1,3)}$. Then $a^{d,w}\mathcal{A} = q\mathcal{A}$, $q^2 = q = q^*$, as required.

(3) $\Rightarrow$ (2) Let $x = a^{d,w}wq$. Then $awx = awa^{d,w}wq = aw[(aw)^d]^2awq = aw(aw)^d q = q$, and so $(awx)^* = q^* = q = awx$. Moreover, we have

$$awx^2 = (awx)x = qa^{d,w}wq = a^{d,w}wq = x.$$

Obviously, $a^{d,w}w(aw) = (aw)a^{d,w}w$, and then we verify that

$$\begin{aligned} &|| (aw)^n - x(aw)^{n+1} || \\ &= || [(aw)^n - (a^{d,w}wq)a^{d,w}w(aw)^{n+2}] - [x((aw)^{n+1} - a^{d,w}w(aw)^{n+2})] || \\ &\leq || (aw)^n - a^{d,w}w(aw)^{n+1} || + || x || || (aw)^{n+1} - a^{d,w}w(aw)^{n+2} || \\ &\leq (1 + || x || || a ||) || (aw)^n - a^{d,w}w(aw)^{n+1} || \\ &= (1 + || x || || a ||) || (aw)^n (1 - a^{d,w}waw)^n || \\ &= (1 + || x || || a ||) || (aw - a^{d,w}w(aw)^2)^n ||. \end{aligned}$$

Since $aw - a^{d,w}w(aw)^2 = aw - (aw)^d(aw)^2 \in \mathcal{A}^{qnil}$, we have

$$\lim_{n \rightarrow \infty} || (aw - a^{d,w}w(aw)^2)^n ||^{\frac{1}{n}} = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} || (aw)^n - x(aw)^{n+1} ||^{\frac{1}{n}} = 0.$$

Then $x = (aw)^\oplus$. In view of Theorem 2.1, $a \in \mathcal{A}_w^\oplus$. In this case, $a_w^\oplus = x = a^{d,w}wq = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}$. $\square$

Corollary 3.6. Let $a, w \in \mathcal{A}$. Then the following are equivalent:

- (1) a has pseudo w -core inverse.
- (2) $a \in \mathcal{A}^{D,w}$ and $a^{D,w} \in \mathcal{A}^{(1,3)}$.
- (3) $a \in \mathcal{A}^{D,w}$ and there exists a projection $q \in \mathcal{A}$ such that $a^{D,w}\mathcal{A} = q\mathcal{A}$.

In this case, $a_w^\oplus = a^{D,w}wa^{D,w}(a^{D,w})^{(1,3)} = a^{D,w}wq$.

Proof. As $a \in \mathcal{A}^D$, we have $a^d = a^D$. Therefore we complete the proof by Theorem 3.5. $\square$

4. Relations with Weighted Core Inverses

The objective of this section is to delineate the connections between the generalized weighted core inverse and other types of weighted generalized inverses. We now proceed to the demonstration that forms the crux of this sections development.

Theorem 4.1. Let $a, w \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}_w^\oplus$.
 (2) $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}_w^\oplus$.

In this case,

$$a_w^\oplus = [a^{d,w}w]^2(a^{d,w})_w^\oplus.$$

Proof. (1) $\Rightarrow$ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. Let $x = a_w^\oplus$. Then we have

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

We verify that

$$\begin{aligned} \|aw(aw)^d - awx(aw)(aw)^d\| &= \|(aw)^n[(aw)^d]^n - awx(aw)^n[(aw)^d]^n\| \\ &\leq \|(aw)^n - awx(aw)^n\| \|(aw)^d\|^n. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n \rightarrow \infty} \|aw(aw)^d - awx(aw)(aw)^d\|^{\frac{1}{n}} = 0.$$

