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Article

A Novel Approach to State-to-State Transformation in Quantum Computing

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Abstract

The article presents a new approach to the problem of transforming one quantum state into another. It is shown that a r -qubit superposition $|x\rangle$ can be obtained from another r -qubit superposition $|y\rangle$, by using only $(2^r - 1)$ rotations, each presented by one controlled rotation gate. The quantum superpositions with real amplitudes are considered. The traditional two-stage approach $U_y^{-1}U_x: |x\rangle \rightarrow |0\rangle^{\oplus r} \rightarrow |y\rangle$ requires twice as many rotations. Here, both transformations to the conventional basis state, $U_x: |x\rangle \rightarrow |0\rangle^{\oplus r}$ and $U_y: |y\rangle \rightarrow |0\rangle^{\oplus r}$, use $(2^r - 1)$ rotations each on two binary planes and many of these rotations require additional sets of CNOTs to be represented as 1- or 2-qubit controlled gates. The proposed method is based on the concept of the discrete signal-induced heap transform (DsiHT) which is unitary and generated by a vector and a set of angular equations with given parameters. The quantum analogue of this transform is described. The main characteristic of the DsiHT is the path of processing the data. It is shown that exist such fast paths that allow for effective computing of the DsiHT, which leads to the simple quantum circuits for state preparation and transformation. Examples of such paths are given and quantum circuits for preparation and transformation of 2-, 3-, and 4-qubits are described in detail. CNOT gates are not used, but only controlled gates of elementary rotations around the y -axis. It is shown that the transformation and, in particular, only rotation gates with control qubits are required for initialization of 4-qubits. The quantum circuits are simple and have a recursive form, which makes them easy to implement for arbitrary r -qubit superposition, with $r \geq 2$. This approach significantly reduces the complexity of quantum state transformations, paving the way for more efficient quantum algorithms and practical implementations on near-term quantum devices.

Keywords: qubit state preparation; quantum state transformation; quantum signal-induced heap transform

1. Introduction

Quantum state transformation - changing a quantum state through quantum operations or gates - is fundamental to many quantum algorithms. It enables the manipulation and evolution of quantum information, facilitating complex computations. Mastery of state transformations is crucial for multi-qubit systems, which utilize specific gates to create entangled states and perform calculations. It is also vital for quantum error correction to ensure the coherence and fidelity of quantum information. Effective and low-noise state preparation procedures are essential for scalable distribution loading, which underpins a wide range of algorithms that provide a quantum advantage. Therefore, the important step in quantum computation in many applications is the multi-qubit superposition transformation. This is the problem of transformation of one quantum state into another one. Many papers have been published related to this problem [1–8]. Analyzing these works, it is important to note the following. The view has been formed that operations on three or more qubits are very

complex for quantum computers. Therefore, when creating quantum circuits, much attention is paid to one- and two-qubit operations, or gates. As a result, each multi-qubit gate, as a unitary transformation, is represented by a chain of single-qubit and dual-qubit gates together with a chain of CNOTs, which may include the long-range CNOTs. The schemes of many simple (for a classical computer) operations have become very complex and are full of such switches of operations from one qubit to another. Therefore, much attention is paid to reducing the number of such gates in quantum circuits [9–11]. As is known, the number of elementary rotation gates is around the number $2(2^r - 1)$ and CNOTs in number $2(2^r - r - 1)$ [12]–[14].

The solution to the problem of multi-qubit state-to-state transformation $|x\rangle \rightarrow |y\rangle$ traditionally lies in the idea of preparing the desired state from the conventional basis state, $|0\rangle^{\oplus r}$. Here, $r \geq 1$ is the dimension of the superposition, or the number of qubits it contains. We note the simple method with Givens rotations, $(x, y) \rightarrow (\pm\sqrt{x^2 + y^2}, 0)$. They rotate points inside the unit circle into the interval $[-1, 1]$ on the X -axis. In matrix theory, such rotations are used in each step of the QR decomposition of a square matrix, when transforming first columns $\mathbf{z}$ of the submatrix into the vector of form of $(\pm\|\mathbf{z}\|, 0, 0, \dots, 0)$. We consider only real vectors with norm 1, as for quantum superpositions with real amplitudes. Since it is possible to perform transformations of two vectors $\mathbf{x}$ and $\mathbf{y}$ into the unit vector $\mathbf{e}_1 = (\pm 1, 0, 0, \dots, 0)$ of same dimension,

$$U_x: \mathbf{x} \rightarrow \mathbf{e}_1 \quad \text{and} \quad U_y: \mathbf{y} \rightarrow \mathbf{e}_1, \quad (1)$$

the transformation $\mathbf{x} \rightarrow \mathbf{y}$, can be fulfilled by the unitary transform $T = U_y^{-1}U_x$. Both transforms U_x and U_y are unitary, and U_y^{-1} is the inverse of U_y . Moreover, each of them requires $(2^r - 1)$ rotations with almost the same number of CNOTs, when implementing calculations with 1- and 2-qubit gates [8]. If CNOTs operate not on the adjacent, or nearest-neighbor, bit planes (BP), they can be implemented by a cascade of CNOTs operating on the nearest neighbor bit planes. The permutations with Gray code can be used for this purpose [14,15]. The number of all CNOTs is estimated as $(10/3)2^r$ [7].

In this work, we analyze the method of rotations and describe the best and fastest, in our opinion, way of elementary rotations for transforming the state. The quantum states, or superpositions of qubits, are considered with real amplitudes. The state-to-state transformation can be performed in one step, which is equivalent to implementing only one transformation which is similar to U_x , instead of two transforms U_x and U_y in the Eq. 1. This is the main goal of this work, not counting and reducing the number of CNOT gates; a lot of work has already been done in this direction. We introduce and describe the new concept of the quantum signal-induced heap transform (QsiHT) which is the analogue of the discrete signal-induced heap transform (DsiHT) [16,17]. With this transform, we show

1. How to implement r -qubit state-to-state transformation with real amplitudes, by using only $(2^r - 1)$ elementary rotations, each with only one angle. This number is half as much as the best-known estimation of $2(2^r - 1)$ rotations.
2. Visual numerical examples of preparing states $\mathbf{x} \rightarrow \mathbf{y}$ for two, three, and four qubits.
3. How to initiate any multi-qubit superposition of qubits (without operation of tensor product).
4. The importance of the path in r -qubit QsiHT and existence of the fast paths for effective computing the QsiHT. For large multi-qubit superpositions, there are various fast paths (with their number increasing with the number of qubits), and we are confident that among them it is possible to choose the most convenient path for implementing state transformations in the topology (architecture) of quantum systems.
5. How to build the simple quantum circuits for the r -qubit QsiHT.

2. One- and Two-Qubit Operation

In this section, we briefly describe the concept of local gates on single qubits in the quantum superposition [18,19]. The 1- and 2-qubit operators are described by 2×2 and 4×4 unitary real or complex matrices, respectively. Let us find out what it means to apply a gate to one qubit in a 4-qubit

superposition. Consider the conventional basis of states. In the general case, a 2-qubit superposition $|z\rangle = z_0|00\rangle + z_1|01\rangle + z_2|10\rangle + z_3|11\rangle$ may be the Kronecker product of two 1-qubits $|x\rangle = x_0|00\rangle + x_1|01\rangle$ and $|y\rangle = y_0|00\rangle + y_1|01\rangle$, or maybe not. In the first case, the 2-qubit $|z\rangle = |x\rangle \otimes |y\rangle$ is non-entangled, and in the second case the 2-qubit $|z\rangle$ is entangled. Let U_2 , V_2 , and U_4 denote some unitary operations, that is, 2×2 , 2×2 , and 4×4 matrices, respectively. If the operation on the non-entangled 2-qubit, that is, 2-qubit superposition, is described by

$$U_4|z\rangle = (U_2|x\rangle, V_2|y\rangle) = (U_2|x\rangle) \otimes (V_2|y\rangle) = (U_2 \otimes V_2)(|x\rangle \otimes |y\rangle), \quad (2)$$

then, it is the qubit-wise operation, or the operation on two qubits $|x\rangle$ and $|y\rangle$ of $|z\rangle$. This operation includes the operations on single qubits, too,

$$U_4|z\rangle = (U_2|x\rangle, |y\rangle) = U_2|x\rangle \otimes |y\rangle \text{ and } U_4|z\rangle = (|x\rangle, V_2|y\rangle) = |x\rangle \otimes V_2|y\rangle. \quad (3)$$

As an example, consider the unitary matrix $U_2 = [a, b; c, d]$, with the real coefficients and $a^2 + b^2 + c^2 + d^2 = 1$. The first operation is described in the matrix form as

$$U_4|z\rangle = U_2|x\rangle \otimes |y\rangle = (U_2 \otimes I_2)(|x\rangle \otimes |y\rangle) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0y_0 \\ x_0y_1 \\ x_1y_0 \\ x_1y_1 \end{bmatrix}. \quad (4)$$

The circuit element of this gate $U_4 = U_2 \otimes I_2$ is shown in Figure 1 in part (a). It is the local gate on the first qubit. Here, I_2 is the 2×2 identity matrix. In the diagram this gate (operator) is connected to the first abstract quantum wire (quantum bit). The circuit element shown in part (b) as the operation on the second qubit. This operation can be written as $I_2 \otimes U_2$. Note that in general the input, or the 2-qubit superposition $|z\rangle$, of the circuit does not necessarily have to be entangled. Any 2-qubit superposition is assumed as an input.

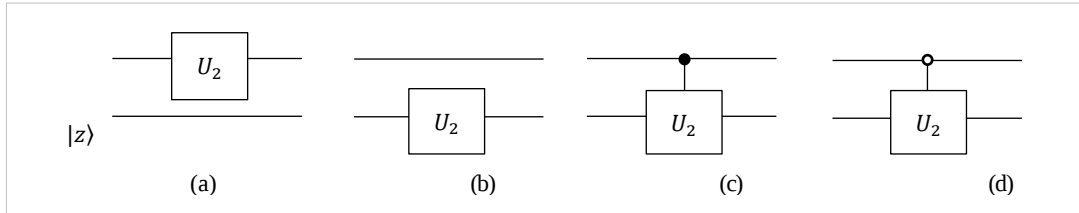


Figure 1. The circuit elements of the local gates (a) $U_2 \otimes I_2$, (b) $I_2 \otimes U_2$, (c) $I_2 \oplus U_2$, and (d) $U_2 \oplus I_2$.

