Hypothesis

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A Potential Proof of Riemann Hypothesis

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Keywords: prime numbers; riemann hypothesis; zeta function; analytic continuation; zeros

Hypothesis

A Potential Proof of Riemann Hypothesis

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Abstract: In this study, a simple approach to solving the Riemann Hypothesis, one of the most prominent unsolved problems in the field of Number Theory in mathematics and one of the "Millenium Prize Problems", is presented. The Riemann Hypothesis, a conjecture about distribution of prime numbers, has remained unsolved since it was first proposed by German mathematician, Bernhard Riemann in 1859. This paper introduces an analytic continuation of the Zeta Function as well as symmetric approach through integration by parts to address this hypothesis. The proposed solution is obtained through analytical continuation in the critical strip $0 < \mathfrak{Re}(s) < 1$ to establish that $\mathfrak{Re}(s) = 1/2$.

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1. INTRODUCTION

Definition 1.1. The Riemann Zeta Function is defined as the Dirichlet series of the form

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

which is valid in the half-plane $\{s \in \mathbb{C} : \mathfrak{Re}(s) > 1\}$.

Theorem 1.1. *The Riemann Zeta Function satisfies the following equation. This result is known as the Euler product formula*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - p^{-s}\right)^{-1}. \tag{1.2}$$

Corollary 1.1. *The Riemann Zeta function has no zeros for* $\mathfrak{Re}(s) > 1$.

Proof. Both sides of equation (1.2) converge for $\mathfrak{Re}(s) > 1$.As the right side of this expression never becomes zero as it is a product of prime numbers, then the left side of this equation never becomes zero for $\mathfrak{Re}(s) > 1$. □

Theorem 1.2. *The Riemann Zeta function, equation (1.1), can be continued analytically through the following functional equation, which is meromorphic for the whole complex plane* $\{s \in \mathbb{C} : s \neq 1\}$ *, where* $s = 1$ *constitutes a simple pole with residue 1, of the form*

$$\zeta(s) = 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{1.3}$$

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Riemann firstly found a slightly different functional equation that was publish in 1859 in his paper "On the Number of Primes Less Than a Given Magnitude". The previous version of equation (1.3) as well as its properties, and the proof of theorem 1.2 can be found in Titchmarsch's work[1].

Corollary 1.2. *The Riemann Zeta function is zero at $s = -2, -4, -6, ...$ These are called the trivial zeros of the Riemann Zeta function.*

Proof. The sine term in equation (1.3) vanishes at $s = -2, -4, -6, ...$ while for the same values, $2^s \pi^{s-1} \neq 0$. If Gamma function is expressed in terms of the Euler reflection formula, $\Gamma(s)\Gamma(1-s) = \pi/sin(\pi s)$, it never becomes zero and only the poles at $\Gamma(-2k)$, $k = 0, 1, 2, ...$ are present. $\zeta(1-s) \neq 0$ at $s = -2, -4, -6, ...$ as per corollary 1.1. □

Corollary 1.3. *There are zeros of the Riemann Zeta function are symmetrically located on the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$around the vertical line $\Re(s) = 1/2$. These are the so called non-trivial zeros.*

Proof. Let ρ be a complex number such as $\Re(\rho) = \sigma$ and $\Im(\rho) = t$ $\{\rho \in \mathbb{C} : 0 < \Re(\rho) < 1,\ 0 < \Im(\rho) < \infty\}$. If $\zeta(\sigma + it) = 0$, as $2^{\rho}\pi^{\rho-1} sin\left(\frac{\pi\rho}{2}\right)\Gamma(1-\rho) \neq 0$ in equation (1.3), then $\zeta(1 - \sigma - it) = 0$. □

Lemma 1.1. *The non-trivial zeros of the Riemann Zeta function are symmetric with respect to the real axis.*

Proof. As $\zeta : \mathbb{R} \to \mathbb{R}$ for $\{s \in \mathbb{C} : \Re(s) \neq 1,\ \Im(s) = 0\}$ is well defined and is holomorphic on the upper half-plane $\{s \in \mathbb{C} : \Re(s) \neq 1,\ \Im(s) > 0\}$, by the Schwarz reflection principle[2], $\overline{\zeta}(s) = \zeta(\overline{s})$, and it implies that $\Re(\zeta(\sigma + it)) = \Re(\zeta(\sigma - it))$ and $\Im(\zeta(\sigma + it)) = -\Im(\zeta(\sigma - it))$. □

2. RIEMANN HYPOTHESIS

Conjecture 2.1. Riemann Hypothesis (R.H.) [3] states that the real part of the non-trivial zeros of the Riemann Zeta function, that is to say, those zeros that are not "trivial", equals 1/2. Let be ρ_n the n-th non-trivial zero of the Riemann Zeta function, then R.H. states that the non-trivial *zeros*[1] *are the set* $\{\rho_n \in \mathbb{C} : \rho_n = \sigma + i\gamma_n,\ \sigma = 1/2,\ \gamma_n \in \mathbb{R}^+,\ n \in \mathbb{N}\}$.

Although some serious attempts have been made on attacking Riemann Hypothesis,it remains unsolved (see, for example, Jensen Polynomials' work from Griffin[4], Montgomery's zeros pair correlation work[5] and Bender, Carl M. et al. on the use of Hamiltonian operators[6]). The present paper is intended to provide an original but simple approach that ultimately leads to one-half value of the real part of every "non-trivial" zero of the Riemann Zeta function.

