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Posted Date: 14 February 2025

doi: 10.20944/preprints202502.1094.v1

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Bäcklund Transformation for Solving a (3+1)-Dimensional Integrable Equation

Binlu Feng [1], Linlin Gui [2,*], Yufeng Zhang [2] and Siqi Han [2]

[1] College of Technology and Data, Yantai Nanshan University, Yantai, Shandong, 265713, China.
[2] School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, China
* Correspondence: guilinlin9806@163.com

Abstract: A new generalized (3+1)-dimensional Kadomtsev-Petviashvil (3dKP) equation is derived from the inverse scattering transform method. This equation can reduce to the standard KP equation and the well-know (3+1)-dimensional equation. Making use of the Lax pair transformation, a Bäcklund transformation of the generalized (3+1)-dimensional KP equation is constructed and some soliton solutions are produced. Finally, a superposition formula by making use of the Bäcklund transformation is singled out as well. As we know that such the work presented in the paper is not studied up to now.

Keywords: 3dKP equation; Bäcklund transformation; soliton solution; superposition formula

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1. Introduction

Searching for Bäcklund transformations (BTS) of nonlinear differential equations has been an important topic in the field of soliton theory and integrable systems. The methods for investigating Bäcklund transformations of (1+1)-dimensional differential equations have widely studied [1–4]. However, for the Bäcklund transformations of high-dimensional integrable equations, only some special approaches were presented. For example, in Ref [5], the Bäcklund transformation connected with the general two-dimensional Gelfand-Dikij-Zakharov-Shabat spectral problems was found. In Refs [6,8], authors studied the well-known KP equation and nonlinear evolution equation starting from a two-dimensional vector field and further obtained the Bäcklund transformation of the KP equation and nonlinear evolution equation. Konopelchenko generalized the AKNS-technique to the two-dimensional arbitrary order matrix spectral problem in [8]. The Bäcklund transformations in (2+1)-dimensional were found as well. For example, the two-dimensional Korteweg-de Vries-like equation [9], sine-Gordon system [10] and Sawada-Kotera equation [11] were exhibited. It has been found that many high-dimensional equations have a wide range of applications in mathematical physics, including neural network algorithms and various architectures, science and engineering, fiber optics and fluid mechanics. In recent years, the exact solutions and properties for high-dimensional equations have become a focus of research, especially for equations with more than two spatial variables. Constructing Bäcklund transformation and exact solutions of nonlinear evolution equations in three dimensions have been important open problems in the field of integrability research. For example, Yin et al. used the Hirota bilinear method to construct a Bäcklund transformation of the (3+1)-dimensional nonlinear evolution equation, which consists of four equations and involves six free parameters [11]. Ma et al. proposed a bilinear Bäcklund transform for the (3+1)-dimensional generalized KP equation and computed two classes of exponential and rational traveling wave solutions with arbitrary wave numbers [13].

In this paper, we study the generalized (3+1)-dimensional Kortweg-de Vries (3dKdV,that is 3dKP) equation starting from a set of vector fields which can reduce to the celebrated KP equation which describes disturbances in a weakly dispersive, weakly nonlinear medium. Again the 3dKP equation obtained in the paper also reduces to the standard three-dimensional KP equation (see [14])

$$u_t + u_{xxx} + 6uu_x + 3\sigma^2\partial_x^{-1}(u_{yy} + u_{zz}) = 0, \tag{1.1}$$

 doi:10.20944/preprints202502.1094.v1

which has an important role in fluid dynamics as well as in fields such as plasma physics. In [15], solitary wave and elliptic functional solutions of the 3dKP equation (1.1) have been studied. And soliton-like solutions and period-form solutions of the equation have been constructed [16]. Where u depends on time variable t and space variables x, y, $\sigma^2 = \pm 1$. At the same time, we need know that ∂_x^{-1} is an inverse operator of ∂_x, $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = 1$. If $\partial_z = 0$, equation (1.1) reduces to the (2+1)-dimensional KP equation [17].

This paper will be organized as follows: Utilizing the Lax pair transformation, the Bäcklund transformation and a type of new soliton solutions are constructed in sections 2 and 3, respectively. In section 4, a superposition formula is obtained via the Bäcklund transformation.