The 1-qubit gate controlled by one qubit (or bit) are described by the operation of the Kronecker sum of matrices. The circuit elements in Figure 1(c) and (d) are described in the matrix form as follows:

$$C_1(U_2) = I_2 \oplus U_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & a & b \\ & & c & d \end{bmatrix} \text{ and } C_0(U_2) = U_2 \oplus I_2 = \begin{bmatrix} a & b & & \\ c & d & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Note that if a 2-qubit superposition $|z\rangle$ is nonentangled, then $C_{0,1}(U_2)|z\rangle$ may be entangled. These operations are performed on the first or last two amplitudes of $|z\rangle = |x\rangle \otimes |y\rangle$, but not on the qubit $|x\rangle$ or $|y\rangle$. The matrices of the local gates $U_2 \otimes I_2$ and $I_2 \otimes U_2$ are more complex than the matrices of the controlled gates $C_0(U_2)$ and $C_1(U_2)$. For instance,

$$I_2 \otimes U_2 = \begin{bmatrix} a & b \\ c & d \\ & a & b \\ & c & d \end{bmatrix} = C_1(U_2)C_2(U_2).$$

3. Definition of the DsiHT

In this section, we describe the concept of the DsiHT with Givens rotations. We note the following feature of many unitary transformations used in engineering and science. The quantum

Fourier transform (QFT) is well-known in quantum computation [20–22]. This transform is used in the solution of such problems, as the Shor's algorithm for integer factorization, the quantum phase estimation algorithm, systems of linear equations and others [23–26]. The QFT is the quantum analogue of the 2^r -point discrete Fourier transform, where integer $r > 1$. The basis functions of the transform are 2^r complex exponential functions on the unit circle. Many other discrete unitary transformations also use sets of defined in advance basis functions. We mention for example, the cosine, Hadamard, hartley, cosine and slant transforms [27–29]. The N -point discrete signal-induced heap transform (DsiHT) generates the basis by a given signal of length N . This signal is called the generator of the transform [30,31]. The DsiHT with its simple and fast algorithm (for any order N of transformation) is successfully used in many applications in grayscale and color image processing. In addition, the DsiHT can be used for QR-decomposition of square matrices [32]. Together with the generator, the transform uses a parameter which defines the order of processing the input signals. This parameter, or path, is very important and allows us to obtain different QR-decompositions, with diagonal matrices R for unitary matrices.

The key of the DsiHT is the generator together with the path. The number of generators can be more than one, but in this work, we focus on the transforms using one pre-selected generator. The selection of the generator depends on the application. For example, in image enhancement, the generators can be defined by the mean or median values along the rows or columns of an image result in very good quality images. The generator-signal is denoted by $\mathbf{x} = (x_0, x_1, x_2, \dots, x_{N-1})$. The length of the signal is considered to be a power of 2, $N = 2^r, r > 1$. The basic building blocks that make up the transform are 2×2 transforms, which can be linear or nonlinear [30]. In this work, we consider 2×2 rotation matrices, or Givens rotations. The transform is calculated by using $(N - 1)$ Givens rotations by angles ϑ_k . In this case, the signal-generator is presented in its angular representation

$$\mathbf{x} \rightarrow A_{\mathbf{x}} = \{\vartheta_1, \vartheta_2, \vartheta_3, \dots, \vartheta_{N-1}\} \quad (5)$$

plus, the norm, $\|\mathbf{x}\|$. The signal is considered real. The DsiHT is a unitary transform, H , and when applying to the generator, it results in the vector $(\|\mathbf{x}\|, 0, 0, \dots, 0)$. For a generator with the unit norm or after normalization, this N -dimensional vector $(1, 0, 0, \dots, 0)$ presents the computational basis state $|00 \dots 0\rangle = |0\rangle^{\oplus r}$. Thus, the following takes place: $H\mathbf{x} = |0\rangle^{\oplus r}$.

To define the DsiHT, $(N - 1)$ rotations are performed on the generator and at each stage of the transformation one of the two rotation outputs can be reset to zero. For instance, the generator data can be processed in order (it is the path) x_0 and x_1 , then the updated x_0 with x_2 , the next updated x_0 with x_3 , and so on. The illustration of the calculation with the 8-point generator is given in Figure 2. Here, $N = 8$ and 2×2 transforms $T_k, k = 1:7$, are rotations by angles $\vartheta_1, \vartheta_2, \vartheta_3, \dots, \vartheta_7$. The DsiHT with this regular path is called *the DsiHT with weak wheel-carriage* [16,31]. The last value is the updated 7 times value of x_0 with the splash of energy of the generator, as if we were collecting energy, $E[\mathbf{x}] = \|\mathbf{x}\|$, in one heap.

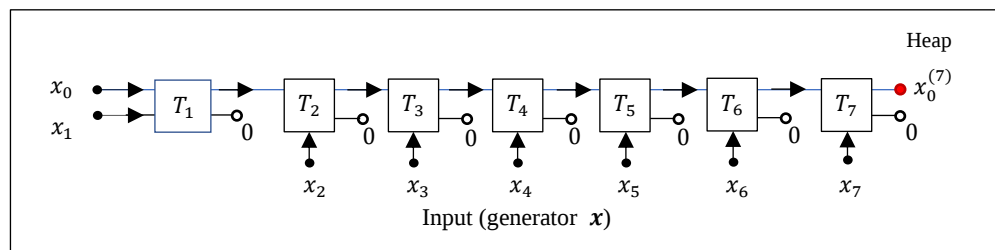


Figure 2. The 8-point DsiHT by using the regular path.

The DsiHT can be considered as a two-level unitary transformation. First, the angles ϑ_k are calculated from the generator. Then, at each stage k of calculation, the transform T_k is applied to the input signal $\mathbf{z} = (z_0, z_1, z_2, \dots, z_{N-1})$ in the way (path) as for the generator. For the DsiHT with the week carriage-wheel, this two-level transform is illustrated in Figure 3, for the $N = 4$ case. The results of calculations are

$$x \rightarrow (x_0^{(3)}, 0, 0, 0) = (\|x\|, 0, 0, 0), \quad z \rightarrow (z_0^{(3)}, z_1', z_2', z_3').$$

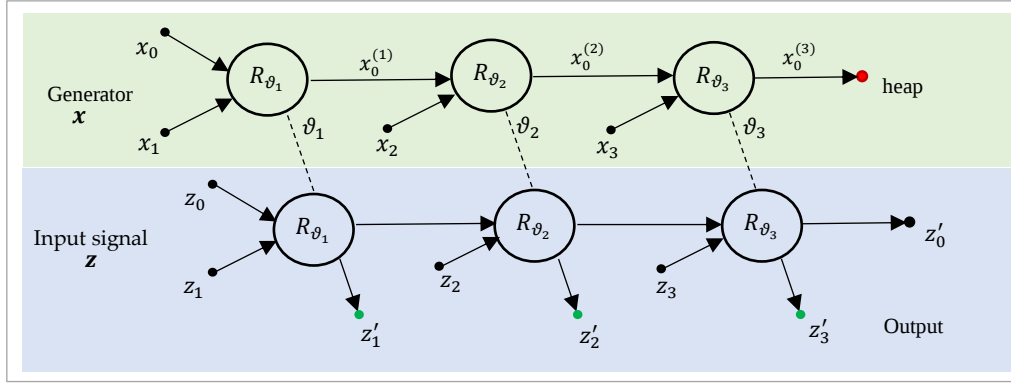


Figure 3. The diagram of the 4-point weak 2-wheel carriage DsiHT applied to a signal $z = (z_0, z_1, z_2, z_3)$.

4. The DsiHT with Strong 2-Wheel Carriage

We will consider the concept of the strong DsiHT, or the *DsiHT with strong 2-wheel carriage* (for more detail, see [16,17]). The path of this DsiHT differs from the path used in the above DsiHT with the weak wheel-carriage. As an example, the block-diagram of the 8-point strong DsiHT on its generator is shown in Figure 4. The case of real signals is considered. In this example, on the first step, the angle of the rotation is calculated from the conditions

$$T_1 \begin{bmatrix} x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} x_0^{(1)} \\ 0 \end{bmatrix} \quad \text{and} \quad |x_0^{(1)}| = \sqrt{x_6^2 + x_7^2}. \quad (6)$$

This transform is the Givens rotation calculated by

$$T_1 = T_{\vartheta_1} = \begin{bmatrix} \cos \vartheta_1 & -\sin \vartheta_1 \\ \sin \vartheta_1 & \cos \vartheta_1 \end{bmatrix}, \quad (7)$$

with the angle $\vartheta_1 = -\arctan(x_7/x_6)$. If $x_6 = 0$, the angle $\vartheta_1 = -\pi/2$, or $\pi/2$. Thus, the angle of rotation is defined from the *angular equation* $x_6 \sin \vartheta_1 + x_7 \cos \vartheta_1 = 0$. Then, the first output, $x_0^{(1)}$, of the transform is calculated in Eq. 6. On the next step, the second angle of rotation, $T_2 = T_{\vartheta_2}$ is calculated from the conditions

$$T_2 \begin{bmatrix} x_5 \\ x_0^{(1)} \end{bmatrix} = \begin{bmatrix} x_0^{(2)} \\ 0 \end{bmatrix} \quad \text{and} \quad |x_0^{(2)}| = \sqrt{x_5^2 + (x_0^{(1)})^2}.$$

Thus, $\vartheta_2 = -\arctan(x_5/x_0^{(1)})$. Continuing similar calculations, the last angle $\vartheta_7 = -\arctan(x_0^{(6)}/x_0)$ of rotation $T_7 = T_{\vartheta_7}$ is calculated and then the last value $x_0^{(7)}$,

$$T_7 \begin{bmatrix} x_0 \\ x_0^{(6)} \end{bmatrix} = \begin{bmatrix} x_0^{(7)} \\ 0 \end{bmatrix}.$$

The value $x_0^{(7)}$ contains information of the energy of the generator.

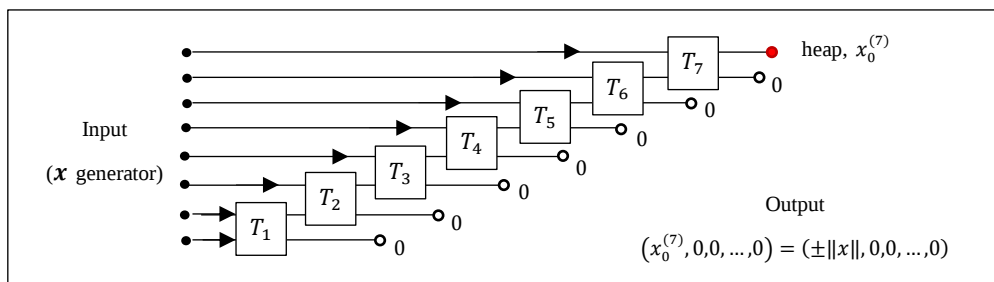


Figure 4. The 8-point strong DsiHT on the generator.

The input 8-point signal $\mathbf{z} = (z_0, z_1, z_2, \dots, z_7)$ is processed by the same seven rotations in the same order (path). This 2-level strong carriage-wheel DsiHT with processing the input signal $\mathbf{z}$ with the generator is shown in Figure 5.