PRELIMINARY DEFINITIONS AND THEOREMS

Definition 2.1. *The Dirichlet eta function is defined, for $\Re(s) > 0$, as*

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \tag{2.1}$$

[1] *In this paper, I will use ρ instead of ρ_n to refer to any n-th "non-trivial" zero unless it is strictly necessary.*

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Definition 2.2. *The Gamma function is defined, for $\mathfrak{Re}(s) > 0$, as*

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx.$$

Theorem 2.1. *The Dirichlet eta function can be written is terms of the Gamma function, for $\mathfrak{Re}(s) > 0$, as*

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1} dx. \tag{2.2}$$

Proof. Let S_∞ be an infinity geometric series of the form, converging for $-1 < r < 1$

$$S_\infty = \sum_{n=0}^{\infty} r^n.$$

As it is well know, this geometric series can be expressed in terms of a closed from

$$S_\infty = \frac{1}{1-r}.$$

And let $r = -e^{-x}$, so

$$S_\infty(x) = \sum_{n=0}^{\infty} (-e^{-x})^n.$$

If I multiply both sides of the last expression I get

$$e^{-x} S_\infty(x) = (-1)^n e^{-x} \sum_{n=0}^{\infty} (-e^{-x})^n,$$

and

$$e^{-x} S_\infty = \frac{e^{-x}}{1 + e^{-x}}.$$

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On one hand, if I use Mellin transform, I get[2]

$$\int_0^\infty x^{s-1}e^{-x}S_\infty(x)dx = \sum_{n=0}^{\infty}(-1)^n \int_0^\infty x^{s-1}e^{-(n+1)x}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^s}\int_0^\infty u^{s-1}e^{-u}du,$$

$$\int_0^\infty x^{s-1}e^{-x}S_\infty(x)dx = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\int_0^\infty u^{s-1}e^{-u}du = \Gamma(s)\eta(s).$$

On the other hand, I have

$$\int_0^\infty x^{s-1}e^{-x}S_\infty(x)dx = \int_0^\infty x^{s-1}\frac{e^{-x}}{1+e^{-x}}dx = \int_0^\infty \frac{x^{s-1}}{e^x+1}dx,$$

so,

$$\Gamma(s)\eta(s) = \int_0^\infty \frac{x^{s-1}}{e^x+1}dx,$$

and we get to equation (2.2)

$$\boxed{\eta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x+1}dx}.$$

□

Theorem 2.2. *Riemann Zeta function can be continued analytically, for $\mathfrak{Re}(s) > 0$, in terms of the Dirichlet eta function as*

$$\zeta(s) = \frac{1}{1-2^{1-s}}\eta(s). \tag{2.3}$$

Proof. Let $\zeta(s)$ be expressed as, valid for $\mathfrak{Re}(s) > 1$, that can be obtained from equation (1.1) by following an analogous procedure for proof of theorem (2.1)

[2] I performed the variable change $u = (n+1)x$.

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$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx,$$

which is valid for $\mathfrak{Re}(s) > 1$. Then I multiply both sides of the previous expression by 2^{-s} to obtain

$$2^{-s}\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx,$$

or more conveniently,

$$2^{-s}\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{[(1/2)x]^s x^{-1}}{e^x - 1} dx.$$

I perform a variable change $(1/2)x = \alpha$, so

$$2^{-s}\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \alpha^{s-1} \frac{1}{e^{2\alpha} - 1} d\alpha.$$

I perform a fractional separation over the term $1/(e^{2\alpha} - 1)$ as follows:

$$\frac{1}{e^{2\alpha} - 1} = \frac{1}{2} \left(\frac{1}{e^\alpha - 1} - \frac{1}{e^\alpha + 1} \right)$$

so, I obtain

$$2^{1-s}\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\alpha^{s-1}}{e^\alpha - 1} d\alpha - \frac{1}{\Gamma(s)} \int_0^\infty \frac{\alpha^{s-1}}{e^\alpha + 1} d\alpha.$$

Recalling equation (2.2) and the integral expression of the Riemann Zeta function used above, I have

$$2^{1-s}\zeta(s) = \zeta(s) - \eta(s)$$

and, reorganizing terms, I come to equation (2.3)

$$\boxed{\zeta(s) = \frac{1}{1 - 2^{1-s}}}$$

□

END OF PRELIMINARY DEFINITIONS AND THEOREMS

Proof. Conjecture 2.1 will be demonstrated from now on up to the end of the document.

I substitute equation (2.2) in equation (2.3) to obtain,

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1} dx, \ \mathfrak{Re}(s) > 0. \tag{2.4}$$

I can multiply both sides of equation (2.4) by s to obtain

$$s\zeta(s) = \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{s x^{s-1}}{e^x + 1} dx, \ \mathfrak{Re}(s) > 0.$$

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Now, it is interesting to see that the integral of the last expression can be intelligently manipulated so that can yield a more useful expression. In effect, I realize that $(d/dx)x^s = sx^{s-1}$. On one hand, I perform the following variable changes, being $u = 1/(e^x+1)$, then $du = -e^x/(e^x+1)^2 dx$ and $dv = (d/dx)(x^s)dx$, then $v = x^s$. Given this, we can write the following

$$s\zeta(s) = \frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s)}\left[\frac{x^s}{e^x+1}\bigg|_0^\infty + \int_0^\infty \frac{x^s e^x}{(e^x+1)^2}dx\right].$$

I conveniently name $A(s) = s\zeta(s)$, $B(s) = 1/((1-2^{1-s}))\Gamma(s)$, $C(s)$ will be the first term in the brackets, and $\mathcal{I}_1(s)$ will be the integral which belongs to the second term in brackets. Then, the domain of definition $\mathcal{D}$ of $A(s)$ can be expressed as $\mathcal{D}_A = \mathcal{D}_B \bigcap (\mathcal{D}_C \bigcup \mathcal{D}_{\mathcal{I}_1})$. The function $1/(1-2^{1-s})$ is defined in the complex plane $\mathbb{C}\backslash\{1\}$ and the reciprocal of the Gamma function, $1/\Gamma(s)$ is defined, according to definition 2.2, in the half-plane $\mathfrak{Re}(s) > 0$, so $\mathcal{D}_B = \{s \in \mathbb{C} : \sigma > 0, -\infty < t < \infty\}\backslash\{1\}$. □

Proposition 2.1. *The limits* $\lim_{x\to\infty} x^s/(e^x+1)$ *and* $\lim_{x\to 0} x^s/(e^x+1)$ *tend to zero whenever* $\mathfrak{Re}(s) > 0$.

