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Finite Time Blowing-Up Solution for a System of Nonlinear Equations Involving Nth Level Fractional Derivative

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Abstract

A system of nonlinear fractional differential equations involving nth level fractional derivative is considered. We prove integration by parts formula for nth level fractional derivative. The result about non-existence of global solution in time has been proved. Our method of proof is based on the weak formulation and judicious choice of test function. Some examples are provided to verify the mathematical analysis.

Keywords: nonlinear equation; weak formulation; fractional integral; nth level fractional derivative

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1. Introduction

In this article, we study the following system of nonlinear fractional differential equations involving nth level fractional derivative

$$u^{'}(t) + D^{\alpha,(r_1,\ldots,r_n)}_{0_+,t} u(t) = e^{v(t)}, \quad t > 0, \quad 0 < \alpha \leq 1. \tag{1}$$

$$v^{'}(t) + D^{\sigma,(\rho_1,\ldots,\rho_n)}_{0_+,t} v(t) = e^{u(t)}, \quad t > 0, \quad 0 < \sigma \leq 1, \tag{2}$$

with initial conditions

$$u(0) = u_0, \ \ \Pi^n_{j=k}\left(I^{r_j}_{0_+,t}\frac{d}{dt}\right) I^{n-\alpha-r_n}_{0_+,t} u(t)\Big|_{t=0} = u_k, \qquad k = 1,2,\ldots n, \tag{3}$$

$$v(0) = v_0, \ \ \Pi^n_{j=k}\left(I^{\rho_j}_{0_+,t}\frac{d}{dt}\right) I^{n-\sigma-r_n}_{0_+,t} v(t)\Big|_{t=0} = v_k, \qquad k = 1,2,\ldots n, \tag{4}$$

where $D^{\alpha,(r_1,\ldots,r_n)}_{0_+,t}$ and $D^{\sigma,(\rho_1,\ldots,\rho_n)}_{0_+,t}$ denote the nth level fractional derivatives of order $\alpha, \ \sigma, \ \ 0 < \alpha, \sigma \leq 1$ and type $(r_1, \ldots, r_n)$ and $(\rho_1, \ldots, \rho_n)$ are defined as

$$D^{\alpha,(r_1,\ldots,r_n)}_{0_+,t} u(t) := \left(\Pi^n_{j=1}(I^{r_j}_{0_+,t}\frac{d}{dt})\right)\left(I^{(n-\alpha-r_n)}_{0_+,t} u\right)(t), \qquad 0 \leq r_j, \qquad \alpha + r_j \leq j, \quad j = 1, \ldots, n.$$

and

$$D^{\sigma,(\rho_1,\ldots,\rho_n)}_{0_+,t} u(t) := \left(\Pi^n_{j=1}(I^{\rho_j}_{0_+,t}\frac{d}{dt})\right)\left(I^{(n-\sigma-r_n)}_{0_+,t} u\right)(t), \qquad 0 \leq \rho_j, \qquad \sigma + \rho_j \leq j, \quad j = 1, \ldots, n,$$

respectively. For simplicity and better understanding we have considered n=2 the following system of nonlinear fractional differential Equations, i.e., (1)-(2) reduces to 2nd level fractional derivative

$$u^{'}(t) + {}^{2L}D^{\alpha,(r_1,r_2)}_{0_+,t} u(t) = e^{v(t)}, \quad t > 0, \quad 0 < \alpha \leq 1. \tag{5}$$

$$v^{'}(t) + {}^{2L}D^{\sigma,(\rho_1,\rho_2)}_{0_+,t} v(t) = e^{u(t)}, \quad t > 0, \quad 0 < \sigma \leq 1, \tag{6}$$

with initial conditions

$$u(0) = u_0,\ I_{0_+,t}^{r_2}\frac{d}{dt}I_{0_+,t}^{(2-\alpha-r_1-r_2)}u(t)\bigg|_{t=0} = u_1,\ I_{0_+,t}^{(2-\alpha-r_1-r_2)}u(t)\bigg|_{t=0} = u_2, \tag{7}$$

$$v(0) = v_0,\ I_{0_+,t}^{\rho_2}\frac{d}{dt}I_{0_+,t}^{(2-\sigma-\rho_1-\rho_2)}v(t)\bigg|_{t=0} = v_1,\ I_{0_+,t}^{(2-\sigma-\rho_1-\rho_2)}v(t)\bigg|_{t=0} = v_2, \tag{8}$$

where $u_i,\ v_i > 0,\ i = 0,1,2$, are given constants and ${}^{2L}D_{0_+,t}^{\alpha,(r_1,r_2)},\ {}^{2L}D_{0_+,t}^{\sigma,(\rho_1,\rho_2)}$ denotes the 2nd level fractional derivatives of order α,σ,

$$0 < \alpha \le 1,\ 0 < \alpha + r_1 \le 1,\ \alpha + r_1 + r_2 \le 2,\ 0 < \sigma \le 1,\ 0 < \sigma + \rho_1 \le 1,\ \sigma + \rho_1 + \rho_2 \le 2, \tag{9}$$

and type (r_1, r_2), (ρ_1, ρ_2), respectively, and are defined as

$${}^{2L}D_{0_+,t}^{\alpha,(r_1,r_2)}f(t) := I_{0_+,t}^{r_1}\frac{d}{dt}I_{0_+,t}^{r_2}\frac{d}{dt}I_{0_+,t}^{(2-\alpha-r_1-r_2)}f(t), \tag{10}$$

and

$${}^{2L}D_{0_+,t}^{\sigma,(\rho_1,\rho_2)}f(t) := I_{0_+,t}^{\rho_1}\frac{d}{dt}I_{0_+,t}^{\rho_2}\frac{d}{dt}I_{0_+,t}^{(2-\sigma-\rho_1-\rho_2)}f(t), \tag{11}$$

respectively.