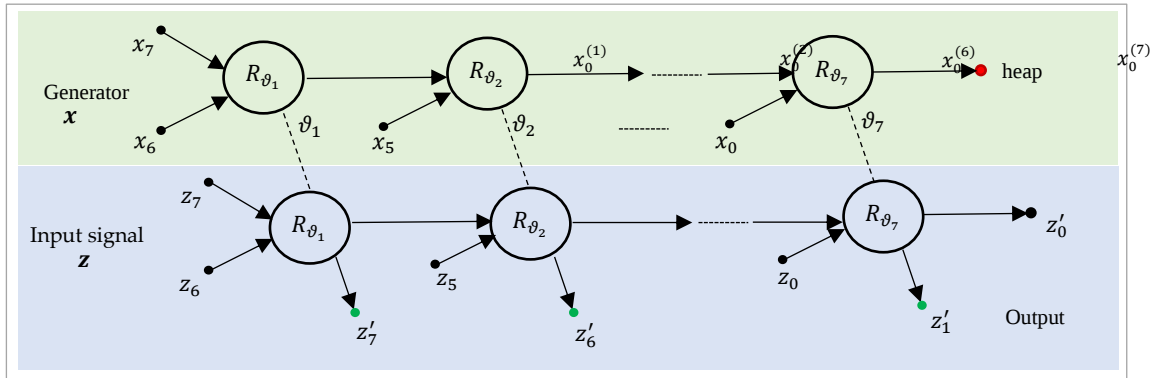


Figure 5. The 8-point DsiHT (with the 2-wheel strong carriage).

In the general case, the N -point DsiHT uses $(N - 1)$ basic transformations $T_{\vartheta_k}, k = 1: (N - 1)$, regardless of the path of the transform. The transform of the signal-generator equals

$$H: \mathbf{x} = (x_0, x_1, x_2, \dots, x_{N-1}) \rightarrow (y_0, 0, 0, \dots, 0), \quad (8)$$

where y_0 is equal to plus/minus energy $E[\mathbf{x}] = \|\mathbf{x}\|$ of the signal. We consider $y_0 = \sqrt{x_0^2 + x_1^2 + \dots + x_{N-1}^2}$; if it is negative, the last angle in rotation can be changed as $\vartheta_{N-1} = \vartheta_{N-1} + \pi$. Thus, the transformation, as well as the signal-generator, are uniquely determined by the data of $E[\mathbf{x}]$ and the angles of rotations. In other words, the transformation is decoded into the vector $(\|\mathbf{x}\|, \vartheta_1, \vartheta_2, \dots, \vartheta_{N-1})$. For the signal-generator with norm 1, the angular representation holds, $\mathbf{x} \rightarrow A_{\mathbf{x}} = \{\vartheta_1, \vartheta_2, \dots, \vartheta_{N-1}\}$.

The output is the transform of the input

$$H(\mathbf{z}) = (z'_0, z'_1, z'_2, \dots, z'_{N-2}, z'_{N-1}), \quad z'_0 = z_0^{(N-1)}. \quad (9)$$

Note that the above algorithm can be split by two parts. In the first part, all angles ϑ_k of rotation can be calculated from the signal-generator. This is about the angular representation of the generator, $A_{\mathbf{x}}$. In the second part, the DsiHT of the input signal $\mathbf{z}$ is calculated, $H(\mathbf{z})$.

We now describe the DsiHT in the matrix form. For $i = 1: (N - 1)$, we denote by $R_{\vartheta; (i, i+1)}$ the block-diagonal matrix with all 1 in the diagonal and the matrix of rotation R_{ϑ} in the cell $(i, i + 1) \times (i, i + 1)$, that is,

$$R_{\vartheta; (i, i+1)} = I_i \oplus R_{\vartheta} \oplus I_{N-i-2} \quad \text{and} \quad R_{\vartheta; (0,1)} = R_{\vartheta} \oplus I_{N-2}. \quad (10)$$

The matrix $R_{\vartheta; (i, i+1)}$ describes the rotation on neighbor bit planes i and $(i + 1)$. For instance, in the $N = 4$ case, consider the rotations

$$R_{\vartheta; (0,1)} = \begin{bmatrix} \cos \vartheta & -\sin \vartheta & 0 & 0 \\ \sin \vartheta & \cos \vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{\vartheta} & 0 \\ 0 & I_2 \end{bmatrix} = R_{\vartheta} \oplus I_2 \quad \text{and} \quad R_{R_{\vartheta; (1,2)}} = 1 \oplus R_{\vartheta} \oplus 1.$$

The bit planes 0 and 1, or 00 and 01, are adjacent planes, that is, the plane numbers differ by only one bit. The bit planes 1 and 2, or 01 and 10, are not adjacent planes.

In the matrix form, the DsiHT, H_N , is written as the multiplication of $(N - 1)$ block-diagonal matrices with rotations on adjacent planes,

$$H_N = R_{\vartheta_{N-1}; (0,1)} R_{\vartheta_{N-2}; (1,2)} R_{\vartheta_{N-3}; (2,3)} \cdots R_{\vartheta_{2; (N-3, N-2)}} R_{\vartheta_{1; (N-2, N-1)}}. \quad (11)$$

In the $N = 4$ case, the 4-point strong DsiHTs with three rotations is written as

$$H_4 = R_{\vartheta_3;(0,1)} R_{\vartheta_2;(1,2)} R_{\vartheta_1;(2,3)} = (R_{\vartheta_3} \oplus I_2)(1 \oplus R_{\vartheta_2} \oplus 1)(I_2 \oplus R_{\vartheta_1}). \quad (12)$$

Now, we consider an example with the 4-point DsiHT generated by a signal $\mathbf{x}$.

Example 1: Let $\mathbf{x}$ be the vector-signal $\mathbf{x} = (1,2,2,1)$ with energy $E[\mathbf{x}] = \sqrt{10}$ and let the input signal be $\mathbf{z} = (1,2,4,2)$ with energy $E[\mathbf{z}] = 5$. The DsiHT with the generator $\mathbf{x}$ has the following matrix:

$$H_4 = \begin{bmatrix} 0.3162 & 0.6325 & 0.6325 & 0.3162 \\ -0.9487 & 0.2108 & 0.2108 & 0.1054 \\ 0 & -0.7454 & 0.5963 & 0.2981 \\ 0 & 0 & -0.4472 & 0.8944 \end{bmatrix} = D \begin{bmatrix} 1 & 2 & 2 & 1 \\ -9 & 2 & 2 & 1 \\ 0 & -2.5 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}. \quad (13)$$

Here, the diagonal matrix $D = \{0.3162, 0.1054, 0.2981, 0.4472\}$. The angles for this transformation are $\{\vartheta_1, \vartheta_2, \vartheta_3\} = \{-26.5651^\circ, -48.1897^\circ, -71.5651^\circ\}$. One can note that the normalized vector-generator $(1,2,2,1)/\sqrt{10}$ lies in the first row of this matrix. The transformation is described by the following decomposition by three rotations:

$$H_4 = R_{\vartheta_3;(0,1)} R_{\vartheta_2;(1,2)} R_{\vartheta_1;(2,3)} = \begin{bmatrix} \cos \vartheta_3 & -\sin \vartheta_3 & 0 & 0 \\ \sin \vartheta_3 & \cos \vartheta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \vartheta_2 & -\sin \vartheta_2 & 0 \\ 0 & \sin \vartheta_2 & \cos \vartheta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \vartheta_1 & -\sin \vartheta_1 \\ 0 & 0 & \sin \vartheta_1 & \cos \vartheta_1 \end{bmatrix}.$$

$\vartheta_3 = -71.56^\circ, \vartheta_2 = -48.19^\circ, \vartheta_1 = -26.56^\circ$

Consider the permutations $P = (0,3,2,1)$ and $P' = (0,1,2,3)$ with the matrices

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

respectively. These two permutations describe the cyclic shifts of a 4-point vector. The following holds for the rotation on the first- and second-bit planes: $R_{\vartheta_2;(1,2)} = PR_{\vartheta_2;(2,3)}P'$, that is,

$$R_{\vartheta_2;(1,2)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \vartheta_2 & -\sin \vartheta_2 \\ 0 & 0 & \sin \vartheta_2 & \cos \vartheta_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, the matrix of the 4-point DsiHT is described by three controlled gates as

$$H_4 = (R_{\vartheta_3} \oplus I_2) P' (I_2 \oplus R_{\vartheta_2}) P' (I_2 \oplus R_{\vartheta_1}). \quad (14)$$

The transform of the generators is $H_4 \mathbf{x}' = (1,0,0,0)'$, and for the input signal, $H_4 \mathbf{z}' = (4.7434, 0.5270, 1.4907, 0)'$. The normalized vectors, $\mathbf{x} = \mathbf{x}'/\sqrt{10}$ and $\mathbf{z} = \mathbf{z}'/5$ can be considered as 2-qubit superpositions

$$|\mathbf{x}\rangle = (|00\rangle + 2|01\rangle + 2|10\rangle + |11\rangle)/\sqrt{10}, \text{ and}$$

$$|\mathbf{z}\rangle = (|00\rangle + 2|01\rangle + 4|10\rangle + 2|11\rangle)/5.$$

The 2-qubit QsiHT of these superpositions are $H_4|\mathbf{x}\rangle = |00\rangle$ and

$$H_4|\mathbf{z}\rangle = (4.7434|00\rangle + 0.5270|01\rangle + 1.4907|10\rangle + 0|11\rangle)/5.$$

The quantum scheme of the 2-qubit quantum QsiHT, is given in Figure 6. The flowchart sorts denote the permutations.

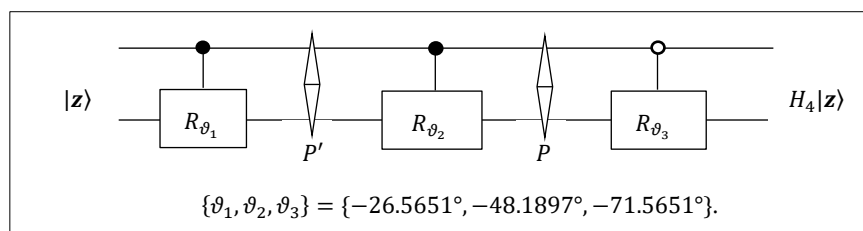


Figure 6. The quantum scheme of the 2-qubit QsiHT with the generator (1,2,2,1).

IBM's Qiskit framework is an open-source library in Python for quantum computing developed by IBM [33,34]. It provides tools for building quantum circuits, simulating their behavior, and running them on actual quantum hardware through IBM's backend. IBM's Qiskit framework was used to construct and simulate the 2-qubit QsiHT quantum circuit designed to prepare the normalized quantum state corresponding to the vector $x = (1,2,2,1)'$. The circuit was built using the decomposition of rotation angles through the QsiHT to encode the target amplitudes and its inverse was applied to map the target state onto a computational basis. The simulation involved extracting both the ideal state vector amplitudes and the corresponding measurement probabilities. These were compared against shot-based measurements from simulated runs to assess the accuracy of the circuit. The mean-square root errors (MSRE) between the expected and observed results were computed to quantify the performance of the state preparation. The results of simulation of this circuit in Qiskit are shown in Table 1.

Table 1. The measured magnitudes for 2-qubit superposition $H_4|z\rangle$ by the 2-qubit QsiHT.