Proof. The following limits can be expressed as

$$\lim_{x\to\infty}\frac{x^{\sigma+it}}{e^x+1} = \lim_{x\to\infty}\frac{x^\sigma cos(tlnx)}{e^x+1} + i\lim_{x\to\infty}\frac{x^\sigma sin(tlnx)}{e^x+1},$$

$$\lim_{x\to 0}\frac{x^{\sigma+it}}{e^x+1} = \lim_{x\to 0}\frac{x^\sigma cos(tlnx)}{e^x+1} + i\lim_{x\to 0}\frac{x^\sigma sin(tlnx)}{e^x+1}.$$

The limit when $x\to\infty$, regardless of the oscillatory behavior of the trigonometric functions, the exponential function in the denominator grows much faster than the polynomial term x^σ, so that

$$\lim_{x\to\infty}\frac{x^{\sigma+it}}{e^x+1} = 0 + i0 = 0.$$

For limit when $x \to 0$, both real and imaginary parts vanish if and only if $\sigma > 0$, so that

$$\frac{x^s}{e^x+1}\bigg|_0^\infty = \lim_{x\to\infty}\frac{x^{\sigma+it}}{e^x+1} - \lim_{x\to 0}\frac{x^{\sigma+it}}{e^x+1} = 0.$$

□

Proposition 2.2. *The integral* $\mathcal{I}_1(s)$ *converges for* $\mathfrak{Re}(s) > -1$.

Proof. For $\mathcal{I}_1(s)$, when $x \gg 1$, then $(e^x+1)^2 \approx e^{2x}$. Then I can approximate the integral as

$$\mathcal{I}_1(s) \approx \int_0^\infty x^s e^{-x} dx.$$

Recalling definition 2.2 and performing $s \mapsto s+1$, I get

$$\Gamma(s+1) = \int_0^\infty x^s e^{-x} dx.$$

By comparing the two integrals above, and taking into account the fact that, as $\Gamma(s)$ is valid for

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$\mathfrak{Re}(s) > 0$, $\Gamma(s+1)$ is valid for $\mathfrak{Re}(s) > -1$ according to definition 2.2, I come to the conclusion that the domain for which the integral converges is $\mathcal{D}_{\mathcal{I}_1} = \{s \in \mathbb{C} : \sigma > -1, 0 < t < \infty\}$. $\square$

Then, $\mathcal{D}_C \bigcup \mathcal{D}_{\mathcal{I}_1} = \{s \in \mathbb{C} : \sigma > 0, 0 < t < \infty\} \bigcup \{s \in \mathbb{C} : \sigma > -1, 0 < t < \infty\} = \{s \in \mathbb{C} : \sigma > -1, 0 < t < \infty\}$ as $x^s/(e^x+1)$ vanishes as per proof of proposition 2.1. As a result, $\mathcal{D}_A = \{s \in \mathbb{C} : \sigma > 0, 0 < t < \infty\} \backslash \{1\} \bigcap \{s \in \mathbb{C} : \sigma > -1, 0 < t < \infty\} = \{s \in \mathbb{C} : \sigma > 0, 0 < t < \infty\} \backslash \{1\}$.

The following then remains valid for $\sigma > 0$ except $s = 1$

$$s\zeta(s) = \frac{1}{1-2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^s e^x}{(e^x+1)^2} dx.$$

I proceed analogously as I did in the proof of theorem 2.2 and I say that $e^x/(e^x+1)^2$ can be split into the following term by performing a fractional separation:

$$\frac{e^x}{(e^x+1)^2} = \frac{A}{e^x+1} + \frac{B}{(e^x+1)^2} = \frac{A(e^x+1)+B}{(e^x+1)^2}.$$

It is straightforward to see that $A = 1$ and $B = -1$. Then I have

$$s\zeta(s) = \frac{1}{1-2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty x^s \left(\frac{1}{e^x+1} - \frac{1}{(e^x+1)^2} \right) dx,$$

or,

$$s\zeta(s) = \frac{1}{1-2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^s}{e^x+1} dx - \frac{1}{1-2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^s}{(e^x+1)^2} dx.$$

Proposition 2.3. *The integral of the second term on the right side converges for $\mathfrak{Re}(s) > -1$.*

Proof. Now, let $\mathcal{I}_2(s)$ be the integral of the second term of the right side of the last expression:

$$\mathcal{I}_2(s) = \int_0^\infty \frac{x^s}{(e^x+1)^2} dx.$$

Proceeding analogously as I did with $\mathcal{I}_1(s)$, when $x \gg 1$, then $(e^x+1)^2 \approx e^{2x}$, so I get

$$\mathcal{I}_2(s) \approx \int_0^\infty x^s e^{-2x} dx,$$

which can be conveniently re-arranged with a variable change $u = 2x$, giving as a result

$$\mathcal{I}_2(s) \approx \frac{1}{2^{s+1}} \int_0^\infty u^s e^{-u} du,$$

or,

$$\mathcal{I}_2(s) \approx \frac{1}{2^{s+1}} \Gamma(s+1),$$

valid for $\mathfrak{Re}(s) > -1$. $\square$

Now, by using the functional equation of the Gamma function $\Gamma(s+1) = s\Gamma(s)$, I can divide both sides of the last expression by s to obtain

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$$\zeta(s) = \frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s}{e^x+1}dx - \frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s}{(e^x+1)^2}dx. \tag{2.5}$$

Now, this last expression is valid for $\Re\mathfrak{e}(s) \in (-1,1)\bigcup(1,\infty)$.