Special Cases: We discuss some particular cases of 2nd level fractional derivative, which have been considered in the literature.

- By taking $r_1 = 1 - \alpha$ and $r_2 = 0$, then 2nd level fractional derivative becomes

$${}^{2L}D_{0_+,t}^{\alpha,(1-\alpha,0)}f(t) = I_{0_+,t}^{(1-\alpha)}\frac{d}{dt}f(t), \tag{12}$$

  which is well known Caputo fractional derivative of order $0 < \alpha < 1$. All initial conditions given in (7)-(8) reduces to the simple integer order type conditions that is $u_0 = u_1 = u_2$ and $v_0 = v_1 = v_2$, in this case.
- By substituting $r_1 = 0$ and $r_2 = 1$, in 2nd level fractional derivative interpolate

$${}^{2L}D_{0_+,t}^{\alpha,(0,1)}f(t) = \frac{d}{dt}I_{0_+,t}^{(1-\alpha)}f(t), \tag{13}$$

  which is Riemann-Liouville fractional derivative of order $0 < \alpha < 1$. Second and third initial conditions are given in (7)-(8) reduces to single condition, i.e., $I_{0_+,t}^{(1-\alpha)}u(t)\Big|_{t=0} = u_1 = u_2$.
- For $r_1 = r_1(1-\alpha)$ and $r_2 = 1$, the 2nd level fractional derivative reduces to

$${}^{2L}D_{0_+,t}^{\alpha,(r_1(1-\alpha),1)}f(t) = \left(I_{0_+,t}^{r_1(1-\alpha)}\frac{d}{dt}I_{0_+,t}^{(1-\alpha)(1-r_1)}f\right)(t), \tag{14}$$

  which is well known Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \le r_1 \le 1$. Second and third initial conditions given in (7)-(8) reduces to single condition, i.e., $I_{0_+,t}^{(1-\alpha)}u(t)\Big|_{t=0} = u_1 = u_2$.

Riemann-Liouville fractional derivative is used in [30] to illustrate the Scott Blair model for the law of deformation. The relaxation that occurs during glass forming is explained using the Hilfer fractional derivative [13,17,29]. Several authors discussed the applications of viscoelastic through fractional derivatives. In the literature [11,36] many authors considered the general fractional derivatives based on these models, i.e., Newtonian dashpot element, Burgers model, Zener model, Maxwell model and Kelvin–Voigt model are thought to best capture the complexity of actual materials. The 2nd and nth level fractional derivatives are generalization of Caputo, Riemann-Liouville and Hilfer fractional derivative. The 2nd level fractional derivative is yet another significant fractional derivative with several physical applications and appealing mathematical characteristics see [20,28]. In addition to having the qualities of the previously described, well-known derivatives, the 2nd level fractional derivative used in this article also has a practical use in linear viscoelasticity [25]. Furthermore, we demonstrate that the solution to the cauchy problem for the fractional relaxation equation with the 2nd level fractional derivative is completely monotone function. According to [33], any relaxation process possesses the property of completely monotonicity. In linear viscoelasticity the completely monotonicity of the relaxation equation solutions, which model the relaxation processes. For more details about applications reader can see the following references [1,4]–[10].

Fractional-order integrals and derivatives are used in the reaction diffusion equations to explain the well known phenomena of anomalous diffusion observed in the experiments. Numerous techniques have been developed to address experimentally observed anomalies in diffusion and transport processes, including continuous time random walks [33], stochastic modelling, Brownian motions, nonlocal differential or integral operators, and others. For instance, the initial condition for a Riemann-Liouville fractional derivative must be expressed in terms of a fractional integral, whereas for a Caputo type derivative, the initial condition can be expressed as in models of integer order. When shifting from microscopic to macroscopic time scales, the infinitesimal generators of time fractional evolutions appear [21]. There are many blow-up solutions exists in literature but here we are going to discuss few of them. A considerable body of literature now exists focusing on qualitative characteristics of solutions for nonlinear evolution equations using either Riemann-Liouville or Caputo derivatives. In [27], Furati et al. discussed the blow-up solution of differential equation involving Hilfer fractional derivative. In their work, Furati and Kirane [26] examined the blow-up solutions for a set of nonlinear fractional differential equations. First example that always exhibits a profile of blow-up arises from the work of Kirane and Malik discussed in [22], where a system of fractional differential equations with exponential nonlinearities is analyzed in [19]. For more details of blow-up results reader can see ([16,18,24,34,35]).