Hence, $awx(aw)(aw)^d = aw(aw)^d$. Let $z = (aw)^2x$. Then

$$\begin{aligned} a^{d,w}wz &= a^{d,w}w(aw)^2x = [(aw)^d]^2aw(aw)^2x = awx, \\ a^{d,w}wz^2 &= (awx)z = (awx)(aw)^2x = aw[x(aw)^2x] = (aw)^2x = z, \\ (a^{d,w}wz)^* &= (awx)^* = awx = a^{d,w}wz, \\ za^{d,w}wa^{d,w} &= (aw)^2xa^{d,w}wa^{d,w} = aw[awx(aw)(aw)^d]a^{d,w}wa^{d,w} \\ &= aw[(aw)(aw)^d]a^{d,w}wa^{d,w} = aw[(aw)(aw)^d][(aw)^d]^2awa^{d,w} \\ &= aw(aw)^d a^{d,w} = a^{d,w}. \end{aligned}$$

Accordingly,

$$a^{d,w}wz^2 = z, (a^{d,w}wz)^* = a^{d,w}wz, za^{d,w}wa^{d,w} = a^{d,w}.$$

Then $a^{d,w} \in \mathcal{A}_w^\oplus$ and $(a^{d,w})_w^\oplus = z = (aw)^2a_w^\oplus$, as desired.

(2) $\Rightarrow$ (1) Set $x = (a^{d,w})_w^\oplus$. Then we have

$$a^{d,w}wxa^{d,w} = a^{d,w}, [a^{d,w}wx]^* = a^{d,w}wx.$$

Hence, $a^{d,w} \in \mathcal{A}^{(1,3)}$. According to Theorem 3.5, $a \in \mathcal{A}_w^\oplus$. Moreover, we have

$$a_w^\oplus = a^{d,w}wa^{d,w}(wx) = a^{d,w}wa^{d,w}w(a^{d,w})_w^\oplus = [a^{d,w}w]^2(a^{d,w})_w^\oplus.$$

□

As an immediate consequence, we provide formulas of the pseudo weighted core inverse of a complex matrix.

Corollary 4.2. Let $A, W \in \mathbb{C}^{n \times n}$. Then

$$\begin{aligned} A_W^\oplus &= [A^{D,W}W]^2(A^{D,W})_W^\oplus \\ &= (AW)^k A[(AW)^{k+1}A]^\dagger, \end{aligned}$$

where $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$.

Proof. By virtue of Theorem 4.1, $A_W^\oplus = [A^{D,W}W]^2(A^{D,W})_W^\oplus$. In view of [9, Theorem 2.10],

$$W^{\oplus, A} = (WA)^k [A(WA)^{k+1}A]^\dagger.$$

According to Lemma 3.1, we get

$$\begin{aligned} A_W^\oplus &= (AW)^\oplus = AW[(AW)^\oplus]^2 \\ &= A[W((AW)^\oplus)^2] = A[W^\oplus, A] \\ &= A[(WA)^k[A(WA)^{k+1}]^\dagger] = (AW)^k A[(AW)^{k+1} A]^\dagger, \end{aligned}$$

as asserted. $\square$

Example 4.3.

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$, $W = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{3 \times 3}$. We take the involution on $\mathbb{C}^{4 \times 4}$ as the conjugate transpose. Then

$$AW = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, WA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Hence, $\max\{\text{ind}(AW), \text{ind}(WA)\} = 1$. Moreover, we have

$$\begin{aligned} A_W^\oplus &= AWA[(AW)^2 A]^\dagger \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^\dagger \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Evidently, we check that

$$AW(A_W^\oplus)^2 = A_W^\oplus, (AWA_W^\oplus)^* = AWA_W^\oplus, AW = AWA_W^\oplus AW.$$

Let $a, x \in \mathcal{A}$. x is called $(1, 4)$ -inverse of a and is denoted by $a^{(1,4)}$ provided that $axa = a$ and $(xa)^* = xa$. We use $\mathcal{A}^{(1,4)}$ to stand for sets of all $(1, 4)$ invertible elements in $\mathcal{A}$.