I recall expression (2.4) and I perform $s \mapsto s+1$, so I get, for $\Re\mathfrak{e}(s) > -1$ except $\Re\mathfrak{e}(s) = 0$ that leads to a simple pole, the following

$$\zeta(s+1) = \frac{1}{1-2^{-s}}\frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s}{e^x+1}dx,$$

or

$$(1-2^{-s})\zeta(s+1) = \frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s}{e^x+1}dx. \tag{2.6}$$

I can then place expression (2.6) on the right side of equation (2.5), replacing the first term to finally obtain

$$\boxed{\zeta(s) = \frac{1-2^{-s}}{1-2^{1-s}}\zeta(s+1) - \frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s}{(e^x+1)^2}dx}, \tag{2.7}$$

valid for $\{s \in \mathbb{C} : \sigma > -1, -\infty < t < \infty\}\setminus(\{0\}\bigcup\{1\})$.

I can then repeat the same process starting from equation. (2.4), but this time by performing a variable change $s \mapsto 1-s$ and multiplying both sides of the expression by $1-s$. In effect, I have

$$(1-s)\zeta(1-s) = \frac{1}{1-2^s}\frac{1}{\Gamma(1-s)}\int_0^\infty \frac{(1-s)x^{-s}e^x}{(e^x+1)^2}dx. \tag{2.8}$$

Proposition 2.4. *Equation (2.8) is valid for all $\Re\mathfrak{e}(s) < 1$.*

Proof. Equation (2.4) is valid for $\sigma > 0$, so, according to the variable change in equation[3] (2.8), $1-\sigma > 0 \Rightarrow \sigma < 1$. Performing integration by parts over (2.8), I use the following variables, $u = 1/(e^x+1)$, $du = -e^x/(e^x+1)^2dx$, $dv = (d/dx)(x^{1-s})dx$, $v = x^{1-s}$, I obtain

$$(1-s)\zeta(1-s) = \frac{1}{1-2^s}\frac{1}{\Gamma(1-s)}\left[\left.\frac{x^{1-s}}{e^x+1}\right|_0^\infty + \int_0^\infty \frac{x^{1-s}e^x}{(e^x+1)^2}dx\right],$$

which remains valid for $\Re\mathfrak{e}(s) < 1$.
□

Proposition 2.5. *The limits $\lim_{x\to\infty} x^{1-s}/(e^x+1)$ and $\lim_{x\to 0} x^{1-s}/(e^x+1)$ tend to zero whenever $\Re\mathfrak{e}(s) < 1$.*

Proof. By following the same reasoning as per proof of proposition (2.1), the limits tend to zero for all $\Re\mathfrak{e}(s) < 1$. □

[3] I use indistinctly σ or $\Re\mathfrak{e}(s)$

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So I end up with

$$(1-s)\zeta(1-s) = \frac{1}{1-2^s}\frac{1}{\Gamma(1-s)}\int_0^\infty \frac{x^{1-s}e^x}{(e^x+1)^2}dx. \tag{2.9}$$

The right side of equation. (2.9) contains three terms. The term $1/(1-2^s)$ is valid for $\mathbb{C}\backslash\{s \in \mathbb{C} : \sigma = 0, t = 2k\pi/ln2, k \in \mathbb{Z}\}$. The term $1/\Gamma(1-s)$ is valid for, according to definition 2.2, $\mathfrak{Re}(s) < 1$. The convergence of the integral term,$\mathcal{I}_3(s)$, can be studied as done in propositions 2.2 and 2.3.

Proposition 2.6. *The integral $\mathcal{I}_3(s)$ converges for $\mathfrak{Re}(s) < 2$.*

Proof. When $x \gg 1$, then $(e^x+1)^2 \approx e^{2x}$, so

$$\mathcal{I}_3(s) \approx \int_0^\infty x^{1-s}e^{-x}dx.$$

According to definition 2.2 and performing a variable change $s \mapsto 2-s$, I have

$$\Gamma(2-s) = \int_0^\infty x^{1-s}e^{-x}dx.$$

As $\Gamma(2-s)$ is valid for $\mathfrak{Re}(s) < 2$, this implies that the integral $\mathcal{I}_3(s)$ converges for $\mathfrak{Re}(s) < 2$. Performing once more fractional separation over the exponential term in the integral, I can also write

$$(1-s)\zeta(1-s) = \frac{1}{1-2^s}\frac{1}{\Gamma(1-s)}\int_0^\infty x^{1-s}\left(\frac{1}{e^x+1} - \frac{1}{(e^x+1)^2}\right)dx,$$

which can also be expanded as

$$(1-s)\zeta(1-s) = \frac{1}{1-2^s}\frac{1}{\Gamma(1-s)}\int_0^\infty \frac{x^{1-s}}{e^x+1}dx - \frac{1}{1-2^s}\frac{1}{\Gamma(1-s)}\int_0^\infty \frac{x^{1-s}}{(e^x+1)^2}dx.$$

□

Proposition 2.7. *The integral of the second term on the right side converges for $\mathfrak{Re}(s) < 2$.*