Motivated by the above work we have generalized the results of [19] by considering nth level fractional derivative with respect to time variable t. In this article particularly, we focus on integration by parts formula of nth level fractional derivative, blow-up solution and profile of blow-up solution of problem (5)-(6) involving 2nd level fractional derivative in time.

The rest of the paper is structured as follows. We shall devote Section 2 to the definitions and basic results which is used to prove the blow-up solution. In Section 3, main result, i.e., blow-up solution of this paper. In Section 4, discussed the profile of blow-up solution. In Section 5, presented the particular cases of initial value problem. Finally, the conclusions are drawn in Section 6.

2. Basic Definitions and Lemmas

In this section we have presented some fundamental definitions and results from fractional calculus for the reader.

Definition 2.1. [15] The two parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad Re(\alpha) > 0, \quad z, \beta \in \mathbb{C}.$$

For $\beta = 1$, $E_{\alpha,\beta}(z)$, reduces to Mittag-Leffler function of a single parameter, i.e.,

$$E_{\alpha,1}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}.$$

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For $0 < \alpha < \beta \leq 1$, the Mittag-Leffler type function

$$\mathcal{E}_{\alpha,\beta}(t,\lambda) := t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha}), \qquad \lambda > 0, \tag{15}$$

is completely monotone.
It is easy to verify that the first derivative of Mittag-Leffler function defined in (15) is given by

$$\mathcal{E}'_{\alpha,\beta}(t,\lambda) := t^{\beta-2}E_{\alpha,\beta-1}(-\lambda t^{\alpha}), \qquad \lambda > 0. \tag{16}$$

Test Function. [26] The proof of our result on non-existence of global solutions for system (5)-(6) is based on the judicious choice of the test function given by

$$\phi(t) = \begin{cases} T^{-\theta}(T-t)^{\theta}, & t \in [0,T], \\ 0, & t > T. \end{cases} \tag{17}$$

where $\theta \geq 2$.
Lemma 2.1. [7] Assume that $0 < \alpha < 1$, and $f, g \in L^p([0,T])$, $1 \leq p \leq \infty$. Then the following formula of integration by parts holds:

$$\int_0^T f(t) I^{\alpha}_{0^+,t} g(t)dt = \int_0^T g(t) I^{\alpha}_{T^-,t} f(t)dt. \tag{18}$$

Definition 2.2. Right sided nth level fractional derivative of order $\alpha, 0 < \alpha \leq 1$ and type $(r_1, ..., r_n)$ is defined as

$$D^{\alpha,(r_1,...,r_n)}_{T_-,t} f(t) := -I^{(n-\alpha-r_n)}_{T_-,t} \frac{d}{dt}\left(\Pi^n_{k=1} I^{r_k}_{T_-,t} f(t)\right). \tag{19}$$

- By putting $n = 2$ in (19) we get right sided 2nd level fractional derivative of order $\alpha, 0 < \alpha \leq 1$ and type (r_1, r_2),

$$^{2L}D^{\alpha,(r_1,r_2)}_{T_-,t} f(t) := -I^{2-r_1-r_2-\alpha}_{T_-,t} \frac{d}{dt} I^{r_2}_{T_-,t} \frac{d}{dt} I^{r_1}_{T_-,t} f(t). \tag{20}$$

- If we fix $r_1 = 1-\alpha$ and $r_2 = 0$, in (20) then we have obtained right sided Riemann-Liouville fractional derivative of order $\alpha, 0 < \alpha \leq 1$,

$$^{2L}D^{\alpha,(1-\alpha,0)}_{T_-,t} f(t) := -\frac{d}{dt} I^{1-\alpha}_{T_-,t} f(t). \tag{21}$$

- Right sided Caputo fractional derivative is obtained by putting $r_1 = 0$ and $r_2 = 1$, in (20)

$$^{2L}D^{\alpha,(0,1)}_{T_-,t} f(t) := -I^{1-\alpha}_{T_-,t} \frac{d}{dt} f(t). \tag{22}$$

- Right sided Hilfer fractional derivative is obtained by fixing $r_1 = r_1(1-\alpha)$ and $r_2 = 1$, in (20)

$$^{2L}D^{\alpha,(r_1,r_2)}_{T_-,t} f(t) := -I^{(1-\alpha)(1-r_1)}_{T_-,t} \frac{d}{dt} I^{r_1(1-\alpha)}_{T_-,t} f(t). \tag{23}$$

The 2nd level fractional derivative, i.e., $^{2L}D^{\alpha,(r_1,r_2)}_{T_-,t}\phi(t)$ and $^{2L}D^{\sigma,(\rho_1,\rho_2)}_{T_-,t}\phi(t)$ of test function $\phi(t)$ are given by

$$^{2L}D^{\alpha,(r_1,r_2)}_{T_-,t}\phi(t) = C_{\alpha,\theta}(1+\theta-\alpha)T^{-\theta}(T-t)^{\theta-\alpha}, \tag{24}$$

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and

$$ {}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi(t) = C_{\sigma,\theta}(1+\theta-\sigma)T^{-\theta}(T-t)^{\theta-\sigma}, \tag{25} $$

respectively.