Basis States	Magnitudes				
	Theoretical	500 shots	1,000 shots	10,000 shots	100,000 shots
00	0.9487	0.9497	0.9602	0.9479	0.9494
01	0.1054	0.1095	0.0837	0.1034	0.1045
10	0.2981	0.2933	0.2665	0.3011	0.2962
MSRE		1.21×10^{-3}	9.55×10^{-3}	7.75×10^{-4}	5.96×10^{-4}

5. Qubits Initiation by the DsiHT

In this section, we consider the inverse DsiHT and describe a simple quantum scheme for initiation of a multi-qubit superposition. Then in Section 6 we show how important it is to choose such a path to make this scheme more efficient.

Let x be the 2^r -D unit vector $x = (x_0, x_1, \dots, x_{2^r-1})$ which represents the r -qubit superposition

$$|x\rangle = x_0|0\rangle + x_1|1\rangle + \dots + x_{2^r-1}|2^r - 1\rangle, \quad |x_0|^2 + |x_1|^2 + \dots + |x_{2^r-1}|^2 = 1,$$

Here, the components of the vector are real and complex, and $\{|k\rangle, k = 0:2^r - 1\}$ is the conventional basis of states. We consider the case when x is real. For the DsiHT generated by this vector, we have

$$Hx' = H(x_0, x_1, \dots, x_{2^r-1})' = (1, 0, \dots, 0)' = |0\rangle^{\otimes r}. \quad (15)$$

In Example 1, such DsiHT is described for the $N = 4$ case, or the 2-qubit superposition $|x\rangle = (|0\rangle + 2|1\rangle + 2|2\rangle + |3\rangle)/\sqrt{10}$. This unitary transformation $H_4 = H$ is real. Therefore, its inverse transformation is defined by the matrix being transpose to H_4 , that is, $H_4^{-1} = H_4'$. We obtain $H_4'|00\rangle = |x\rangle$. Using the property of the inverse operation (transposition), the matrix H_4' can be composed by the matrices of rotation by angles $\{-\vartheta_3, -\vartheta_2, -\vartheta_1\}$ as follows:

$$\begin{aligned} H_4' &= [R_{\vartheta_3;(0,1)} R_{\vartheta_2;(1,2)} R_{\vartheta_1;(2,3)}]' = [R_{\vartheta_1;(2,3)}]' [R_{\vartheta_2;(1,2)}]' [R_{\vartheta_3;(0,1)}]' = R_{-\vartheta_1;(2,3)} R_{-\vartheta_2;(1,2)} R_{-\vartheta_3;(0,1)} \\ &= (I_2 \oplus R_{-\vartheta_1})(P(I_2 \oplus R_{-\vartheta_2})P')(R_{-\vartheta_3} \oplus I_2). \end{aligned}$$

To initiate the 2-qubit superposition $|x\rangle$ from the first basis state $|00\rangle$, the quantum circuit of the inverse 2-qubit DsiHT shown in Figure 7 can be used. The angular representation of the signal x (or the superposition $|x\rangle$) thus allows this superposition to be initiated by three rotations.

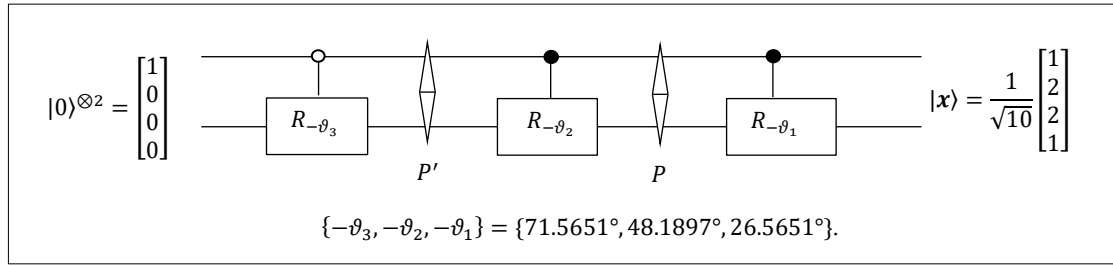


Figure 7. The quantum scheme for initiation of the 2-qubit superposition $|x\rangle$ by the 2-qubit inverse QsiHT.

In the general case of $r > 1$, the unitary matrix of the DsiHT generated by the real signal is real. Therefore, the inverse DsiHT is described by the matrix transpose to the matrix of the DsiHT and

$$H^{-1}(|0\rangle^{\otimes r}) = H'(1, 0, \dots, 0)' = (x_0, x_1, \dots, x_{2^r-1})' = \mathbf{x}' = |\mathbf{x}\rangle.$$

Example 2: Consider the 8-D unit vector $\mathbf{x} = (1, 1, 2, -2, -1, 3, 1, 2)/5$ and the corresponding 3-qubit superposition

$$|\mathbf{x}\rangle = \frac{1}{5}(1|000\rangle + 1|001\rangle + 2|010\rangle - 2|011\rangle - 1|100\rangle + 3|101\rangle + 1|110\rangle + 2|111\rangle).$$

For the 8-point strong DsiHT with this generator, the transform matrix can be written as

$$H_8 = \left(\prod_{i=0}^6 R_{\vartheta_{7-i;(i,i+1)}} \right) = (R_{\vartheta_{7;(0,1)}} R_{\vartheta_{6;(1,2)}} R_{\vartheta_{5;(2,3)}} R_{\vartheta_{4;(3,4)}} R_{\vartheta_{3;(4,5)}} R_{\vartheta_{2;(5,6)}} R_{\vartheta_{1;(6,7)}}). \quad (16)$$

Seven rotations by the following angles (in degrees) are used:

$$\{\vartheta_k; k = 1:7\} = \{-78.46^\circ, -78.22^\circ, 65.35^\circ, -62.68^\circ, 75.03^\circ, -36.69^\circ, -63.43^\circ\}. \quad (17)$$

The transform in the matrix form can be written as

$$H_8 \mathbf{x}' = D \begin{bmatrix} 1 & 1 & 2 & -2 & -1 & 3 & 1 & 2 \\ -24 & 1 & 2 & -2 & -1 & 3 & 1 & 2 \\ 0 & -23 & 2 & -2 & -1 & 3 & 1 & 2 \\ 0 & 0 & -9.5 & -2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 7.5 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 14 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -5/3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix} \times \frac{1}{5} \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |000\rangle. \quad (18)$$

Here, the diagonal matrix

$$D = \text{diag}\{0.2, 0.0408, 0.0426, -0.0957, -0.1185, 0.0690, 0.3586, 0.4472\}.$$

The matrix of the inverse 8-point DsiHT is calculated by seven rotations in reverse order, as

$$H'_8 = R_{-\vartheta_{1;(6,7)}} R_{-\vartheta_{2;(5,6)}} R_{-\vartheta_{3;(4,5)}} R_{-\vartheta_{4;(3,4)}} R_{-\vartheta_{5;(2,3)}} R_{-\vartheta_{6;(1,2)}} R_{-\vartheta_{7;(0,1)}}.$$

The inverse transform of the first basis state $|000\rangle$ can be written with the diagonal matrix D as

$$H'_8 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -23 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & -9.5 & 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & -2 & 7.5 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 14 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 & 3 & -5/3 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \end{bmatrix} D \times \frac{1}{5} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} = |\mathbf{x}\rangle. \quad (19)$$

This diagonal matrix D was introduced to better illustrate the process of constructing this matrix H_8 . The first basis function of this transformation is the generator $\mathbf{x}$ itself (the first column of the matrix). Each next basis function is result of motion of the previous basis functions (for more detail, see [31]).

Any r -qubit superposition $|\mathbf{x}\rangle$ can be obtained from the zero state $|0\rangle^{\otimes r}$ by the unitary inverse heap transform H'_{2^r} generated by $\mathbf{x}$. The DsiHT requires only $(2^r - 1)$ rotations and it is defined by the angular representation of $\mathbf{x}$. Not all of these Givens rotations act on adjacent bit planes, what is defined by the path of the transformation. However, there are paths that allow us to perform all calculations only on adjacent bit planes.

6. Path in the DsiHT

In this section, we discuss the path which is an important characteristic of the DsiHT. The path, or the order in which components of both generator and input signal are processed, can be chosen in different ways (for more detail, see [17]). Two more example paths are shown in Figure 8. They are paths with partitioning into pairs. The path shown in part (a) resembles the path that is used in the 8-point fast Fourier transform (FFT) with decimation-in-frequency [35]. Only the 8-point DsiHT uses 7 rotations (as butterflies) rather than $4 \cdot 3 = 12$ butterflies in the FFT. These rotations T_k are denoted by the circles with numbers k in the figure. The angles ϑ_k of these rotations' changes when the path of the DsiHT changes. We will call the path in part (a) fast path #1 (or simply fast path) and fast path #2 is the path in part (b). The small open circles in these diagrams are used to indicate that these transform outputs are zero when the input is the generator $\mathbf{x}$.

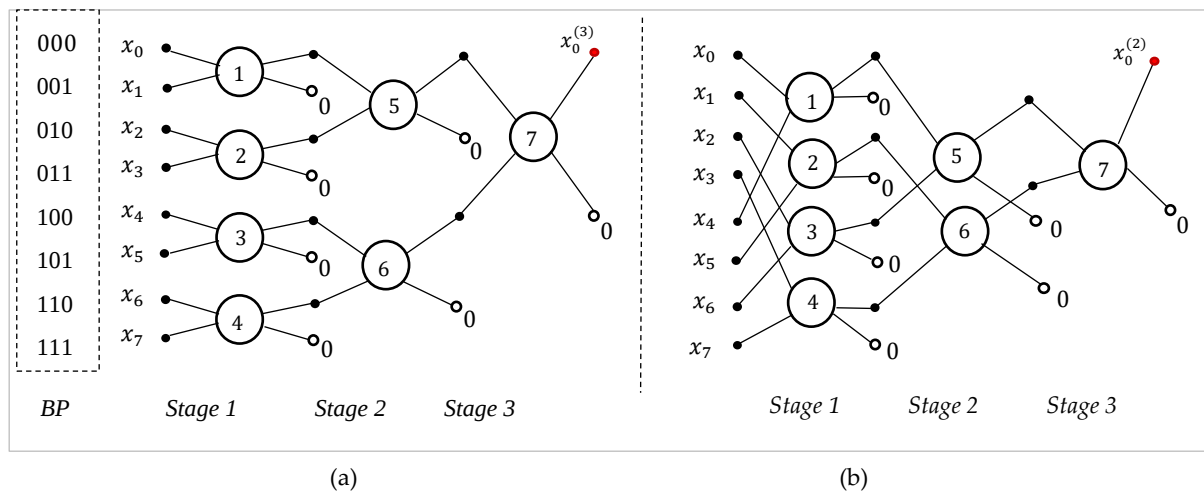


Figure 8. The 8-point DsiHTs using two paths with splitting into pairs.