Lemma 4.4. Let $f \in \mathcal{A}$ be an idempotent. Then the following are equivalent:

- (1) $f \in \mathcal{A}^{(1,3)}$.
- (2) $f^\pi \in \mathcal{A}^{(1,4)}$.

Proof. See [20, Lemma 3.2]. $\square$

Theorem 4.5. Let $a, w \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}_w^\oplus$.
- (2) $a \in \mathcal{A}^{d,w}$ and $awa^{d,w}w \in \mathcal{A}^{(1,3)}$.
- (3) $a \in \mathcal{A}^{d,w}$ and $(aw)^\pi \in \mathcal{A}^{(1,4)}$.

In this case, $a_w^\oplus = a^{d,w}w(awa^{d,w}w)^{(1,3)} = a^{d,w}w[1 - (a^\pi)^{(1,4)}a^\pi]$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. For any $m \in \mathbb{N}$, we check that

$$\begin{aligned} \|awa^{d,w}w - awa_w^\oplus awa^{d,w}w\| &= \|awa^{d,w}w - awa_w^\oplus awa^{d,w}w\| \\ &\leq \|(aw)^m - awa_w^\oplus (aw)^m\| \|((aw)^d)^m a^{d,w}w\|. \end{aligned}$$

Since

$$\lim_{m \rightarrow \infty} \|(aw)^m - awa_w^\oplus (aw)^m\|^{\frac{1}{m}} = 0,$$

we have

$$\lim_{m \rightarrow \infty} \|awa^{d,w}w - awa_w^\oplus awa^{d,w}w\|^{\frac{1}{m}} = 0.$$

Hence $awa_w^\oplus awa^{d,w}w = awa^{d,w}w$, and then

$$\begin{aligned} [awa^{d,w}w][awa_w^\oplus] &= aw[(aw)^d]^2 awa_w^\oplus = awa_w^\oplus, \\ ((awa^{d,w}w)(awa_w^\oplus))^* &= (awa_w^\oplus)^* = awa_w^\oplus = (awa^{d,w}w)(awa_w^\oplus), \\ (awa^{d,w}w)(awa_w^\oplus)(awa^{d,w}w) &= awa_w^\oplus (awa^{d,w}w) = awa_w^\oplus awa^{d,w}w \\ &= awa^{d,w}w. \end{aligned}$$

Accordingly, $awa^{d,w}w \in \mathcal{A}^{(1,3)}$, as desired.

(2) $\Rightarrow$ (1) Let $x = a^{d,w}w(awa^{d,w}w)^{(1,3)}$. Then we verify that

$$\begin{aligned} awx &= awa^{d,w}w(awa^{d,w}w)^{(1,3)} = aw(aw)^d(aw(aw)^d)^{(1,3)}, \\ (awx)^* &= awx, \\ awx^2 &= aw(aw)^d(aw(aw)^d)^{(1,3)}a^{d,w}w(awa^{d,w}w)^{(1,3)} \\ &= aw(aw)^d(aw(aw)^d)^{(1,3)}aw[(aw)^d]^2(awa^{d,w}w)^{(1,3)} \\ &= (aw)^d(awa^{d,w}w)^{(1,3)} = x, \end{aligned}$$

$$\begin{aligned} &\|(aw)^n - awx(aw)^n\| \\ &= \|(aw)^n - aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n\| \\ &\leq \|(aw)^n - (aw)^d(aw)^{n+1}\| + \|(aw)(aw)^d(aw)^n - aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n\| \\ &\leq \|(aw)^n - (aw)^d(aw)^{n+1}\| + \|(aw)(aw)^d(aw(aw)^d)^{(1,3)}(aw)(aw)^d(aw)^n \\ &\quad - aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n\| \\ &\leq \|(aw)^n - (aw)^d(aw)^{n+1}\| + \|(aw)(aw)^d(aw(aw)^d)^{(1,3)}\| \|(aw)^d(aw)^{n+1} - (aw)^n\| \\ &= \|(aw)^n - (aw)^d(aw)^{n+1}\| [1 + \|(aw)(aw)^d(aw(aw)^d)^{(1,3)}\|]. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Therefore $a \in \mathcal{A}_w^\oplus$. In this case, $a_w^\oplus = a^{d,w}w(awa^{d,w}w)^{(1,3)}$.