Definition 2.3. [25] The nth level fractional derivative of order $\alpha, 0 < \alpha \leq 1$ and type $(r_1, ..., r_n)$ is defined as

$$ D_{0_+,t}^{\alpha,(r_1,...,r_n)}f(t) := \left(\Pi_{k=1}^{n}(I_{0_+,t}^{r_k}\frac{d}{dt})\right)\left(I_{0_+,t}^{(n-\alpha-r_n)}f\right)(t), \qquad 0 \leq r_k, \qquad \alpha + r_k \leq k. $$

Before we proceed further let us provide integration by parts formula for nth level fractional derivative.

Lemma 2.2. Assume that $0 < \alpha < 1$, $f \in AC([0,T])$ and $g \in L^p([0,T])$, $1 \leq p \leq \infty$. Then the following formula of integration by parts holds:

$$ \int_0^T f(t)D_{0_+,t}^{\alpha,(r_1,...,r_n)}g(t)dt = \int_0^T g(t)D_{T^-,t}^{\alpha,(r_1,...,r_n)}f(t)dt + \left(\Pi_{k=2}^{n-1}(I_{0_+,t}^{r_k}\frac{d}{dt})\right)\left(I_{0_+,t}^{(n-\alpha-r_n)}g\right)(t)I_{T_-,t}^{r_1}f(t)\Big|_0^T. \tag{26} $$

Proof. Consider

$$ \int_0^T f(t)D_{0_+,t}^{\alpha,(r_1,...,r_n)}g(t)dt = \int_0^T f(t)\left(I_{0_+,t}^{r_1}\frac{d}{dt}\Pi_{k=2}^{n-1}(I_{0_+,t}^{r_k}\frac{d}{dt})\right)\left(I_{0_+,t}^{(n-\alpha-r_n)}g\right)(t)dt. $$

Using Lemma 2.1, we have

$$ \int_0^T f(t)D_{0_+,t}^{\alpha,(r_1,...,r_n)}g(t)dt = \int_0^T \frac{d}{dt}\left(\Pi_{k=2}^{n-1}(I_{0_+,t}^{r_k}\frac{d}{dt})\right)\left(I_{0_+,t}^{(n-\alpha-r_n)}g\right)(t)I_{T_-,t}^{r_1}f(t)dt. $$

Integration by parts, we have

$$ \int_0^T f(t)D_{0_+,t}^{\alpha,(r_1,...,r_n)}g(t)dt = \left(\Pi_{k=2}^{n-1}(I_{0_+,t}^{r_k}\frac{d}{dt})\right)\left(I_{0_+,t}^{(n-\alpha-r_n)}g\right)(t)I_{T_-,t}^{r_1}f(t)\Big|_0^T - \int_0^T \left(\Pi_{k=2}^{n-1}(I_{0_+,t}^{r_k}\frac{d}{dt})\right)\left(I_{0_+,t}^{(n-\alpha-r_n)}g\right)(t)\frac{d}{dt}I_{T_-,t}^{r_1}f(t)dt. \tag{27} $$

Using $n-1$ times Lemma 2.1 and integration by parts on second term of (27), we get

$$ \int_0^T f(t)D_{0_+,t}^{\alpha,(r_1,...,r_n)}g(t)dt = \left(\Pi_{k=2}^{n-1}(I_{0_+,t}^{r_k}\frac{d}{dt})\right)\left(I_{0_+,t}^{(n-\alpha-r_n)}g\right)(t)I_{T_-,t}^{r_1}f(t)\Big|_0^T - \int_0^T g(t)I_{T_-,t}^{(n-\alpha-r_n)}\frac{d}{dt}\left(\Pi_{k=1}^{n}I_{T_-,t}^{r_k}f(t)\right)dt. $$

Consequently, we obtain (26).

Remark 2.1. If $n = 2$ then formula (26) becomes integration by parts formula for 2nd level fractional derivative i.e.,

$$ \int_0^T f(t){}^{2L}D_{0+,t}^{\alpha,(r_1,r_2)}g(t)dt = \int_0^T g(t){}^{2L}D_{T-,t}^{\alpha,(r_1,r_2)}f(t)dt + \left(\frac{d}{dt}I_{T-,t}^{r_1}f(t)\right)I_{0+,t}^{2-\alpha-r_1-r_2}g(t)\Big|_0^T - \left(I_{T-,t}^{r_2}\frac{d}{dt}I_{T-,t}^{r_1}f(t)\right)I_{0+,t}^{2-\alpha-r_1-r_2}g(t)\Big|_0^T. \tag{28} $$

We have the following particular results about Equation (28).