The path can be defined in such a way that the computation complexity of the transformation and its inverse will be minimized. For instance, it is possible to define the path that allows for simplification of quantum circuits of the r -qubit QsiHT with only controlled gates of rotation. No permutations are required. The representation of the QsiHT matrix is still in the form shown in Eq. 11, only the order of processing the input data and therefore the angles of rotations must be changed accordingly. The r -qubit QsiHT always requires maximum $(2^r - 1)$ rotations on different bit-planes (some of 2×2 -rotations may be trivial). In the case of the strong QsiHT, for many rotations, these bit planes do not differ by only one bit and require additional permutations, as shown in the $R_{\vartheta_2;1,2}$ rotation example above. Other paths require rotations in other planes. Therefore, it is necessary to find better ways to simplify the calculations [36]. To illustrate this fact, we first consider the fast path which is shown in Figure 8(a), for the 8-point DsiHT, or 3-qubit QsiHT. Then, the example for 2-qubit QsiHT is presented and compared with Example 1.

The notation $R_{\vartheta;(i,j)}$ will be used for the rotation R_{ϑ} by the angle ϑ in planes number i and j , where integer $j \neq i \in \{0, 1, \dots, r-1\}$. The 3-qubit DsiHT with the fast path is calculated by seven

rotations in the following way. On stage 1, four rotations are used, as shown below with the corresponding bit-planes

$$R_{\theta_1;(0,1)} \rightarrow \begin{pmatrix} 000 \\ 001 \end{pmatrix}, \quad R_{\theta_2;(2,3)} \rightarrow \begin{pmatrix} 010 \\ 011 \end{pmatrix}, \quad R_{\theta_3;(4,5)} \rightarrow \begin{pmatrix} 100 \\ 101 \end{pmatrix}, \quad R_{\theta_4;(6,7)} \rightarrow \begin{pmatrix} 110 \\ 111 \end{pmatrix}. \quad (20)$$

The circuit elements for these controlled rotation gates are shown on Figure 9.

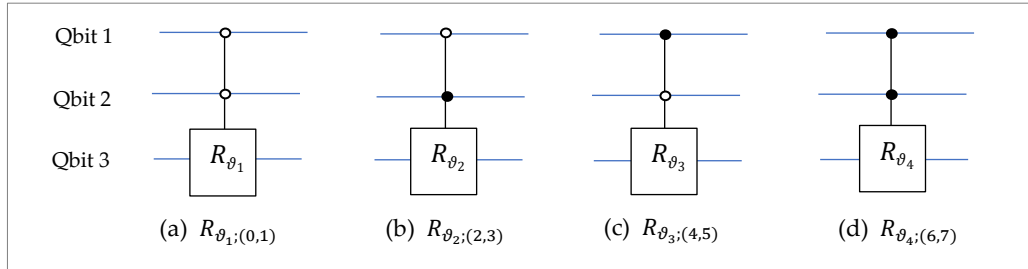


Figure 9. Four 3-qubit rotation gates with two control bits.

The second stage uses two rotations, and the last stage uses one rotation, as shown below

$$R_{\theta_5;(0,2)} \rightarrow \begin{pmatrix} 000 \\ 010 \end{pmatrix}, \quad R_{\theta_6;(4,6)} \rightarrow \begin{pmatrix} 100 \\ 110 \end{pmatrix}, \quad \text{and} \quad R_{\theta_7;(0,4)} \rightarrow \begin{pmatrix} 000 \\ 100 \end{pmatrix}. \quad (21)$$

The circuit elements of these rotation gates are shown in Figure 10.

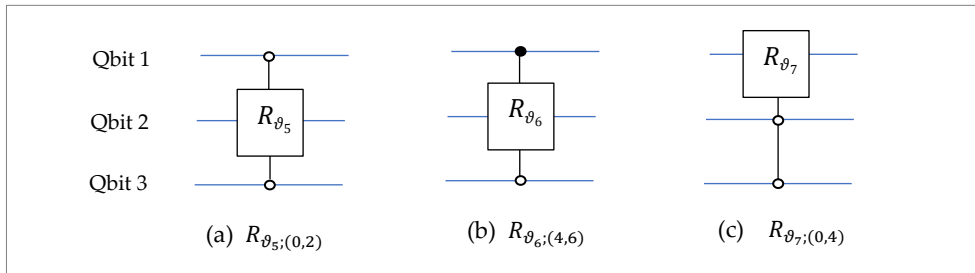


Figure 10. Three 3-qubit gates with two control qubits in the second stage.

Summarizing the above reasonings, we obtain the quantum circuit for the 3-qubit QsiHT with the fast path which is shown in Figure 11. Seven controlled gates of rotations with 2 control qubits are used: no additional permutations. All rotations operate on adjacent bit planes, that is, on bit planes that differ by only one bit. Such bit planes are also called adjacent qubits.

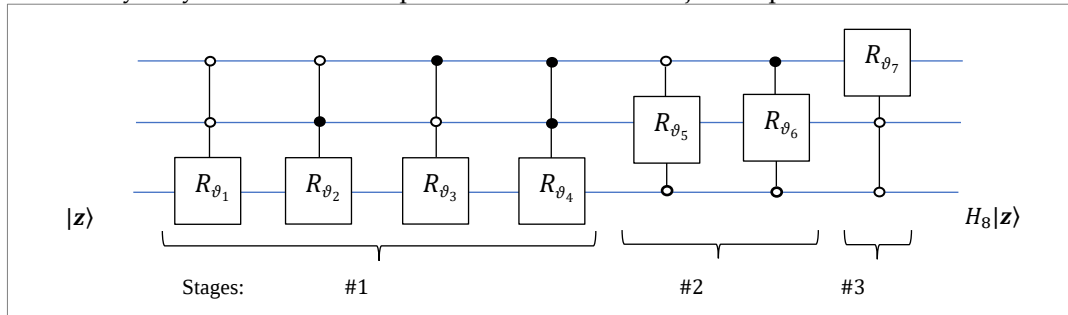


Figure 11. The circuit for the 3-qubit QsiHT with seven controlled rotation gates.

Based on this scheme, the circuit for the inverse 3-qubit QsiHT can be easily composed and it is shown in Figure 12. The signs of all angles change, and seven controlled gates of rotation are executed in reverse order. Here, the input is a 3-qubit superposition $|\mathbf{y}\rangle$ and the output is $H'_8|\mathbf{y}\rangle$. In the case when the input is the first basis state, $|\mathbf{y}\rangle = |000\rangle$, the output of the circuit is the 3-qubit

superposition, $|\mathbf{x}\rangle$. Thus, to initiate a 3-qubit superposition maximum of seven rotation gates are only required.

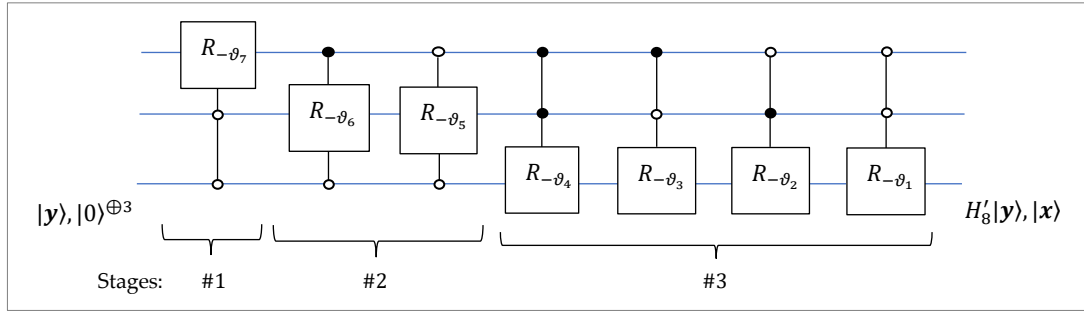


Figure 12. The circuit for the inverse 3-qubit QsiHT with seven controlled rotation gates.

Now, let us analyze this circuit and compare it with the circuit of the strong 3-qubit DsiHT. **Example 3:** Consider the quantum circuit of the 3-qubit QsiHT with the fast path, which is given in Figure 13. The generator $\mathbf{x} = (1, 1, 2, -2, -1, 3, 1, 2)$ and its 3-qubit $|\mathbf{x}\rangle$ is the same as in Example 2. The angular representation of the generator with the fast path equals to

$$A_{\mathbf{x}} = \{\vartheta_1, \vartheta_2, \dots, \vartheta_7\} = \{-45^\circ, 45^\circ, 71.5651^\circ, -63.4349^\circ, -63.4349^\circ, 35.2644^\circ, 50.7685^\circ\}.$$

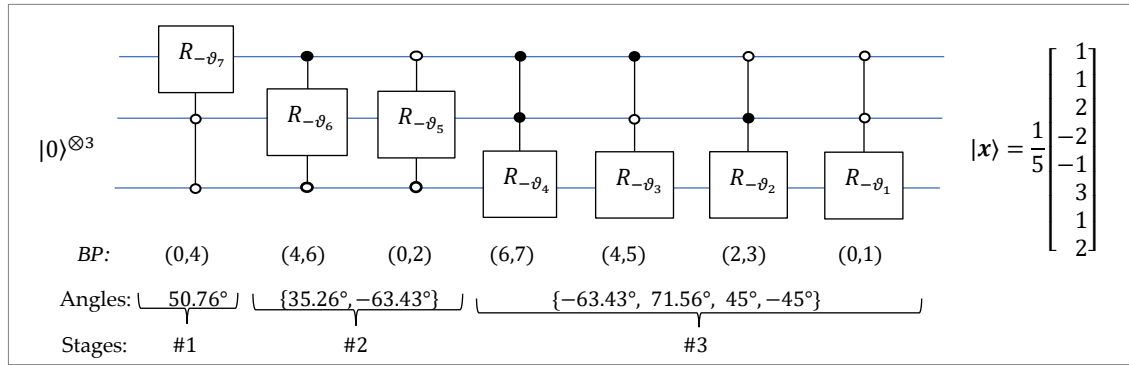


Figure 13. The circuit for the initiation of the 3-qubit state $|\mathbf{x}\rangle$.

The matrix of this transformation can be written as

$$H_{8;\text{fast}} = D \begin{bmatrix} 1 & 1 & 2 & -2 & -1 & 3 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 3 & -3 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix} \quad (22)$$

where $D = \text{diag}\{0.2, 0.7071, 0.3162, 0.3536, 0.1633, 0.3162, 0.1826, 0.4472\}$. This matrix has 32 zero coefficients compared to 21 zeros in the matrix H_8 for the strong DsiHT, which is given in Eq. 18.

Now we describe the quantum circuit for the 3-qubit strong DsiHT. This transform is composed of seven rotations

$$H_8 = R_{\vartheta_{7;(0,1)}} R_{\vartheta_{6;(1,2)}} R_{\vartheta_{5;(2,3)}} R_{\vartheta_{4;(3,4)}} R_{\vartheta_{3;(4,5)}} R_{\vartheta_{2;(5,6)}} R_{\vartheta_{1;(6,7)}}.$$

The angles $\vartheta_k; k = 1:7$ are from the set in Eq. 17. The block-diagram of the transform is shown in Figure 14.