Proof. Let $\mathcal{I}_4(s)$ be the integral of the second term on the right side of the last expression:

$$\mathcal{I}_4(s) = \int_0^\infty \frac{x^{1-s}}{(e^x+1)^2}dx.$$

Proceeding analogously as before, when $x \gg 1$, then $(e^x+1)^2 \approx e^{2x}$, so I get

$$\mathcal{I}_4(s) \approx \int_0^\infty x^{1-s}e^{-2x}dx,$$

which can be conveniently re-arranged with a variable change $u = 2x$, giving as a result

$$\mathcal{I}_2(s) \approx \frac{1}{2^{2-s}}\int_0^\infty u^{1-s}e^{-u}du,$$

or

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$$\mathcal{I}_2(s) \approx \frac{1}{2^{2-s}}\Gamma(2-s),$$

valid for $\Re(s) < 2$. □

Last expression above proposition 2.7 can be divided both sides by $1-s$ and, by applying again functional equation of Gamma function $(1-s)\Gamma(1-s) = \Gamma(2-s)$, I obtain

$$\zeta(1-s) = \frac{1}{1-2^s}\frac{1}{\Gamma(2-s)}\int_0^\infty \frac{x^{1-s}}{e^x+1}dx - \frac{1}{1-2^s}\frac{1}{\Gamma(2-s)}\int_0^\infty \frac{x^{1-s}}{(e^x+1)^2}dx. \tag{2.10}$$

Equation (2.4) is used here again by now setting $s \mapsto 2-s$ and is turned into

$$\zeta(2-s) = \frac{1}{1-2^{s-1}}\frac{1}{\Gamma(2-s)}\int_0^\infty \frac{x^{1-s}}{e^x+1}dx,$$

or

$$(1-2^{s-1})\zeta(2-s) = \frac{1}{\Gamma(2-s)}\int_0^\infty \frac{x^{1-s}}{e^x+1}dx. \tag{2.11}$$

I perform a substitution of the first term of the right side of expression 2.10 by 2.11, obtaining finally

$$\boxed{\zeta(1-s) = \frac{1-2^{s-1}}{1-2^s}\zeta(2-s) - \frac{1}{1-2^s}\frac{1}{\Gamma(2-s)}\int_0^\infty \frac{x^{1-s}}{(e^x+1)^2}dx}. \tag{2.12}$$

Proposition 2.8. *The equation (2.12) is valid for $\Re(s) < 2$ except $s = 1$, where there is a pole for $\zeta(2-s)$, and $s = 0$, where there is a pole for $\zeta(1-s)$.*

Proof. See proof of proposition 2.6 for $1/\Gamma(2-s)$, proofs of propositions 2.3 and 2.6 for integral convergence of the term $x^{1-s}/(e^x+1)^2$.
□

Definition 2.3. *Let $\Lambda(s)$ be a complex-valued function of the form*

$$\Lambda(s) := \frac{1-2^{-s}}{1-2^{1-s}}. \tag{2.13}$$

Lemma 2.1. *The function $\Lambda(s)$ satisfies $\Lambda(\rho) \neq 0$.*

Proof. Let $f(s)$ be the numerator in (2.13), which can be expressed as $f(s) = 1 - e^{(-\sigma-it)ln2}$, with real part $\Re(f(s)) = 1 - 2^{-\sigma}cos(tln2)$ and imaginary part $\Im(f(s)) = 2^{-\sigma}sin(tln2)$. $\Re(f(\rho)) = 0 \iff cos(\gamma ln2) = 2^\sigma$ and $\Im(f(\rho)) = 0 \iff \gamma = k\pi/ln2\ \forall\sigma \in (0,1)$ with $k = 0, \pm1, \pm2, \pm3, ...$. As the values of γ have to be the same for both real and imaginary parts, then $\cos(k\pi) = 2^\sigma$. For $k = 2j+1$, with $j = 0, \pm1, \pm2, \pm3, ...$, $cos((2j+1)\pi) = -1$, and $2^\sigma > 0 \forall\sigma \in \mathbb{R}$, so $cos((2j+1)\pi) \neq 2^\sigma\ \forall\sigma \in \mathbb{R}$. For $k = 2l$, with $l = 0, \pm1, \pm2, \pm3, ...$, $cos(2l\pi) = 1$, and $2^\sigma \neq 1\ \forall\sigma \in (0,1)$, so $cos(2l\pi) \neq 2^\sigma\ \forall\sigma \in (0,1)$. I can conclude that $\nexists\sigma \in (0,1) : cos(k\pi) = 2^\sigma \implies f(s) \neq 0\ \forall\sigma \in (0,1)$.

As $f(s) \neq 0\ \forall\sigma \in (0,1) \implies \Lambda(\rho) \neq 0$. □

Lemma 2.2. *The function $\Lambda(s)$ is well defined for $s = \rho$.*

Proof. Let $g(s)$ be the denominator in equation (2.13), which can be expressed as $g(s) = 1 - e^{(1-\sigma-it)ln2}$, with real part $\mathfrak{Re}(g(s)) = 1 - 2^{1-\sigma}cos(tln2)$ and imaginary part $\mathfrak{Im}(g(s)) = 2^{1-\sigma}sin(tln2)$. $\mathfrak{Re}(g(\rho)) = 0 \iff cos(\gamma ln2) = 2^{\sigma-1}$ and $\mathfrak{Im}(g(\rho)) = 0 \iff \gamma = k\pi/ln2\ \forall\sigma \in (0,1)$ with $k = 0, \pm1, \pm2, \pm3, ...$. As the values of γ have to be the same for both real and imaginary parts, then $\cos(k\pi) = 2^{\sigma-1}$. For $k = 2j+1$, with $j = 0, \pm1, \pm2, \pm3, ...$, $cos((2j+1)\pi) = -1$, and $2^{\sigma-1} > 0\ \forall\sigma \in \mathbb{R}$, so $cos((2j+1)\pi) \neq 2^{\sigma-1}\ \forall\sigma \in (0,1)$. For $k = 2l$, with $l = 0, \pm1, \pm2, \pm3, ...$, $cos(2l\pi) = 1$, and $2^{\sigma-1} \neq 1\ \forall\sigma \in (0,1)$, so $cos(2l\pi) \neq 2^{\sigma-1}\ \forall\sigma \in (0,1)$. I can conclude that $\nexists\sigma \in (0,1) : cos(k\pi) = 2^{\sigma-1} \implies g(s) \neq 0\ \forall\sigma \in (0,1)$.