- If we fix $r_1 = 1-\alpha$ and $r_2 = 0$, in (28) then we obtain well known integration by parts formula of Caputo fractional derivative, i.e.,

$$ \int_0^T f(t){}^cD_{0_+,t}^{\alpha}g(t)dt = \int_0^T g(t)D_{T_-,t}^{\alpha}f(t)dt + g(t)I_{T_-,t}^{1-\alpha}f(t)\Big|_0^T. \tag{29} $$

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- By putting $r_1 = 0$ and $r_2 = 1$, in (28) then we get integration by parts formula of Riemann-Liouville fractional derivative, i.e.,

$$\int_0^T f(t) D^{\alpha}_{0^+,t} g(t) dt \quad = \quad \int_0^T g(t) {}^{c}D^{\alpha}_{T_-,t} f(t) dt - \left(I^{1-\alpha}_{0+,t} g(t) \right) f(t) \Big|_0^T. \tag{30}$$

- If we fix $r_1 = r_1(1-\alpha)$ and $r_2 = 1$, in (28) then we obtain integration by parts formula of Hilfer fractional derivative, i.e.,

$$\int_0^T f(t) D^{\alpha, r_1(1-\alpha)}_{0^+,t} g(t) dt = \int_0^T g(t) D^{(1-\alpha), r_1}_{T-,t} f(t) dt + \left(I^{(1-r_1)(1-\alpha)}_{0+,t} g(t) \right) I^{r_1(1-\alpha)}_{T-,t} f(t) \Big|_0^T. \tag{31}$$

For further details of Riemann-Liouville, Caputo and Hilfer fractional derivatives see the following references [7], [12].

Lemma 2.3. [20] The Laplace transform of the 2nd level fractional derivative is given by

$$\mathcal{L}\left\{ {}^{2L}D^{\alpha,(r_1,r_2)}_{0_+,t} f(t); s \right\} = s^{\alpha} \mathcal{L}\{f(t); s\} - s^{-r_1} I^{r_2}_{0_+,t} \frac{d}{dt} I^{2-\alpha-r_1-r_2}_{0_+,t} f(t) \Big|_{t=0} - s^{-r_1-r_2+1} I^{2-\alpha-r_1-r_2}_{0_+,t} f(t) \Big|_{t=0}. \tag{32}$$

Lemma 2.4. The components of the solution (u, v) to the system (5)-(6) satisfy the integral equations.

$$u(t) \quad = \quad u_0 E_{1-\alpha,1}(-t^{1-\alpha}) + u_1 t^{r_1} E_{1-\alpha,r_1+1}(-t^{1-\alpha}) + u_2 t^{r_1+r_2-1} E_{1-\alpha,r_1+r_2}(-t^{1-\alpha}) + \int_0^t E_{1-\alpha,1}(-(t-\tau)^{1-\alpha}) e^{v(\tau)} d\tau. \tag{33}$$

$$v(t) \quad = \quad v_0 E_{1-\sigma,1}(-t^{1-\sigma}) + v_1 t^{\rho_1} E_{1-\sigma,\rho_1+1}(-t^{1-\sigma}) + v_2 t^{\rho_1+\rho_2-1} E_{1-\sigma,\rho_1+\rho_2}(-t^{1-\sigma}) + \int_0^t E_{1-\sigma,1}(-(t-\tau)^{1-\sigma}) e^{u(\tau)} d\tau. \tag{34}$$

Proof. Taking Laplace of Equation (5), we have

$$\mathcal{L}\{u'(t); s\} + \mathcal{L}\{D^{\alpha,(r_1,r_2)}_{0^+,t} u(t); s\} = \mathcal{L}\{e^{v(t)}; s\}. \tag{35}$$

After simplifying Equation (35), we get

$$u(s) = \frac{u_0}{s(1+s^{\alpha-1})} + \frac{u_1 s^{-r_1}}{s(1+s^{\alpha-1})} + \frac{u_2 s^{-r_1-r_2+1}}{s(1+s^{\alpha-1})} + \frac{\mathcal{L}\left\{ e^{v(t)}; s \right\}}{s(1+s^{\alpha-1})}. \tag{36}$$

Taking Laplace inverse of Equations (36), we get Equation (33). Similarly, we can have Equation (34).

Lemma 2.5. For the system (5)-(6), the functions $v'(t)$ and $u'(t)$ satisfy the integral equations,

$$u'(t) \quad = \quad u_0 E'_{1-\alpha,1}(-t^{1-\alpha}) + u_1 t^{r_1-1} E_{1-\alpha,r_1}(-t^{1-\alpha}) + u_2 t^{r_1+r_2-2} E_{1-\alpha,r_1+r_2-1}(-t^{1-\alpha}) + \int_0^t E'_{1-\alpha,1}(-(t-\tau)^{1-\alpha}) e^{v(\tau)} d\tau + e^{v(t)}. \tag{37}$$

$$v'(t) \quad = \quad v_0 E'_{1-\sigma,1}(-t^{1-\sigma}) + v_1 t^{\rho_1-1} E_{1-\sigma,\rho_1}(-t^{1-\sigma}) + v_2 t^{\rho_1+\rho_2-2} E_{1-\sigma,\rho_1+\rho_2-1}(-t^{1-\sigma}) + \int_0^t E'_{1-\sigma,1}(-(t-\tau)^{1-\sigma}) e^{u(\tau)} d\tau + e^{u(t)}. \tag{38}$$

Proof. Proof is straightforward. Equations (37) and (38) are obtained by taking derivative of (33) and (34).