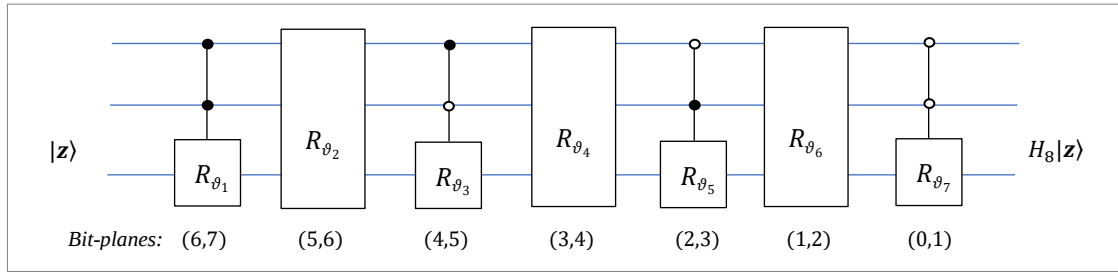


Figure 14. The circuit for the 3-qubit QsiHT with seven rotations.

Four matrices from the above list, namely $R_{\vartheta_7;(0,1)}$, $R_{\vartheta_5;(2,3)}$, $R_{\vartheta_3;(4,5)}$ and $R_{\vartheta_1;(6,7)}$ are described by the controlled rotation gates similar to those shown in Figure 9, only with different angles $\vartheta_7, \vartheta_5, \vartheta_3$, and ϑ_1 in parts (a)-(d), respectively. Three rotations by the angles ϑ_2, ϑ_4 , and ϑ_6 operate on the nonadjacent bit-planes 5 and 6, 3 and 4, and 1 and 2, respectively. For example, in binary representation, the numbers $5 = (101)$ and $6 = (110)$ differ by two bits. Therefore, these three rotations should be fulfilled by the rotation gates on nearest-neighbor bit-planes. The circuit elements for other three rotation gates $R_{\vartheta_2;(5,6)}$, $R_{\vartheta_4;(3,4)}$, and $R_{\vartheta_6;(1,2)}$ can be described as follows.

- 1) By the permutation (4,5), the matrix of the rotation on bit-planes 5 and 6 is presented as

$$R_{\vartheta_2;(5,6)} = \begin{bmatrix} I_4 & & \\ & X & \\ & & I_2 \end{bmatrix} R_{\vartheta_2;(4,6)} \begin{bmatrix} I_4 & & \\ & X & \\ & & I_2 \end{bmatrix}. \quad (23)$$

The permutation (4,5) is the controlled NOT, that is, $I_4 \oplus X \oplus I_2$. The controlled gate for the rotation on bit-planes 4 and 6 is shown in Figure 10 in part (b). Therefore, the operation $R_{\vartheta_2;(5,6)}$ can be fulfilled by using the rotation gate $R_{\vartheta_2;(4,6)}$. The circuit element of the operation in Eq. 19 is shown in Figure 15.

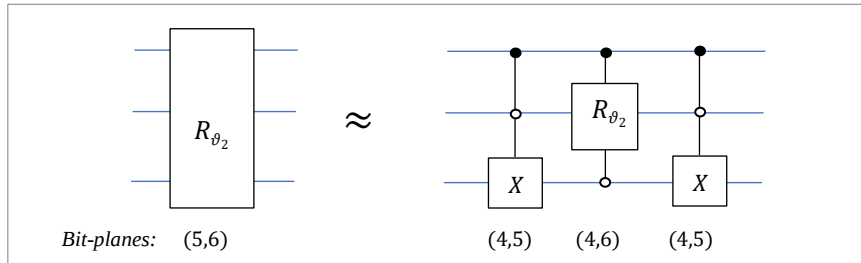


Figure 15. The circuit for the 3-qubit rotation on the bit-planes 5 and 6 with two controlled NOTs.

- 2) By two permutations (2,3) and (4,6), the matrix of the rotation on bit-planes 3 and 4 is presented as

$$R_{\vartheta_4;(3,4)} = \begin{bmatrix} I_2 & & \\ & X & \\ & & P_{0,2} \end{bmatrix} R_{\vartheta_4;(2,6)} \begin{bmatrix} I_2 & & \\ & X & \\ & & P_{0,2} \end{bmatrix}, \quad P_{0,2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (24)$$

Here, $P_{0,2}$ is the permutation (0,2). The circuit element of this rotation is shown in Figure 16. The operation is reduced to the rotation gate on the adjacent bit-planes $2 = (010)$ and $6 = (110)$.

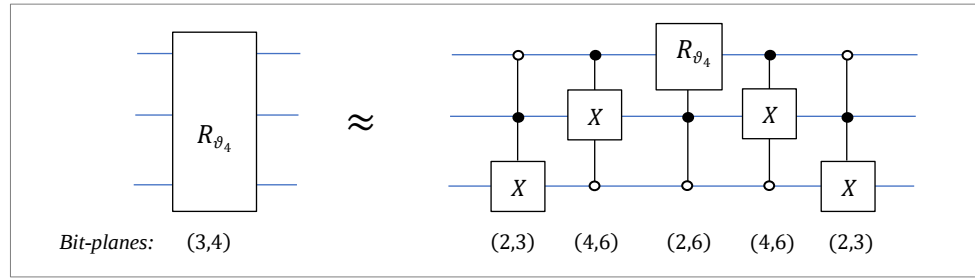


Figure 16. The circuit for the 3-qubit rotation on the bit-planes 3 and 4 with four controlled NOTs.

3) By the permutation $(0,1)$, the matrix of the rotation on bit-planes 1 and 2 is presented as

$$R_{\theta_{6;(1,2)}} = \begin{bmatrix} X & & \\ & I_2 & \\ & & I_4 \end{bmatrix} R_{\theta_{6;(0,2)}} \begin{bmatrix} X & & \\ & I_2 & \\ & & I_4 \end{bmatrix}. \quad (25)$$

This operation is fulfilled by using the rotation gate $R_{\theta_{6;(0,2)}}$. This controlled gate of the rotation on adjacent bit-planes 0 and 2 is shown in Figure 10 in part (a). The permutation $(0,1)$ is the controlled NOT, $X \oplus I_2 \oplus I_4$. The circuit element of the operation in Eq. 25 is shown in Figure 17.

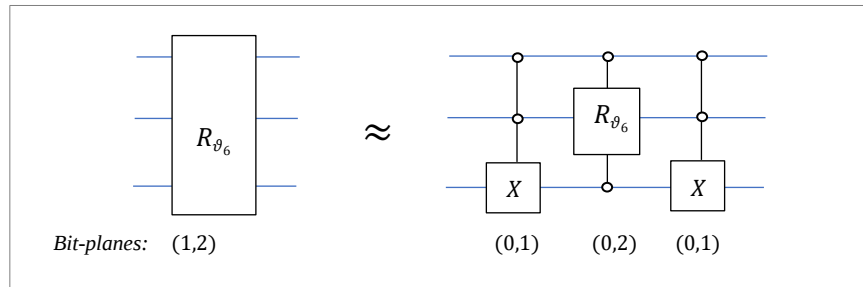


Figure 17. The circuit for the 3-qubit rotation on the bit-planes 1 and 2 with two controlled NOTs.

The entire quantum circuit for the 3-qubit QsiHT is shown on Figure 18. Together with seven controlled rotation gates, eight controlled NOTs are used. All these gates are controlled by two bits (qubits).

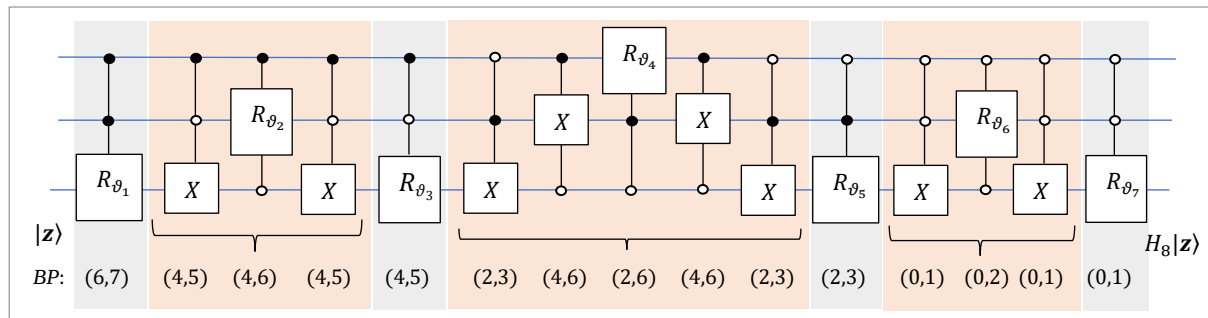


Figure 18. The circuit for the 3-qubit strong QsiHT.

The circuit for the inverse 3-qubit strong QsiHT is shown in Figure 19. Here, the rotation matrix R'_{θ_k} denotes the inverse matrix to R_{θ_k} , that is, $R'_{\theta_k} = R_{-\theta_k}$ for $k = 1:7$. There is some symmetry in this scheme (as well as in Figure 18). The change of each controlled NOT from one rotation operation to the next one is accomplished by adding $(2,2)$ to the bit-plane numbers. Indeed, these changes are $(0,1) \rightarrow (2,3) \rightarrow (4,5) \rightarrow (6,7)$. Quantum circuits for r -qubit strong QsiHT can be described similarly for the number of qubits $r \geq 4$.

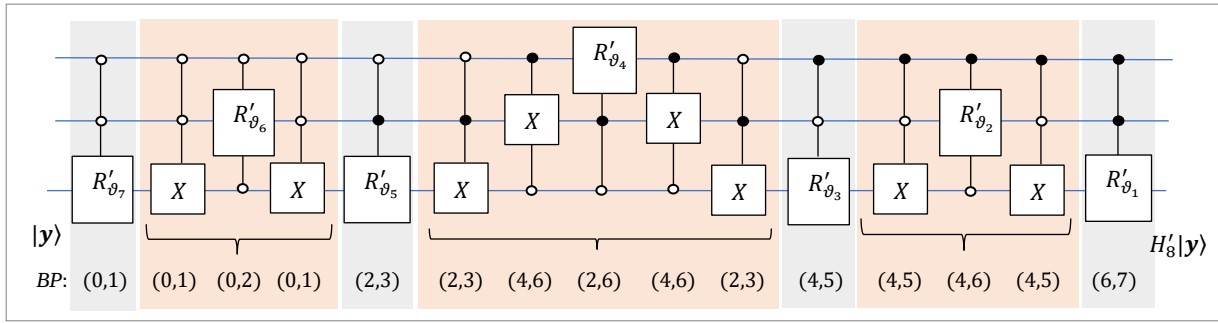


Figure 19. The circuit for the inverse 3-qubit strong QsiHT.

The above comparison of 3-qubit QsiHTs shows how important it is to choose a path that leads to a simplified circuit. The fast path for the 3-qubit QsiHT allows the effective quantum circuit to build. This circuit is without permutations. The results of simulation of this circuit in Qiskit for the 3-qubit superposition $|x\rangle$ in Example 3 are shown in Table 2. For comparison, the results of using the circuit in Figure 13 with the QsiHT by the fast path are also given in this table. One can note from this table that the DsiHT with the fast path improves (reduces) the computational errors.

Table 2. The measured magnitudes of the 3-qubit state $|x\rangle$ by the 3-qubit QsiHTs.