(2) $\Leftrightarrow$ (3) In view of Lemma 4.4, $aw(aw)^d \in \mathcal{A}^{(1,3)}$ if and only if $(aw)^\pi = 1 - (aw)(aw)^d \in \mathcal{A}^{(1,4)}$, as desired. $\square$

Corollary 4.6. Let $a, w \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}^\oplus$.
- (2) $a \in \mathcal{A}^d$ and $aa^d \in \mathcal{A}^{(1,3)}$.
- (3) $a \in \mathcal{A}^d$ and $a^\pi \in \mathcal{A}^{(1,4)}$.

In this case, $a^\oplus = a^d(aa^d)^{(1,3)} = a^d[1 - (a^\pi)^{(1,4)}a^\pi]$.

Proof. This is obvious by choosing $w = 1$ in Theorem 4.6. $\square$

5. Generalized w -Core Orders

This section is dedicated to exploring the generalized weighted core order for pairs of elements within a Banach $*$ -algebra. The subsequent result will be a recurrent tool in examining the characteristics of generalized weighted core orders.

Lemma 5.1. *Let $a, b \in \mathcal{A}_w^\oplus$. Then the following are equivalent:*

- (1) $a \leq_w^\oplus b$.
- (2) $awa^{d,w} = bwa^{d,w}, a^*a^{d,w} = b^*a^{d,w}$.

Proof. (1) $\Rightarrow$ (2) Since $a \leq_w^\oplus b$, we have

$$awa_w^\oplus = bwa_w^\oplus, a_w^\oplus a = a_w^\oplus b.$$

By virtue of 4.5, $a_w^\oplus = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}$. Then

$$awa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = bwa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)},$$

and so

$$awa^{d,w}wa^{d,w} = bwa^{d,w}wa^{d,w}.$$

Since $a^{d,w} = (a^{d,w}w)^2a$, we have $awa^{d,w} = [awa^{d,w}wa^{d,w}]wa = [bwa^{d,w}wa^{d,w}]wa = bwa^{d,w}$.

Since $a_w^\oplus a = a_w^\oplus b$, we have

$$a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}b.$$

As $a^{d,w} = a(wa^{d,w})^2$, we deduce that

$$a^{d,w}(a^{d,w})^{(1,3)}a = a^{d,w}(a^{d,w})^{(1,3)}b.$$

This implies that

$$a^*a^{d,w}(a^{d,w})^{(1,3)} = b^*a^{d,w}(a^{d,w})^{(1,3)}.$$

Therefore $a^*a^{d,w} = b^*a^{d,w}$, as required.

(2) $\Rightarrow$ (1) Since $awa^{d,w} = bwa^{d,w}$, by virtue of Theorem 3.5, we have

$$\begin{aligned} awa_w^\oplus &= awa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ &= bwa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ &= bwa_w^\oplus. \end{aligned}$$

Since $a^*a^{d,w} = b^*a^{d,w}$, we have $a^*a^{d,w}(a^{d,w})^{(1,3)} = b^*a^{d,w}(a^{d,w})^{(1,3)}$, and then $a^{d,w}(a^{d,w})^{(1,3)}a = a^{d,w}(a^{d,w})^{(1,3)}b$. Therefore we derive

$$\begin{aligned} a_w^\oplus a &= a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a \\ &= a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}a] \\ &= a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}b] \\ &= a_w^\oplus b, \end{aligned}$$

thus yielding the result. $\square$

Let $a \in \mathcal{A}_w^\oplus, b \in \mathcal{A}$. Recall that $a \leq_w^\oplus b$ if $awa_w^\oplus = bwa_w^\oplus$ and $a_w^\oplus a = a_w^\oplus b$ (see [32]). We are now ready to prove:

Theorem 5.2. Let $a, b \in \mathcal{A}_w^\oplus$. If $a = a_1 + a_2, b = b_1 + b_2$ are generalized w -core decompositions of a and b . Then the following are equivalent:

- (1) $a \leq_w^\oplus b$.
- (2) $a_1 \leq_w^\oplus b_1$.