The result about the existence of local solution for the system (5)-(8) can be obtained via equivalent integral Equations (33) and (34) and following the same strategy as discussed in [19].

3. Blow-Up Solution

In this section we are going to discuss main result of blow-up solution of system (5)-(6).

Theorem 3.1. Suppose $0 < \alpha, \sigma < 1$ and $0 < r_1, r_2, \rho_1, \rho_2 < 1$, then the solution of the system (5)-(6)

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subject to $u_i, v_i > 0$,
$i = 0, 1, 2$, blow-up in a finite time.

Proof. Proof of this theorem is obtained by contradiction. Suppose (u, v) is a global solution of the system (5)-(6). Multiplying both sides of the two equations of the system (5)-(6) by ϕ and integration by parts from (28) leads to

$$-\int_0^T u\phi' + \int_0^T u\,{}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi = u_0 - \frac{u_1 T^{r_1}}{\Gamma(r_1+1)} + \frac{u_2 T^{r_1+r_2-1}}{\Gamma(r_1+r_2)} + \int_0^T e^v\phi, \tag{39}$$

$$-\int_0^T v\phi' + \int_0^T v\,{}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi = v_0 - \frac{v_1 T^{\rho_1}}{\Gamma(\rho_1+1)} + \frac{v_2 T^{\rho_1+\rho_2-1}}{\Gamma(\rho_1+\rho_2)} + \int_0^T e^u\phi, \tag{40}$$

where ϕ is Test function such that ${}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi$, ${}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi$ are evaluated by given formula in (24)-(25) and $\phi(T) = 0$. We will use the Young-type inequality $\forall\ a, b, d \geq 0$,

$$ab \leq de^a + b\ln\left(\frac{b}{ed}\right). \tag{41}$$

By using inequality (41), we have

$$\int_0^T u|\phi'| \leq \int_0^T \frac{1}{4}\phi e^u + \int_0^T |\phi'|\ln\left(\frac{4|\phi'|}{e\phi}\right), \qquad \int_0^T v|\phi'| \leq \int_0^T \frac{1}{4}\phi e^v + \int_0^T |\phi'|\ln\left(\frac{4|\phi'|}{e\phi}\right), \tag{42}$$

$$\int_0^T u\,{}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi \leq \int_0^T \frac{1}{4}\phi e^u + \int_0^T {}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi\ln\left(\frac{4\,{}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi}{e\phi}\right), \tag{43}$$

$$\int_0^T v\,{}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi \leq \int_0^T \frac{1}{4}\phi e^v + \int_0^T {}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi\ln\left(\frac{4D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi}{e\phi}\right), \tag{44}$$

where $\int_0^T u\,{}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi > 0$, $\int_0^T v\,{}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi > 0$, i.e., ${}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi$, ${}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi > 0$ from (24) and (25), and u and v are increasing functions that's why integral of both terms are greater than zero. Using (42)-(44) in (39)-(40) leads to

$$\int_0^T e^v\phi \leq \frac{1}{2}\int_0^T e^v\phi + \int_0^T |\phi'|\ln\left(\frac{4|\phi'|}{e\phi}\right) + \int_0^T {}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi\ln\left(\frac{4\,{}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi}{e\phi}\right), \tag{45}$$

and

$$\int_0^T e^u\phi \leq \frac{1}{2}\int_0^T e^u\phi + \int_0^T |\phi'|\ln\left(\frac{4|\phi'|}{e\phi}\right) + \int_0^T {}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi\ln\left(\frac{4\,{}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi}{e\phi}\right), \tag{46}$$

respectively. We have fixed some notions, i.e.,

$$I := \int_0^T e^u\phi, \quad J := \int_0^T e^v\phi, \quad X := \int_0^T |\phi'|\ln\left(\frac{4|\phi'|}{e\phi}\right). \tag{47}$$

$$Y := \int_0^T {}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi\ln\left(\frac{4\,{}^{2L}D_{T_-,t}^{\alpha,(r_1,r_2)}\phi}{e\phi}\right), \qquad Z := \int_0^T {}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi\ln\left(\frac{4\,{}^{2L}D_{T_-,t}^{\sigma,(\rho_1,\rho_2)}\phi}{e\phi}\right), \tag{48}$$

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where ${}^{2L}D^{\alpha,(r_1,r_2)}_{T_-,t}\phi$ and ${}^{2L}D^{\sigma,(\rho_1,\rho_2)}_{T_-,t}\phi$ are defined in (24) and (25), respectively. The integrals X, Y and Z are evaluated as follows:

$$X = \ln\left(\frac{4\theta}{eT}\right) + \frac{1}{\theta}, \quad Y = \frac{\Gamma(1+\theta)}{\Gamma(1+\theta-\alpha)}\frac{T^{1-\alpha}}{1+\theta-\alpha}\left[\ln\left(\frac{4}{e}\frac{\Gamma(1+\theta)}{\Gamma(1+\theta-\alpha)}\frac{1}{T^{\alpha}}\right) + \frac{\alpha}{1+\theta-\alpha}\right],$$

and

$$Z = \frac{\Gamma(1+\theta)}{\Gamma(1+\theta-\alpha)}\frac{T^{1-\alpha}}{1+\theta-\alpha}\left[\ln\left(\frac{4}{e}\frac{\Gamma(1+\theta)}{\Gamma(1+\theta-\alpha)}\frac{1}{T^{\alpha}}\right) + \frac{\alpha}{1+\theta-\alpha}\right].$$