Basis States	Magnitudes				
	Theoretical	Inverse strong QsiHT		Inverse fast QsiHT	
		10,000 shots	100,000 shots	10,000 shots	100,000 shots
000	0.2000	0.1969	0.2005	0.2004	0.1989
001	0.2000	0.1978	0.1995	0.2008	0.2019
...	...	...	...	...	...
110	0.2000	0.2012	0.2007	0.2003	0.1995
111	0.4000	0.4012	0.3986	0.3982	0.3994
MSRE		4.24×10^{-4}	2.56×10^{-4}	4.04×10^{-4}	1.92×10^{-4}

Example 4: Now we consider the 4-point DsiHT and its inverse transform with the fast path shown in Figure 20. This is the two-stage path with partitioning into 3 pairs.

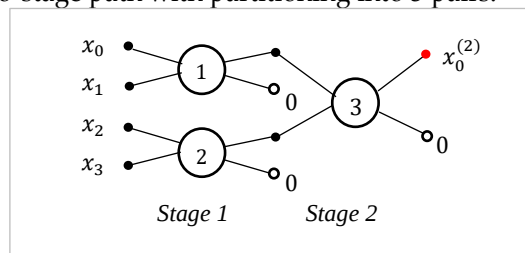


Figure 20. The 4-point DsiHT using the path with splitting into pairs.

On these two stages of calculation, the rotations are described as

$$R_{\theta_1;(0,1)} \rightarrow \begin{pmatrix} 00 \\ 01 \end{pmatrix}, \quad R_{\theta_2;(2,3)} \rightarrow \begin{pmatrix} 10 \\ 11 \end{pmatrix}, \quad \text{and} \quad R_{\theta_3;(0,2)} \rightarrow \begin{pmatrix} 00 \\ 10 \end{pmatrix}. \quad (26)$$

The corresponding circuit elements of these controlled rotation gates are shown in Figure 21. Two gates are with 0-control bit and the gate for the second rotation is with 1-control bit. Connected together these gates compose the 2-qubit QsiHT.

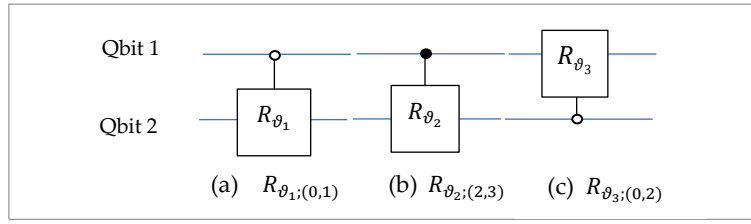


Figure 21. The 2-qubit gates with control bit 0 or 1.

It directly follows from this circuit that the inverse 2-qubit DsiHT with the fast path is described by the circuit given in Figure 22. Thus, this circuit can be used to initiate the 2-qubit superposition $|\mathbf{x}\rangle$ from the basis state $|00\rangle$.

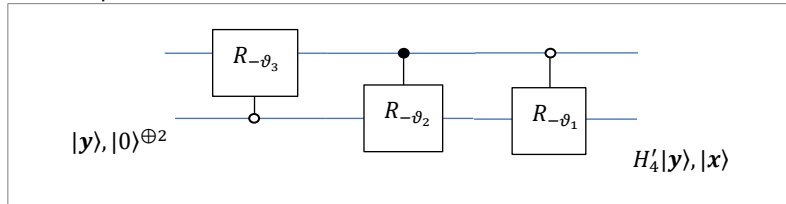


Figure 22. The circuit for the inverse 2-qubit QsiHT with 3 controlled rotation gates.

Compared with the quantum scheme in Figure 7, the above circuit does not require permutations, which makes this circuit more effective. The angular representations $A_{\mathbf{x}}$ of the same signal $\mathbf{x}$ are different, due to their different paths. For the fast path, this representation is $A_{\mathbf{x}} = \{\vartheta_1, \vartheta_2, \vartheta_3\} = \{-63.4349^\circ, -26.5651^\circ, -45^\circ\}$. The matrix of the DsiHT is equal to

$$H_{4;\text{fast}} = \begin{bmatrix} 0.3162 & 0.6325 & 0.6325 & 0.3162 \\ -0.8944 & 0.4472 & 0 & 0 \\ -0.3162 & -0.6325 & 0.6325 & 0.3162 \\ 0 & 0 & -0.4472 & 0.8944 \end{bmatrix} = D \begin{bmatrix} 1 & 2 & 2 & 1 \\ -2 & 1 & 0 & 0 \\ -1 & -2 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad (27)$$

where the diagonal matrix $D = \text{diag}\{0.3162, 0.4472, 0.3162, 0.4472\}$. (from: example3_paper.m)

The new circuit for calculating this 2-qubit superposition $|\mathbf{x}\rangle = (|0\rangle + 2|1\rangle + 2|2\rangle + |3\rangle)/\sqrt{10}$ is given in Figure 23. The rotation by $-\vartheta_3 = 45^\circ$ is $R_{45} = 1/\sqrt{2}[1 \ -1; 1 \ 1]$. Also, note that the 2-qubit $|\mathbf{x}\rangle$ is not the tensor product of qubits, that is, $|\mathbf{x}\rangle$ is an entangled superposition.

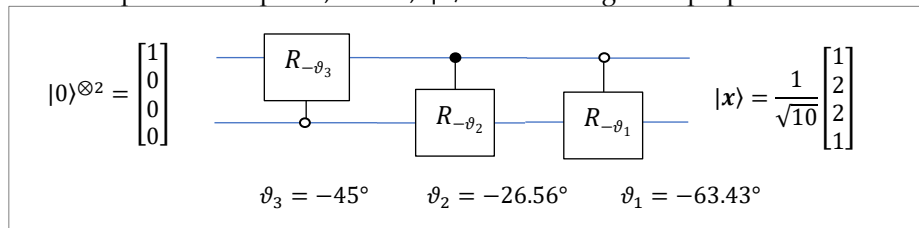


Figure 23. The circuit for the initiation of the 2-qubit state $|\mathbf{x}\rangle = (1,2,2,1)'/\sqrt{10}$.

The results of simulation of this circuit in Qiskit are shown in Table 3. For comparison, the results of using the inverse QsiHT with strong path (with the circuit in Figure 7) are also given in this table.

Table 3. Qiskit measured magnitudes for 2-qubit state $|\mathbf{x}\rangle$ by the 2-qubit QsiHTs.

Basis States	Magnitudes of $ \mathbf{y}\rangle$				
	Theoretical	Inverse strong QsiHT		Inverse fast QsiHT	
		10,000 shots	100,000 shots	10,000 shots	100,000 shots
00	0.3162	0.3211	0.3169	0.3135	0.3188
01	0.6325	0.6288	0.6309	0.6356	0.6320
10	0.6325	0.6333	0.6337	0.6298	0.6315

11	0.3162	0.3167	0.3163	0.3178	0.3164
MSRE		1.42×10^{-3}	6.42×10^{-4}	1.39×10^{-3}	5.31×10^{-4}

We can also consider the transformation of two 2-qubits superpositions $|y\rangle$ to $|x\rangle$, by using two 2-qubit QsiHT.

Example 5: Consider the same 2-qubit $|x\rangle = (|0\rangle + 2|1\rangle + 2|2\rangle + |3\rangle)/\sqrt{10}$. Let the second superposition be $|y\rangle = (1, 3, -4, 1)/\sqrt{27}$. Denoting by the H_y and H_x the 2-qubit QsiHTs generated by $|y\rangle$ and $|x\rangle$, respectively, we obtain the 2-qubit $|x\rangle$, by two steps: 1) $H_y|y\rangle = |00\rangle$, and 2) $H'_x|00\rangle = |x\rangle$. Calculating the angular representation of $|y\rangle$, $A_y = \{\theta_1, \theta_2, \theta_3\} = \{-71.5651^\circ, 14.0362^\circ, 52.5131^\circ\}$, we obtain

$$|x\rangle = H'_x H_y |y\rangle = \underbrace{R_{-\vartheta_1;(0,1)} R_{-\vartheta_2;(2,3)} R_{-\vartheta_3;(0,2)}}_{R_{\theta_3-\vartheta_3}} \underbrace{R_{\vartheta_3;(0,2)} R_{\vartheta_1;(2,3)} R_{\vartheta_1;(0,1)}}_{R_{\theta_1}} |y\rangle, \quad (28)$$

or

$$|x\rangle = \underbrace{R_{-\vartheta_1;(0,1)} R_{-\vartheta_2;(2,3)} R_{\vartheta_3-\vartheta_3;(0,2)}}_{R_{\theta_3-\vartheta_3}} \underbrace{R_{\vartheta_1;(2,3)} R_{\vartheta_1;(0,1)}}_{R_{\theta_1}} |y\rangle. \quad (29)$$

The quantum circuit for this 2-qubit transform with five controlled rotation gates is given in Figure 24.

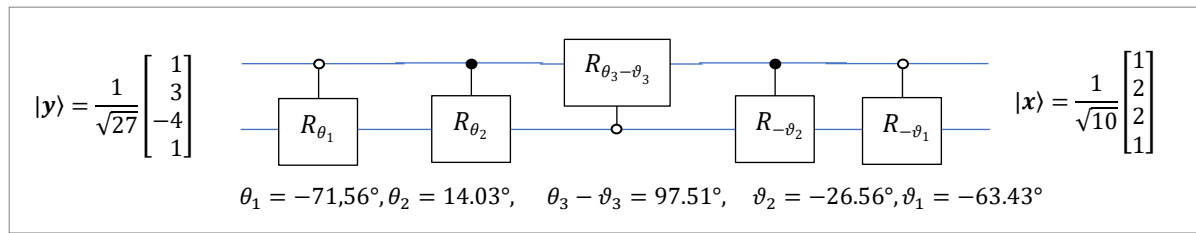


Figure 24. The circuit for the initiation of the 2-qubit state $|x\rangle$ from $|y\rangle$.