2. A Bäcklund Transformation of Three Dimensions

Consider the three-dimensional vector fields addtoresetequationsection

$$\begin{aligned} L_1 =& \partial_x^2 + \sigma\partial_y + \sigma\partial_z + u, \\ L_2 =& \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x - 3\sigma\partial_x^{-1}u_y - 3\delta\partial_x^{-1}u_z + \gamma\partial_x, \end{aligned} \tag{2.1}$$

where σ, δ and γ are constants independent of x, y, z and t. It is easy calculate the commutativity condition of the Lax pair

$$\begin{aligned} L_1\psi =& \alpha\psi, \\ L_2\psi =& \beta\psi, \end{aligned} \tag{2.2}$$

implies that

$$u_{xt} + u_{xxxx} + \gamma u_{xx} + 3(u^2)_{xx} + 3\sigma^2 u_{yy} + 6\sigma\delta u_{yz} + 3\delta^2 u_{zz} = 0, \tag{2.3}$$

where α, β are eigenvalues independent of $\vec{x} = (x, y, z, t)$. When $\gamma = 0$, $\sigma = 1$, $\delta = 0$ equation (2.3) reduces to well-known KP equation

$$u_{xt} + u_{xxxx} + 3(u^2)_{xx} + 3u_{yy} = 0. \tag{2.4}$$

When $\gamma = 0$, $\sigma = \pm i$, $\delta = 0$ equation (2.3) becomes

$$u_{xt} + u_{xxxx} + 3(u^2)_{xx} - 3u_{yy} = 0. \tag{2.5}$$

The positive sign refers to negative dispersion in (2.4), while the negative sign refers to the positive dispersion in (2.5)

Let $\phi = \ln\psi$, $u = v_x$, we can obtain that equation (2.2) has the following form

$$\begin{aligned} &\phi_{xx} + \phi_x^2 + \sigma\phi_y + \sigma\phi_z + v_x = \alpha, \\ &\phi_t + 4(\phi_{xxx} + 3\phi_x\phi_{xx} + \phi_x^3) + 6v_x\phi_x + 3v_{xx} - 3\sigma(v_y + v_z) + \gamma\phi_x = \beta. \end{aligned} \tag{2.6}$$

Eliminating v in equations (2.6), we get the following nonlinear equation

$$\begin{aligned} &\phi_t + \phi_{xxx} - 2\phi_x^3 + 6\alpha\phi_x + 6\sigma\delta\int^x \phi_{yz}dx + 3\sigma^2\int^x \phi_{yy}dx \\ &+ 3\sigma^2\int^x \phi_{zz}dx - 6\sigma\int^x \phi_y\phi_{xx}dx - 6\sigma\int^x \phi_z\phi_{xx}dx + \gamma\phi_x = \beta. \end{aligned} \tag{2.7}$$

 doi:10.20944/preprints202502.1094.v1

We observe that for every $(\phi, \beta, \sigma, \delta, \gamma)$ which satisfy equation (2.7), the $(-\phi, -\beta, -\sigma, -\delta, \gamma)$ satisfy the equation as well. For this new sequence $(-\phi, -\beta, -\sigma, -\delta, \gamma)$, there is a corresponding solution $u' \equiv v'_x$ of equation (2.3) such that

$$\begin{aligned} &-\phi_{xx} + \phi_x^2 + \sigma\phi_y + \sigma\phi_z + v'_x = \alpha, \\ &-\phi_t + 4(-\phi_{xxx} + 3\phi_x\phi_{xx} - \phi_x^3) - 6v'_x\phi_x + 3v'_{xx} - 3\sigma(v'_y + v'_z) - \gamma\phi_x = -\beta. \end{aligned} \tag{2.8}$$

Taking the difference and sum of equation (2.6) and equation (2.8), one gets that

$$\begin{aligned} &2\phi_{xx} + v_x - v'_x = 0, \\ &2\phi_t + 8\phi_{xxx} + 8\phi_x^3 + 6(v + v')_x\phi_x + 3(v - v')_{xx} \\ &\quad - 3\sigma(v' - v)_y + 3\sigma(v' - v)_z + 2\gamma\phi_x = 2\beta, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &2\phi_x^2 + 2\sigma\phi_y + 2\sigma\phi_z + v_x + v'_x = 2\alpha, \\ &24\phi_x\phi_{xx} + 6(v - v')_x\phi_x + 3(v - v')_{xx} - 3\sigma(v + v')_y - 3\sigma(v + v')_z = 0. \end{aligned} \tag{2.10}$$