Using (47) and (48) in (45) and (46), we get

$$J \leqslant 2X + \frac{2}{3}Z + \frac{4}{3}Y, \quad \text{and} \quad I \leqslant 2X + \frac{4}{3}Z + \frac{2}{3}Y. \tag{49}$$

Now, using equality (49) in (39), we have

$$u_0 - \frac{u_1 T^{r_1}}{\Gamma(r_1+1)} + \frac{u_2 T^{r_1+r_2-1}}{\Gamma(r_1+r_2)} \leqslant 2X + \frac{4}{3}Y + \frac{2}{3}Z, \tag{50}$$

using positivity of $u_0,\ u_2 > 0$ inequality (50) becomes

$$u_1 \leqslant \Gamma(r_1+1)T^{-r_1}\left[2X + \frac{4}{3}Y + \frac{2}{3}Z\right], \tag{51}$$

which implies $u_1 \leq -\infty$. This provides the contradiction $0 \leq u_1 < 0$. Similarly, we can obtain the contradiction of $0 \leq v_1 < 0$.

Remark. The proof of blow-up solution of nth level fractional derivative of system (1)-(2) is obtained on the similar lines as we have did in Theorem 3.1.

4. Profile of Blow-Up Solution

In this section, we are going to discuss the profile of blow-up solution of three systems.
System 1. First system is defined in (5)-(6).
System 2. Consider the following system of nonlinear ordinary differential equations

$$u^{'}(t) = e^{v(t)}, \quad t > 0, \tag{52}$$

$$v^{'}(t) = e^{u(t)}, \quad t > 0, \tag{53}$$

with initial conditions, i.e., $u(0) = u_0 > 0,\ \ v(0) = v_0 > 0$, where u_0 and v_0 are given constants. Blow-up solution of system (52)-(53) is given by

$$u(t) = \ln\left(\frac{c_2}{1-e^{c_2(t-c_1)}}\right), \quad v(t) = \ln\left(\frac{c_2 e^{c_2(t-c_1)}}{1-e^{c_2(t-c_1)}}\right), \tag{54}$$

where $c_1 = \left(\frac{v_0-u_0}{e^{v_0}-e^{u_0}}\right) > 0, \quad c_2 = e^{u_0} - e^{v_0}$.

System 3. Consider the following system of nonlinear fractional differential equations

$${}^{2L}D^{\alpha,(r_1,r_2)}_{0_+,t}u(t) = e^{v(t)}, \quad t > 0, \tag{55}$$

$${}^{2L}D^{\sigma,(\rho_1,\rho_2)}_{0_+,t}v(t) = e^{u(t)}, \quad t > 0, \tag{56}$$

with initial conditions, i.e., $u(0) = u_0 > 0,\ \ v(0) = v_0 > 0$, where u_0 and v_0 are given constants.

Blow-up solution of system (55)-(56) is obtained on the same strategy as discussed in Theorem 3.1.

Lemma 4.1. The components of the solution (u, v) to the system (55)-(56) satisfy the integral equations.

$$u(t) = \frac{u_0 t^{\alpha+\gamma_1-1}}{\Gamma(\alpha+r_1)} + \frac{u_1 t^{\alpha+r_1+r_2-2}}{\Gamma(\alpha+r_1+r_2-1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{v(\tau)} d\tau. \tag{57}$$

$$v(t) = \frac{v_0 t^{\sigma+\rho_1-1}}{\Gamma(\sigma+\rho_1)} + \frac{v_1 t^{\sigma+\rho_1+\rho_2-2}}{\Gamma(\sigma+\rho_1+\rho_2-1)} + \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} e^{u(\tau)} d\tau. \tag{58}$$

Proof. Taking Laplace of Equation (55), we have

$$\mathcal{L}\{{}^{2L}D^{\alpha,(r_1,r_2)}_{0+,t} u(t); s\} = \mathcal{L}\{e^{v(t)}; s\}. \tag{59}$$

After simplifying Equation (59), we get

$$u(s) = u_0 s^{-\alpha-r_1} + u_1 s^{-\alpha-r_1-r_2+1} + s^{-\alpha} \mathcal{L}\{e^{v(t)}; s\}. \tag{60}$$

Similarly, Equation (56) becomes,

$$v(s) = v_0 s^{-\sigma-\rho_1} + v_1 s^{-\sigma-\rho_1-\rho_2+1} + s^{-\sigma} \mathcal{L}\{e^{u(t)}; s\}. \tag{61}$$

Taking Laplace inverse of Equations (60)-(61) we get Equations (57)-(58).
Solution (u, v) to the system (5)-(6) is discussed in (33)-(34).