Proof. (1) $\Rightarrow$ (2) Since $a \leq_w^\oplus b$, we have $awa_w^\oplus = bwa_w^\oplus$ and $a_w^\oplus a = a_w^\oplus b$. For any $m \in \mathbb{N}$, we derive

$$\begin{aligned} a_1 w(a_1)_w^\oplus &= (a_1 + a_2) w(a_1)_w^\oplus = a w a_w^\oplus = b w a_w^\oplus \\ &= b w a w (a_w^\oplus)^2 = b w [a w a_w^\oplus] a_w^\oplus = b w [b w a_w^\oplus] a_w^\oplus \\ &= (b w)^2 (a_w^\oplus)^2 = \dots = (b w)^m (a_w^\oplus)^m, \\ b_1 w(a_1)_w^\oplus &= b w b_w^\oplus b w a_w^\oplus = b w b_w^\oplus b w a w (a_w^\oplus)^2 = b w b_w^\oplus (b w)^2 (a_w^\oplus)^2 \\ &= \dots = b w b_w^\oplus (b w)^m (a_w^\oplus)^m. \end{aligned}$$

Thus, we have

$$\begin{aligned} &||a_1 w(a_1)_w^\oplus - b_1 w(a_1)_w^\oplus|| \\ &= ||(b w)^m (a_w^\oplus)^m - b w b_w^\oplus (b w)^m (a_w^\oplus)^m|| \\ &\leq ||(b w)^m - b w b_w^\oplus (b w)^m|| ||(a_w^\oplus)^m||. \end{aligned}$$

In view of Theorem 2.1,

$$\lim_{m \rightarrow \infty} ||(b w)^m - b w b_w^\oplus (b w)^m||^{\frac{1}{m}} = 0.$$

Hence,

$$\lim_{m \rightarrow \infty} ||a_1 w(a_1)_w^\oplus - b_1 w(a_1)_w^\oplus||^{\frac{1}{m}} = 0.$$

Therefore $a_1 w(a_1)_w^\oplus = b_1 w(a_1)_w^\oplus$.

Since $b_1 = b w b_w^\oplus b$, we verify that

$$a w a_w^\oplus = a_1 w(a_1)_w^\oplus = b_1 w(a_1)_w^\oplus = b w b_w^\oplus b w a_w^\oplus = b w b_w^\oplus a w a_w^\oplus.$$

Thus,

$$[a w a_w^\oplus]^* = [b w b_w^\oplus a w a_w^\oplus]^*,$$

and so

$$a w a_w^d = a w a_w^\oplus b w b_w^\oplus.$$

Then we see that

$$\begin{aligned} (a_1)_w^\oplus a_1 &= a_w^\oplus (a w a_w^\oplus a) = a_w^\oplus (a w a_w^\oplus) a \\ &= a_w^\oplus (a w a_w^\oplus) b \\ &= a_w^\oplus (a w a_w^\oplus b w b_w^\oplus) b \\ &= (a_w^\oplus a w a_w^\oplus) b w b_w^\oplus b \\ &= a_w^\oplus (b w b_w^\oplus b) = (a_1)_w^\oplus b_1. \end{aligned}$$

Therefore $a_1 \leq_w^\oplus b_1$.