As $g(s)$ are well defined for $\forall\sigma \in (0,1) \implies \Lambda(\rho)$ is well defined. □

Theorem 2.3. *The Riemann Zeta function satisfies the following equation for $s = \rho$, which has no zeros and is well defined*

$$\boxed{(1-2^{-\rho})\zeta(\rho+1) = \frac{1}{\Gamma(\rho+1)}\int_0^\infty \frac{x^\rho}{(e^x+1)^2}dx}. \tag{2.14}$$

Proof. If I set $\zeta(\rho) = 0$ in equation (2.7) and I recall lemmas 2.1 and 2.2 to ensure $\Lambda(\rho)$ has no zeros and is well defined, respectively, and I recall corollary 1.1 to state that $\zeta(\rho+1) \neq 0$, in consequence, $\mathcal{I}_2(\rho) \neq 0$, and I multiply both sides of equation (2.7) by $1 - 2^{1-\rho}$, I obtain 2.14. □

Definition 2.4. *Let $\Delta(s)$ be a complex-valued function of the form*

$$\Delta(s) := \frac{1-2^{s-1}}{1-2^s}. \tag{2.15}$$

Lemma 2.3. *The function $\Delta(s)$ satisfies $\Delta(\rho) \neq 0$.*

Proof. Let $F(s)$ be the numerator in equation 2.15, which can be expressed as $F(s) = 1 - e^{(\sigma+it-1)ln2}$, with real part $\mathfrak{Re}(F(s)) = 1 - 2^{\sigma-1}cos(tln2)$ and imaginary part $\mathfrak{Im}(F(s)) = -2^{\sigma-1}sin(tln2)$. $\mathfrak{Re}(F(\rho)) = 0 \iff cos(\gamma ln2) = 2^{1-\sigma}$ and $\mathfrak{Im}(F(\rho)) = 0 \iff \gamma = k\pi/ln2\ \forall\sigma \in (0,1)$ with $k = 0, \pm1, \pm2, \pm3, ...$. Again, as the values of γ have to be the same for both real and imaginary parts, then $\cos(k\pi) = 2^{1-\sigma}$. For $k = 2j+1$, with $j = 0, \pm1, \pm2, \pm3, ...$, $cos((2j+1)\pi) = -1$, and $2^{1-\sigma} > 0$ $\forall\sigma \in \mathbb{R}$, so $cos((2j+1)\pi) \neq 2^{1-\sigma}\ \forall\sigma \in \mathbb{R}$. For $k = 2l$, with $l = 0, \pm1, \pm2, \pm3, ...$, $cos(2l\pi) = 1$, and $2^\sigma \neq 1$ $\forall\sigma \in (0,1)$, so $cos(2l\pi) \neq 2^\sigma\ \forall\sigma \in (0,1)$. I can conclude that $\nexists\sigma \in (0,1) : cos(k\pi) = 2^\sigma \implies F(s) \neq 0$ $\forall\sigma \in (0,1)$.

As $F(s) \neq 0\ \forall\sigma \in (0,1) \implies \Delta(\rho) \neq 0$. □

Lemma 2.4. *The function $\Delta(s)$ is well defined for $s = \rho$.*

Proof. Let $G(s)$ be the denominator in equation (2.15), which can be expressed as $G(s) = 1 - e^{(\sigma+it)ln2}$, with real part $\mathfrak{Re}(G(s)) = 1 - 2^\sigma cos(tln2)$ and imaginary part $\mathfrak{Im}(G(s)) = -2^\sigma sin(tln2)$. $\mathfrak{Re}(G(\rho)) = 0 \iff cos(\gamma ln2) = 2^{-\sigma}$ and $\mathfrak{Im}(G(\rho)) = 0 \iff \gamma = k\pi/ln2\ \forall\sigma \in (0,1)$ with $k = 0, \pm1, \pm2, \pm3, ...$. Following an identical reasoning, as the values of γ have to be the same for both real and imaginary parts, then $\cos(k\pi) = 2^{-\sigma}$. For $k = 2j+1$, with $j = 0, \pm1, \pm2, \pm3, ...$, $cos((2j+1)\pi) = -1$, and $2^{-\sigma} > 0$ $\forall\sigma \in \mathbb{R}$, so $cos((2j+1)\pi) \neq 2^{-\sigma}\ \forall\sigma \in (0,1)$. For $k = 2l$, with $l = 0, \pm1, \pm2, \pm3, ...$, $cos(2l\pi) = 1$, and $2^{-\sigma} \neq 1\ \forall\sigma \in (0,1)$, so $cos(2l\pi) \neq 2^{-\sigma}\ \forall\sigma \in (0,1)$. I can conclude that $\nexists\sigma \in (0,1) : cos(k\pi) = 2^{-\sigma} \implies G(s) \neq 0\ \forall\sigma \in (0,1)$.