From the first equation in (2.9), we have

$$\phi = \frac{1}{2}\int^x (v - v')dx. \tag{2.11}$$

Inserting equation (2.11) into equations (2.9) and (2.10) yields the Bäcklund transformation

$$\begin{aligned} &(v' - v)^2 + 2\int^x (v - v')_y dx + 2\int^x (v - v')_z dx + 2(v + v')_x = 0, \\ &24\phi_x\phi_{xx} + 6(v - v')_x\phi_x + 3(v - v')_{xx} - 3\sigma(v + v')_y - 3\sigma(v + v')_z = 0. \end{aligned} \tag{2.12}$$

When taking $v = 0$ in equation (2.1), it becomes

$$\begin{aligned} &\psi_{xx} + \sigma\psi_y + \sigma\psi_z = 0, \\ &\psi_t + 4\psi_{xxx} + \gamma\psi_x = 0. \end{aligned} \tag{2.13}$$

A solution to (2.13) can be taken as

$$\begin{aligned} \psi &= \sum_l \Lambda_l e^{-lx - \frac{l^2}{2\delta}y - \frac{l^2}{2\delta}z + (4l^3 + \gamma l)t} \\ &\equiv \sum_l \Lambda_l e^{\theta_l}. \end{aligned} \tag{2.14}$$

The summation runs over all complex values of l and Λ_l is a spectral function. A new solution is given by

$$\begin{aligned} u' &= v'_x = 2\phi_{xx} = 2(\ln\phi)_{xx} \\ &= 2\frac{(-\sum_l \Lambda_l l^2 e^{\theta_l})(\sum_l \Lambda_l e^{\theta_l}) - (\sum_l l\Lambda_l e^{\theta_l})^2}{(\sum_l \Lambda_l e^{\theta_l})^2}, \end{aligned} \tag{2.15}$$

where

$$\theta_i = -l_i x - \frac{l_i^2}{2\delta}y - \frac{l_i^2}{2\delta}z + (4l_i^3 + \gamma l_i)t, \quad i = 1, 2. \tag{2.16}$$

 doi:10.20944/preprints202502.1094.v1

Special choice of Λ_l will result in special solutions, then we consider two different cases. When $\Lambda_l = \delta_{l,l_1} + \delta_{l,l_2}$, from (2.15) one infers that

$$u' = \frac{1}{2}(l_1 - l_2)^2 \operatorname{sech}^2 \frac{1}{2}[(l_2 - l_1)x + \frac{l_2^2 - l_1^2}{2\sigma}y + \frac{l_2^2 - l_1^2}{2\delta}z + (4(l_1^3 - l_2^3 + \gamma(l_1 - l_2))t], \tag{2.17}$$

which is a three-dimensional soliton solution with amplitude $\frac{1}{2}(l_1 - l_2)^2$ and velocity

$$P_x =: -4(l_1^2 + l_2^2 + l_1 l_2 + \gamma), \quad P_y =: -\frac{8\sigma(l_1^2 + l_2^2 + l_1 l_2 + \gamma)}{l_1 + l_2}, \quad P_z =: -\frac{8\delta(l_1^2 + l_2^2 + l_1 l_2 + \gamma)}{l_1 + l_2}. \tag{2.18}$$

When $\Lambda_l = \delta_{l,l_1} - \delta_{l,l_2}$, from (2.15) we have

$$u' = \frac{1}{2}(l_1 - l_2)^2 \csc h^2 \frac{1}{2}[(l_2 - l_1)x + \frac{l_2^2 - l_1^2}{2\sigma}y + \frac{l_2^2 - l_1^2}{2\delta}z + (4(l_1^3 - l_2^3 + \gamma(l_1 - l_2))t], \tag{2.19}$$

which is a singular solution.

3. A Type of New Soliton Solutions

The spectral problem (2.13) not only possesses the soliton solutions such as (2.17) and (2.19), but also does the following solution addtoresetequationsection

$$\psi = \sum_l \frac{\Lambda_l e^{\theta_l}}{1 + \Delta e^{\theta_l}}, \tag{3.1}$$

where Δ is a constant. When $\Lambda_l = \delta_{l,l_1} + \delta_{l,l_2}$, it is easy to find

$$\begin{aligned} &\psi = P + Q, \\ &\psi_x = l_1 \Delta P^2 + l_2 \Delta Q^2 - l_1 P - l_2 Q, \\ &\psi_{xx} = 2l_1^2 \Delta^2 P^3 + 2l_2^2 \Delta Q^3 - 3l_1^2 \Delta P^2 - 3l_2^2 \Delta Q^2 + l_1^2 P + l_2^2 Q, \end{aligned} \tag{3.2}$$

where

$$P = \frac{\Lambda_l e^{\theta_1}}{1 + \Delta e^{\theta_1}}, \quad Q = \frac{\Lambda_l e^{\theta_2}}{1 + \Delta e^{\theta_2}}. \tag{3.3}$$