(2) $\Rightarrow$ (1) Obviously, we have

$$a w a_w^\oplus = (a_1 + a_2) w a_1^\oplus = a_1 w a_1^\oplus = b_1 w a_1^\oplus = b w b_w^\oplus b w a_w^\oplus.$$

Then

$$a_w^\oplus = a w (a_w^\oplus)^2 = b w b_w^\oplus b w (a_w^\oplus)^2.$$

Since $\lim_{n \rightarrow \infty} ||(b w)^n - b w b_w^\oplus (b w)^n||^{\frac{1}{n}} = 0$, we deduce that

$$b w b_w^\oplus b w a_w^\oplus = b w a_w^\oplus.$$

This implies that

$$awa_w^{\oplus} = bwa_w^{\oplus}.$$

Clearly, $a_w^{\oplus}a_2 = (a_1)_w^{\oplus}a_2 = (a_1)_w^{\oplus}a_1w(a_1)_w^{\oplus}a_2 = (a_1)_w^{\oplus}(a_1w(a_1)_w^{\oplus})^*a_2 = (a_1)_w^{\oplus}[w(a_1)_w^{\oplus}]^*(a_1)^*a_2 = 0$.

Moreover, we have

$$\begin{aligned} awa_w^{\oplus} &= bwb_w^{\oplus}bwa_w^{\oplus} \\ &= (bwb_w^{\oplus})(awa_w^{\oplus}). \end{aligned}$$

Then

$$\begin{aligned} awa_w^{\oplus} &= (awa_w^{\oplus})^* \\ &= (awa_w^{\oplus})^*(bwb_w^{\oplus})^* \\ &= awa_w^{\oplus}bwb_w^{\oplus}. \end{aligned}$$

Hence, $a_w^{\oplus} = a_w^{\oplus}awa_w^{\oplus} = a_w^{\oplus}awa_w^{\oplus}bwb_w^{\oplus} = a_w^{\oplus}bwb_w^{\oplus}$. Accordingly, $a_w^{\oplus}b = a_w^{\oplus}bwb_w^{\oplus}b = (a_1)_w^{\oplus}b_1 = (a_1)_w^{\oplus}a_1 = a_w^{\oplus}(a_1 + a_2) = a_w^{\oplus}a$, thus yielding the result. $\square$

Corollary 5.3. The relation $\leq_w^{\oplus}$ for generalized w -core invertible elements is a pre-order on $\mathcal{A}$.

Proof. Step 1. $a \leq_w^{\oplus} a$. Let $a = a_1 + a_2$ be the generalized w -core decomposition. In view of [32, Theorem 2.3], $a_1 \leq_w^{\oplus} a_1$. By using Theorem 5.1, $a \leq_w^{\oplus} a$.

Step 2. Assume that $a \leq_w^{\oplus} b$ and $b \leq_w^{\oplus} c$. We claim that $a \leq_w^{\oplus} c$. Let $a = a_1 + a_2$, $b = b_1 + b_2$ and $c = c_1 + c_2$ be the generalized w -core decompositions of a, b and c , respectively. By virtue of Lemma 5.1, we have $a_1 \leq_w^{\oplus} b_1$ and $b_1 \leq_w^{\oplus} c_1$. In view of [32, Theorem 2.3], we have $a_1 \leq_w^{\oplus} c_1$. By using Lemma 5.1 again, $a \leq_w^{\oplus} c$.

Therefore the relation $\leq_w^{\oplus}$ for generalized w -core invertible elements is a pre-order. $\square$

The relation $\leq_w^{\oplus}$ for generalized w -core invertible elements is a pre-order, while it is not partial order as the following shows.

Example 5.4.

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{3 \times 3}$. We take the involution on $\mathbb{C}^{3 \times 3}$ as the conjugate transpose. Then $A^{D,W} = [(AW)^D]^2A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = [(BW)^D]^2B = B^{D,W}$. By using Lemma 5.1, we directly verify that $A \leq_W^{\oplus} B$ and $B \leq_W^{\oplus} A$. But $A \neq B$.