As $G(s)$ are well defined for $\forall\sigma \in (0,1) \implies \Delta(\rho)$ is well defined. □

Theorem 2.4. *The Riemann Zeta function satisfies the following equation for $s = \rho$, which has no zeros and is well defined*

$$\boxed{(1-2^{\rho-1})\zeta(2-\rho) = \frac{1}{\Gamma(2-\rho)}\int_0^\infty \frac{x^{1-\rho}}{(e^x+1)^2}dx}. \tag{2.16}$$

Proof. If I set $\zeta(\rho) = 0$ in equation (2.12) and I recall lemmas 2.3 and 2.4 to ensure $\Delta(\rho)$ has no zeros and is well defined, respectively, and I recall again corollary 1.1 to state that $\zeta(2-\rho) \neq 0$, in consequence, $\mathcal{I}_4(\rho) \neq 0$, and I multiplying both sides of (15) by $1-2^\rho$, I get to (2.16). □

Now, I multiply expressions (2.14) and (2.16) to get

$$\boxed{\zeta(\rho+1)\zeta(2-\rho) = \frac{2}{3-2^\rho-2^{1-\rho}}\frac{1}{\Gamma(\rho+1)}\frac{1}{\Gamma(2-\rho)}\int_0^\infty \frac{x^\rho}{(e^x+1)^2}dx\int_0^\infty \frac{x^{1-\rho}}{(e^x+1)^2}dx}.$$

Let $\mathcal{D}_{\zeta+}$ be the domain of the Riemann Zeta function in expression (2.7) and let$\mathcal{D}_{\zeta-}$be the domain of the Riemann Zeta function in expression (2.12). Let $\mathcal{D}^*$ be the domain of the multiplication of expressions (2.7) and (2.12), so that $\mathcal{D}^* = \mathcal{D}_{\zeta+}\bigcap\mathcal{D}_{\zeta-} = \{s \in \mathbb{C} : -1 < \mathfrak{Re}(s) < 2, 0 < \mathfrak{Im}(s) < \infty\}$. If I let $\mathcal{D}^*_\rho$ be the domain the last boxed expression, it is easy to see that $\mathcal{D}^*_\rho \subset D^*$ as $\rho \in \{\sigma + i\gamma : 0 < \sigma < 1, 0 < \gamma < \infty\}$. Now, I re-arrange the last boxed expression by applying again the properties of the functional equation of the gamma function. By doing so, one can obtain

$$\zeta(\rho+1)\zeta(2-\rho) = \frac{2}{3-2^\rho-2^{1-\rho}}\frac{1}{\rho\Gamma(\rho)}\frac{1}{(1-\rho)\Gamma(1-\rho)}\int_0^\infty \frac{x^\rho}{(e^x+1)^2}dx\int_0^\infty \frac{x^{1-\rho}}{(e^x+1)^2}dx,$$

or

$$\rho-\rho^2 = \frac{2}{3-2^\rho-2^{1-\rho}}\frac{1}{\zeta(\rho+1)\zeta(2-\rho)}\frac{1}{\Gamma(\rho)\Gamma(1-\rho)}\int_0^\infty \frac{x^\rho}{(e^x+1)^2}dx\int_0^\infty \frac{x^{1-\rho}}{(e^x+1)^2}dx.$$

As it is known by the Euler's reflection formula, $\Gamma(s)\Gamma(1-s) = \pi/sin(\pi s)$, $s \notin \mathbb{Z}$, therefore, the last expression can be written as

$$\rho-\rho^2 = \frac{2}{3-2^\rho-2^{1-\rho}}\frac{1}{\zeta(\rho+1)\zeta(2-\rho)}\frac{sin(\pi\rho)}{\pi}\int_0^\infty \frac{x^\rho}{(e^x+1)^2}dx\int_0^\infty \frac{x^{1-\rho}}{(e^x+1)^2}dx. \tag{2.17}$$

Definition 2.5. *Let $\Omega(\rho)$ be the right side of equation* (2.17)

$$\boxed{\Omega(\rho) = \frac{2}{3-2^\rho-2^{1-\rho}}\frac{1}{\zeta(\rho+1)\zeta(2-\rho)}\frac{sin(\pi\rho)}{\pi}\int_0^\infty \frac{x^\rho}{(e^x+1)^2}dx\int_0^\infty \frac{x^{1-\rho}}{(e^x+1)^2}dx}. \tag{2.18}$$

Lemma 2.5. *Omega function satisfies $\Omega(\rho) = \Omega(1-\rho)$.*

Proof. If I place $1-\rho$ in expression (2.18), then the Omega function leads to

$$\frac{2}{3-2^{1-\rho}-2^{1-(1-\rho)}}\frac{1}{\zeta(1-\rho+1)\zeta(2-(1-\rho))}\frac{sin(\pi(1-\rho))}{\pi}\int_0^\infty \frac{x^{1-\rho}}{(e^x+1)^2}dx\int_0^\infty \frac{x^{1-(1-\rho)}}{(e^x+1)^2}dx$$

as the sine term can be broken down as $sin(\pi-\pi\rho) = sin(\pi)cos(\pi\rho) - sin(\pi\rho)cos(\pi)$, then I get back to (2.18).
□

I therefore come to

$$\rho-\rho^2 = \Omega(\rho). \tag{2.19}$$

Now, left side of equation (2.19) can be expanded as

$$\rho-\rho^2 = (\sigma+i\gamma)-(\sigma+i\gamma)^2 = (\sigma-\sigma^2+\gamma^2)+i\gamma(1-2\sigma).$$

As two complex-valued functions are equal if and only if their real and imaginary parts are equal, I can state that

$$(\sigma-\sigma^2+\gamma^2) = \mathfrak{Re}(\Omega(\rho)) \tag{2.20}$$

$$\gamma(1-2\sigma) = \mathfrak{Im}(\Omega(\rho)) \tag{2.21}$$

Lemma 2.6. *Omega function satisfies* $\overline{\Omega}(\rho) = \Omega(\overline{\rho})$.