Therefore, a new solution to equation (2.1) is given by

$$u' = 2[\frac{l_1^2 P - 3l_1^2 \Delta P^2 + 2l_1^2 \Delta^2 P^3 + l_2^2 Q - 3l_2^2 \Delta Q^2 + 2l_2^2 \Delta Q^3}{P + Q} - (\frac{P - Q + l_1 \Delta P^2 + l_2 \Delta Q^2)}{P + Q})^2]. \tag{3.4}$$

When $\Lambda_l = \delta_{l,l_1} + \delta_{l,l_2}$, equation (3.1) becomes

$$\psi = \frac{\Lambda_l e^{\theta_1}}{1 + \Delta e^{\theta_1}} - \frac{\Lambda_l e^{\theta_2}}{1 + \Delta e^{\theta_2}} = P - Q. \tag{3.5}$$

Similar to the above calculation, we obtain another new solution to equation (2.1)

$$u' = 2[\frac{l_1^2 P - 3l_1^2 \Delta P^2 + 2l_1^2 \Delta^2 P^3 - l_2^2 Q + 3l_2^2 \Delta Q^2 + 2l_2^2 \Delta Q^3}{P - Q} - (\frac{-l_1 P + l_2 Q + l_1 \Delta P^2 - l_2 \Delta Q^2)}{P - Q})^2]. \tag{3.6}$$

4. Superposition Formula

The advantage of the Bäcklund transformation is that a superposition of solutions can be derived. This superposition allows us to construct more complex solutions only using algebraic methods. To get the superposition formula, let v_1 be a solution given by Bäcklund transformation from a known solution v_0 with some spectral function $\Lambda_{1,l}$, v_2 be a second solution generated from v_0 with spectral

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function $\Lambda_{2,l}$, v_3 a third solution obtained from v_1 with spectral functions $\Lambda_{2,l}$. According to the definition and Bäcklund transformation (2.12), one gets that addtoresetequationsection

$$\begin{aligned}
&(v_1-v_0)^2+2\int^x (v_1-v_0)_y dx+2\int^x (v_1-v_0)_z dx+2(v_1+v_0)_x=0,\\
&(v_2-v_0)^2+2\int^x (v_2-v_0)_y dx+2\int^x (v_2-v_0)_z dx+2(v_2+v_0)_x=0,\\
&(v_3-v_1)^2+2\int^x (v_3-v_1)_y dx+2\int^x (v_3-v_1)_z dx+2(v_3+v_1)_x=0.
\end{aligned}\tag{4.1}$$

Form (4.1) we can deduce that

$$v_3+v_0-2(\ln|v_1-v_2|)_x=v_1+v_2,\tag{4.2}$$

which is just right the superposition formula. Starting form $v_0=0$, we have v_1 and v_2 as given in Eq. (2.17) and (2.19) with spectral functions $\Lambda_{1,l}$ and $\Lambda_{2,l}$, respectively, which can be referred to as single-spectrum solutions. Substituting them into formula (4.2), we can get a solution v_3 which contains two spectral function $\Lambda_{1,l}$ and $\Lambda_{2,l}$. Going on in terms of the calculation, we can get many new solutions to Eq. (2.3).

5. Conclusions

In this paper, the Bäcklund transformation (2.12) of 3dKP was constructed by means of the Lax pair (2.2), which in turn yielded several types of soliton solutions and superposition formulas for the equation. Extending the low-dimensional equations to high-dimensional equations and investigating their various properties and solutions, such as multi-solitons, multi-breathers, rational solutions etc., via Bäcklund transformation method may be one of the future research topics. Various types of exact solutions of the KP equation (1.1) in three spatial dimensions were being analyzed. The methods and results presented in this paper may provide a good inspiration for dealing with similar high-dimensional nonlinear equations [18–22].

Acknowledgments: This work was supported by the National Natural Science Foundation of China grant No.12371256; the National Natural Science Foundation of China grant No.11971475.

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