Analysis. Mittag-Leffler functions

$$E_{1-\alpha,1}\left(-t^{1-\alpha}\right),\ t^{r_1} E_{1-\alpha,r_1+1}\left(-t^{1-\alpha}\right),\ t^{r_1+r_2-1} E_{1-\alpha,r_1+r_2}\left(-t^{1-\alpha}\right), \tag{62}$$

and

$$E_{1-\sigma,1}\left(-t^{1-\sigma}\right),\ t^{\rho_1} E_{1-\sigma,\rho_1+1}\left(-t^{1-\sigma}\right),\ t^{\rho_1+\rho_2-1} E_{1-\sigma,\rho_1+\rho_2}\left(-t^{1-\sigma}\right), \tag{63}$$

are completely monotone for $0 \le r_1, \rho_1 \le 1,\ \ 1 \le r_1 + r_2, \rho_1 + \rho_2 \le 2$ given in [25]. In particular first derivative of completely monotone functions are given below

$$E'_{1-\alpha,1}(-t^{1-\alpha}) \le 0, \qquad t^{r_1-1} E_{1-\alpha,r_1}\left(-t^{1-\alpha}\right) \le 0, \quad t^{r_1+r_2-2} E_{1-\alpha,r_1+r_2-1}\left(-t^{1-\alpha}\right) \le 0, \tag{64}$$

and

$$E'_{1-\sigma,1}(-t^{1-\sigma}) \le 0, \qquad t^{\rho_1-1} E_{1-\sigma,\rho_1}\left(-t^{1-\sigma}\right) \le 0, \quad t^{\rho_1+\rho_2-2} E_{1-\sigma,\rho_1+\rho_2-1}\left(-t^{1-\sigma}\right) \le 0, \tag{65}$$

for $t > 0$ in Equations (37) and (38) becomes

$$u'(t) \le e^{v(t)}, \quad v'(t) \le e^{u(t)}, \quad t > 0. \tag{66}$$

The differential inequalities (66) leads to the estimates from above of the solution (u, v) to the system (5)-(6) for $0 < t < T_{\max} < +\infty$, is discussed in [19]. We notice that the system can be written as

$${}^{2L}D^{\alpha,(r_1,r_2)}_{0+,t} u(t) = e^{v(t)} - u'(t), \quad {}^{2L}D^{\sigma,(\rho_1,\rho_2)}_{0+,t} v(t) = e^{u(t)} - v'(t). \tag{67}$$

Using inequalities (66), we get

$$ {}^{2L}D_{0+,t}^{\alpha,(r_1,r_2)} u(t) \geq 0, \quad {}^{2L}D_{0+,t}^{\sigma,(\rho_1,\rho_2)} v(t) \geq 0. \tag{68} $$

5. Particular Cases

- Let $r_1 = r_1(1-\alpha)$ and $r_2 = 1$, then system (5)-(6) becomes

$$ \begin{cases} u'(t) + {}^{C}D_{0+,t}^{\alpha,r_1(1-\alpha)} u(t) = e^{v(t)}, & t > 0, \quad 0 < \alpha < 1, \quad 0 < r_1 \leq 1, \\ v'(t) + {}^{C}D_{0+,t}^{\sigma,\rho_1(1-\sigma)} v(t) = e^{u(t)}, & t > 0, \quad 0 < \sigma < 1, \quad 0 < \rho_1 \leq 1, \\ I_{0+,t}^{1-\alpha} u(t)\big|_{t=0} = u_0 > 0, & I_{0+,t}^{1-\sigma} v(t)\big|_{t=0} = v_0 > 0 \end{cases} \tag{69} $$

- If $r_1 = 1-\alpha$ and $r_2 = 0$, then system (5)-(6) becomes

$$ \begin{cases} u'(t) + {}^{C}D_{0+,t}^{\alpha} u(t) = e^{v(t)}, & t > 0, \quad 0 < \alpha < 1, \\ v'(t) + {}^{C}D_{0+,t}^{\sigma} v(t) = e^{u(t)}, & t > 0, \quad 0 < \sigma < 1, \\ u(0) = u_0 > 0, & v(0) = v_0 > 0 \end{cases} \tag{70} $$

  In [19] authors explored the blow-up of the solutions and determined lower bounds for the maximum time of the system (70).
- Suppose $\alpha = \sigma = 1$ in (70), we obtain

$$ \begin{cases} u'(t) = \frac{1}{2} e^{v(t)}, & t > 0, \\ v'(t) = \frac{1}{2} e^{u(t)}, & t > 0, \\ u(0) = u_0 > 0, & v(0) = v_0 > 0. \end{cases} \tag{71} $$

  Blow-up solution of system (71) is obtained on the same technique discussed in [19].

6. Concluding Remarks

The paper is concerned with system of nonlinear equations involving nth level fractional derivative alongside fractional initial conditions. We first dealt with integration by parts formula of nth level fractional derivative. We have recovered some well known results by fixing some parameters in nth level fractional derivative. Furthermore, we have fixed the value of $n = 2$ in nth level fractional derivative then calculate the existence of blow-up solution and profile of blow-up solution. Moreover, particular cases of system of nonlinear equations taken into consideration in this paper, we recovered the results of [19]. The extension of our work to higher dimensions with more general time fractional derivatives will be explored in future.

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