Proof. The conjugate of the Omega function is

$$\overline{\Omega}(\rho) = \overline{\left(\frac{2}{3-2^{\rho}-2^{1-\rho}}\right)\left(\frac{1}{\zeta(\rho+1)\zeta(2-\rho)}\right)\left(\frac{sin(\pi\rho)}{\pi}\right)\int_0^\infty \frac{x^{\rho}}{(e^x+1)^2}dx}\,\overline{\int_0^\infty \frac{x^{1-\rho}}{(e^x+1)^2}dx}.$$

By means of 1.1, $\overline{\zeta}(\rho+1) = \zeta(\overline{\rho}+1)$ and $\overline{\zeta}(2-\rho) = \zeta(2-\overline{\rho})$, and $\overline{sin}(\pi\rho) = \left(\overline{e^{i\rho}} - \overline{e^{-i\rho}}\right)/\overline{2i} = (e^{i\overline{\rho}}-e^{-i\overline{\rho}})/2i$. Equally, the integrals can be broken down as

$$\int_0^\infty \frac{x^{\sigma+i\gamma}}{(e^x+1)^2}dx = \int_0^\infty \frac{x^{\sigma}cos(\gamma lnx)}{(e^x+1)^2}dx + i\int_0^\infty \frac{x^{\sigma}sin(\gamma lnx)}{(e^x+1)^2}dx,$$

$$\int_0^\infty \frac{x^{1-\sigma-i\gamma}}{(e^x+1)^2}dx = \int_0^\infty \frac{x^{1-\sigma}cos(\gamma lnx)}{(e^x+1)^2}dx - i\int_0^\infty \frac{x^{1-\sigma}sin(\gamma lnx)}{(e^x+1)^2}dx.$$

By conjugating both previous expressions, one can easily see that the conjugate of each integral equals the integral of each conjugate, so I finally have

$$\overline{\Omega}(\rho) = \frac{2}{3-2^{\overline{\rho}}-2^{1-\overline{\rho}}}\frac{1}{\zeta(\overline{\rho}+1)\zeta(2-\overline{\rho})}\frac{sin(\pi\overline{\rho})}{\pi}\int_0^\infty \frac{x^{\overline{\rho}}}{(e^x+1)^2}dx\int_0^\infty \frac{x^{1-\overline{\rho}}}{(e^x+1)^2}dx,$$

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and it implies that

$$\overline{\Omega}(\rho) = \Omega(\overline{\rho}). \tag{2.22}$$

□

Therefore, the imaginary part of omega function is $\mathfrak{Im}(\Omega(\rho)) = (\Omega(\rho) - \Omega(\overline{\rho}))/(2i)$. Now, I conjugate both sides of equation (2.19) so that I have

$$\overline{\rho - \rho^2} = \overline{\Omega(\rho)}.$$

By noting that the conjugate of a subtraction is the subtraction of the conjugates, the conjugate of a product is the product of the conjugates, and by applying lemma, 2.6 I realize that the above expression can be written as

$$\overline{\rho} - \overline{\rho}^2 = \overline{\Omega(\rho)}. \tag{2.23}$$

The left side of (2.23) is just

$$\overline{\rho} - \overline{\rho}^2 = (\sigma - i\gamma) - (\sigma - i\gamma)^2 = (\sigma - \sigma^2 + \gamma^2) + i\gamma(2\sigma - 1).$$

Proceeding analogously, I can say that

$$(\sigma - \sigma^2 + \gamma^2) = \mathfrak{Re}(\Omega(\overline{\rho})) \tag{2.24}$$

$$-\gamma(1 - 2\sigma) = \mathfrak{Im}\{\Omega(\overline{\rho})\} \tag{2.25}$$

Proposition 2.9. *Both* (2.21) *and* (2.25) *have to satisfy the same σ-values of the zeros ρ and $\overline{\rho}$, as the conjugation does not affect* (2.20) *and* (2.24).

Proof. I recall equations and (2.25) and, by lemma 2.5, I know that $\Omega(\sigma + i\gamma) = \Omega((1-\sigma) - i\gamma)$ and by 2.6, I also know that $\Omega((1-\sigma) - i\gamma) = \overline{\Omega}((1-\sigma) + i\gamma)$. In the same way, by, again, 2.5, $\Omega((1-\sigma) + i\gamma) = \Omega(\sigma - i\gamma)$ and again, by 2.6, it turns out that $\Omega(\sigma - i\gamma) = \overline{\Omega}(\sigma + i\gamma)$. If I recall equation (2.21), its right side is also $\mathfrak{Im}(\Omega(\sigma + i\gamma)) = (\Omega(\sigma + i\gamma) - \overline{\Omega}(\sigma + i\gamma))/(2i)$. But at the same time and according to the development of this proof, it can also be expressed as $\mathfrak{Im}(\Omega(\sigma + i\gamma)) = (\Omega((1 - \sigma) - i\gamma) - \overline{\Omega}((1-\sigma) - i\gamma))/(2i)$. If I substitute the real part σ of the left side of equation (2.21) by $1 - \sigma$, then $\gamma(1 - 2(1-\sigma)) = \gamma(-1 + 2\sigma))$. As $\Omega(\sigma + i\gamma) - \overline{\Omega}(\sigma + i\gamma) = \Omega((1-\sigma) - i\gamma) - \overline{\Omega}((1-\sigma) - i\gamma)$, then $\mathfrak{Im}(\Omega(\sigma + i\gamma)) = \mathfrak{Im}(\Omega((1-\sigma) + i\gamma))$, and then $\gamma(1 - 2\sigma) = \gamma(-1 + 2\sigma))$. So, $1 - 2\sigma = -1 + 2\sigma$, or

$$\boxed{\sigma = \frac{1}{2}}. \tag{2.26}$$

□

The last value of σ is consistent with equations (2.21) and 2.25, which leads to $\Omega(\rho) = \Omega(\overline{\rho})$, which means that $\mathfrak{Im}(\Omega(\rho) = 0$ and, therefore, the Omega function is real.

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