Theorem 5.5. Let $a, b \in \mathcal{A}_w^{\oplus}$. Then the following are equivalent:

- (1) $a \leq_w^{\oplus} b$.
- (2) a and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{(p,q)}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 + (b - a) \end{pmatrix}_{(p,q)},$$

where $p = awa_w^{\oplus}$ and $q = (wa)(wa)^d$.

Proof. (1) $\Rightarrow$ (2) Let $p = awa_w^{\oplus}$ and $q = (wa)(wa)^d$. Then $p^2 = p = p^* \in \mathcal{A}$ and $q^2 = q \in \mathcal{A}$. We check that

$$\begin{aligned}(1-p)aq &= [1 - awa_w^{\oplus}]a(wa)(wa)^d \\ &= [1 - awa_w^{\oplus}]awa(wa)^n[(wa)^d]^n \\ &= [1 - aw(aw)^{\oplus}](aw)^{n+1}a[(wa)^d]^n \\ &= [(aw)^{n+1} - (aw)a_w^{\oplus}(aw)^{n+1}]a[(wa)^d]^n \\ &= 0;\end{aligned}$$

then $(1-p)ap = 0$. Moreover, we verify that

$$\begin{aligned}(1-p)bq &= [1 - awa_w^{\oplus}]b(wa)(wa)^d \\ &= [1 - awa_w^{\oplus}]bwa^{d,w}wa \\ &= [1 - awa_w^{\oplus}]awa^{d,w}wa \\ &= 0.\end{aligned}$$

Write $b = \begin{pmatrix} b_1 & b_{12} \\ 0 & b_2 \end{pmatrix}_p$. Clearly, we have

$$\begin{aligned}pbq &= awa_w^{\oplus}b(wa)(wa)^d \\ &= awa_w^{\oplus}bwa^{d,w}wa \\ &= awa_w^{\oplus}awa^{d,w}wa \\ &= awa_2^{\oplus}a(wa)(wa)^d \\ &= paq,\end{aligned}$$

and so $a_1 = b_1$.

Also we have

$$\begin{aligned}pb(1-q) &= awa_w^{\oplus}b[1 - (wa)(wa)^d] \\ &= aw(a_w^{\oplus}b)[1 - (wa)(wa)^d] \\ &= aw(a_w^{\oplus}a)[1 - (wa)(wa)^d] \\ &= pa(1-p).\end{aligned}$$

Moreover, $(1-p)b(1-q) = (1-p)b = b - pb = b - a(a_w^{\oplus}b) = b - aa_w^{\oplus}a = b - pa = (1-p)a + (b-a) = a_2 + (b-a)$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, $paq = pbq$ and $pa(1-q) = pb(1-q)$. Then $pa = pb$. Hence, $awa_w^{\oplus}a = awa_w^{\oplus}b$. This implies that $a_w^{\oplus}a = a_w^{\oplus}b$.

Moreover, we have $(1-p)aq = 0 = (1-p)bq$. As $paq = pbq$, we have $aq = bq$, and so $a(wa)(wa)^d = b(wa)(wa)^d$. Then $awa_w^{\oplus} = bwa_w^{\oplus}$. In light of Lemma 5.1, $a \leq_w^{\oplus} b$, as asserted. $\square$

The generalized core-EP inverse for a Banach algebra element was introduced in [5]. $a \leq^{\oplus} b$ if and only if $aa^{\oplus} = ba^{\oplus}$ and $a^{\oplus}a = a^{\oplus}b$. As an immediate consequence of Theorem 5.5, we derive

Corollary 5.6. Let $a, b \in \mathcal{A}^{\oplus}$. Then the following are equivalent:

- (1) $a \leq^{\oplus} b$.
- (2) a and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{(p,q)}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 + (b-a) \end{pmatrix}_{(p,q)},$$

where $p = aa^{\oplus}$ and $q = aa^d$.

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