From the circuits in Figures 11,12,18,19, and 24, it is not difficult to notice a recursive structure of the direct and inverse QsiHTs with the fast path and construct the quantum circuits, to initiate any r -qubit superposition by $(2^r - 1)$ rotations, for $r \geq 2$. To show this, we describe the 4-qubit QsiHT. **Example 6:** Let $|x\rangle$ be the 4-qubit superposition $x_0|0\rangle + x_1|1\rangle + \dots + x_{15}|15\rangle$ with real amplitudes $x_k, k = 0:15$. The 4-qubit QsiHT by this generator $|x\rangle$ can be calculated by the quantum circuit given in Figure 25. Fifteen angles are used; the angles of the angular representation of the generator, $A_x = \{\vartheta_k, k = 1:15\}$. To simplify the notations in this circuit, all rotations are denoted by $R_k = R_{\vartheta_k}$. Four stages of calculations are shown with a total of 15 gates with 3 control bits. On the first stage, eight rotation gates and the corresponding bit-planes are

$$\begin{aligned} R_{\vartheta_1;(0,1)} &\rightarrow \begin{pmatrix} 0000 \\ 0001 \end{pmatrix}, & R_{\vartheta_2;(2,3)} &\rightarrow \begin{pmatrix} 0010 \\ 0011 \end{pmatrix}, & R_{\vartheta_3;(4,5)} &\rightarrow \begin{pmatrix} 0100 \\ 0101 \end{pmatrix}, \\ R_{\vartheta_4;(6,7)} &\rightarrow \begin{pmatrix} 0110 \\ 0111 \end{pmatrix}, \\ R_{\vartheta_5;(8,9)} &\rightarrow \begin{pmatrix} 1000 \\ 1001 \end{pmatrix}, & R_{\vartheta_6;(10,11)} &\rightarrow \begin{pmatrix} 1010 \\ 1011 \end{pmatrix}, & R_{\vartheta_7;(12,13)} &\rightarrow \begin{pmatrix} 1100 \\ 1101 \end{pmatrix}, \\ R_{\vartheta_8;(14,15)} &\rightarrow \begin{pmatrix} 1110 \\ 1111 \end{pmatrix}. \end{aligned}$$

On the 2nd stage, 4 rotation gates have the last control bit 0,

$$\begin{aligned} R_{\vartheta_9;(0,2)} &\rightarrow \begin{pmatrix} 0000 \\ 0010 \end{pmatrix}, & R_{\vartheta_{10};(4,6)} &\rightarrow \begin{pmatrix} 0100 \\ 0110 \end{pmatrix}, & R_{\vartheta_{11};(8,10)} &\rightarrow \begin{pmatrix} 1000 \\ 1010 \end{pmatrix}, \\ R_{\vartheta_{12};(12,14)} &\rightarrow \begin{pmatrix} 1100 \\ 1110 \end{pmatrix}. \end{aligned}$$

Two rotation gates with the last two 0-control bits are used on stage 3, and the last rotation with three control bits 0 on stage 4,

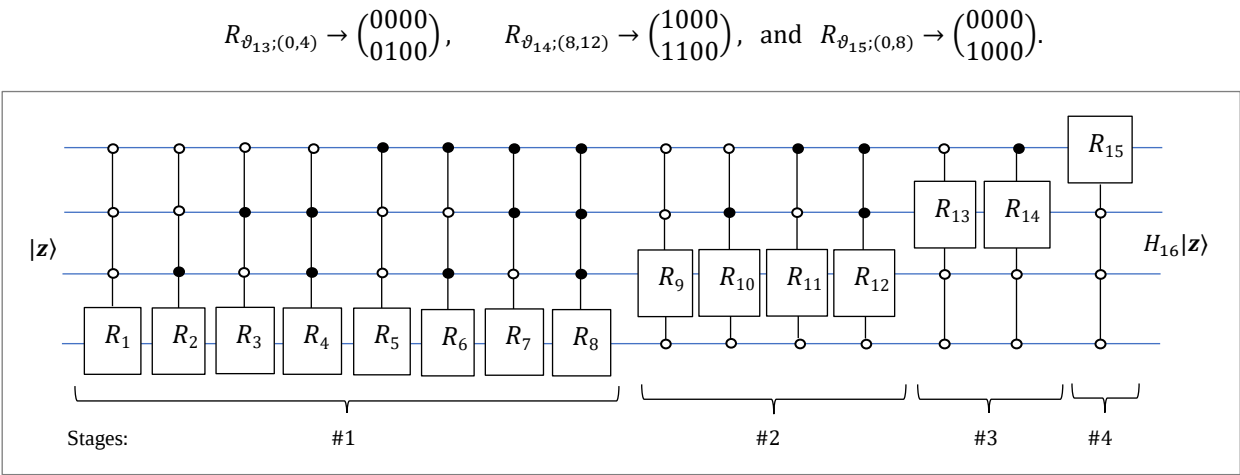


Figure 25. The circuit for the 4-qubit QsiHT by the fast path #1.

The circuit for the inverse 4-qubit DsiHT is obtained directly from the above circuit and is shown in Figure 26. All rotations are performed in reverse order and the notations $R'_k=R_{-\vartheta_k}$ are used, for $k = 1:15$.

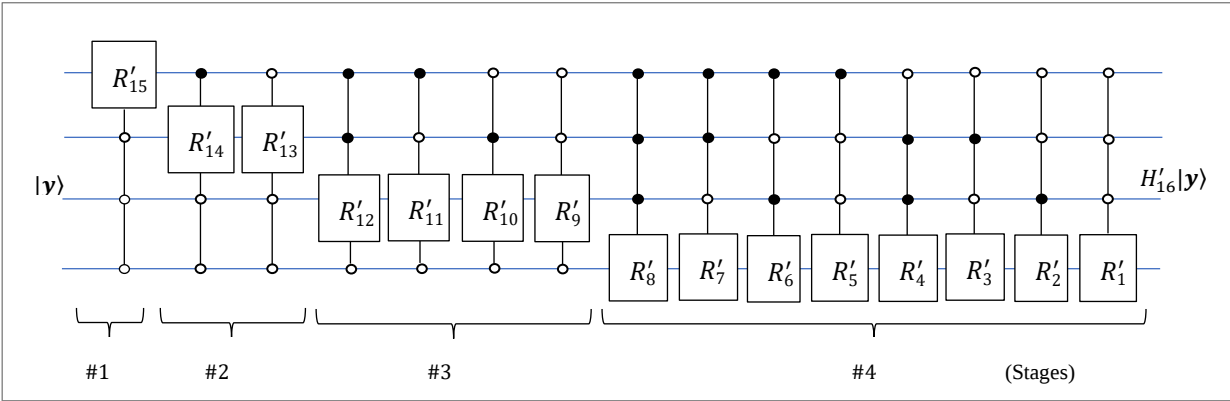


Figure 26. The circuit for the 4-qubit inverse QsiHT by the fast path #1. (from: example7_paper.m).

Let us consider for example the 4-qubit superposition $|x\rangle$ with amplitudes of the normalized vector

$$x = (1, -1, 2, 4, 2, -1, 2, 6, 1, 1, 4, -2, 4, -1, 1, 2)/\sqrt{111}.$$

With the fast path, the angular representation of this vector is the set of angles (rounded to two decimals)

$$A_x = \left\{ \underbrace{45^\circ, -63.43^\circ, 26.56^\circ, -71.56^\circ, -45^\circ, 26.56^\circ, 14.04^\circ, -63.43^\circ}_{-72.45^\circ, -70.53^\circ, -72.45^\circ, -28.47^\circ}, \underbrace{-55.04^\circ, -45^\circ, -39.02^\circ} \right\}.$$

The results of simulation of the above circuit for the 4-qubit superposition $|x\rangle$ are given in Table 4.

Table 4. Measured magnitudes for the 4-qubit $|x\rangle$ through the DsiHT with the fast path.

Basis States	Magnitudes				
	Theoretical	500 shots	1,000 shots	10,000 shots	100,000 shots
0000	0.0949	0.0775	0.0943	0.0936	0.0950
0001	0.0949	0.1095	0.0872	0.0976	0.0963
...	...	...	...	...	...

1110	0.0949	0.1225	0.0894	0.0970	0.0952
1111	0.1898	0.1844	0.1913	0.1891	0.1887
MSRE		1.83×10^{-3}	5.97×10^{-4}	1.75×10^{-4}	6.56×10^{-5}

Now, we consider the circuit symbols used in [37], for the k -fold uniformly controlled 1-qubit gate, which is a full sequence of k -fold controlled gates of rotation. Figure 27 shows the equivalent circuit of Figure 25.

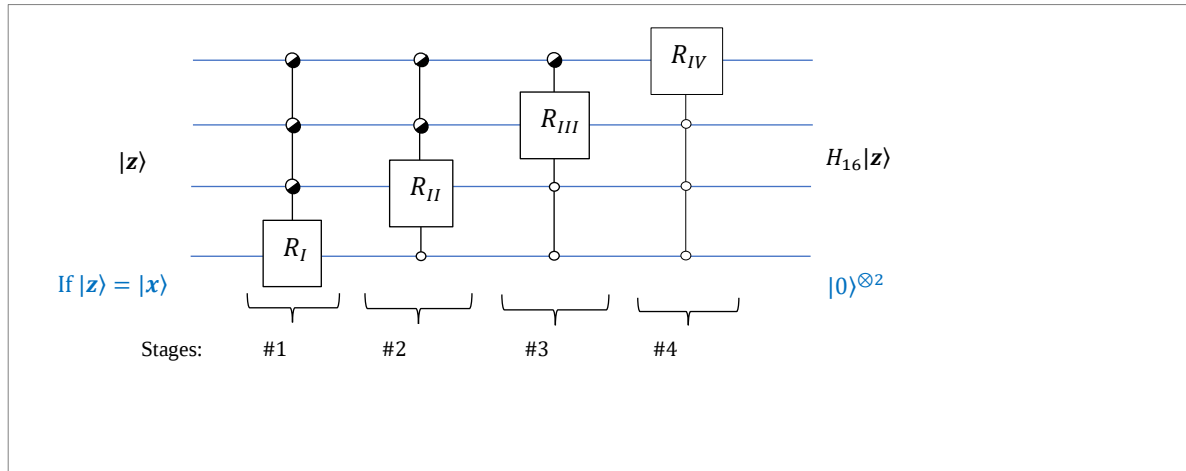


Figure 27. The circuit for the 4-qubit QsiHT by the fast path #1.

Here, the notation R_I stands for the set of controlled rotations $R_1, R_2, \dots, R_8$. Also, R_{II} stands for rotations $R_9, R_{10}, R_{11}, R_{12}$, and R_{III} for two rotations R_{13}, R_{14} , and $R_{IV} = R_{15}$. This circuit does not use any permutations. With these notations for the k -fold uniformly controlled 1-qubit gates, the quantum scheme in Figure 26 can be presented, as shown in Figure 28.

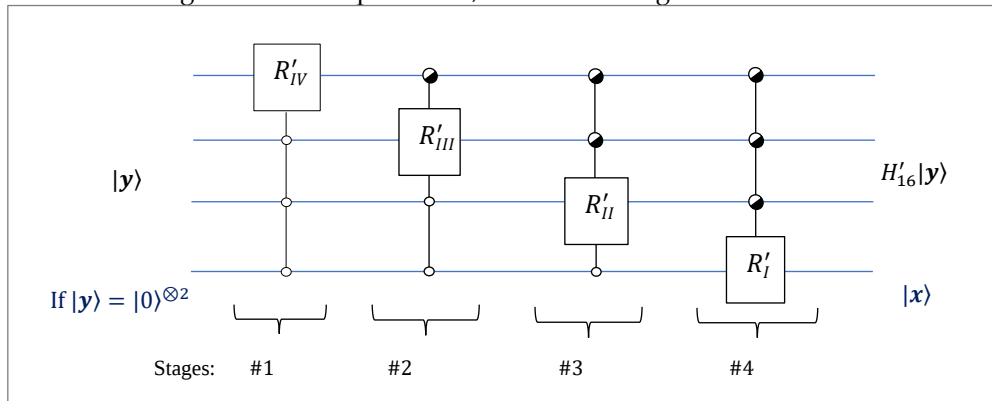


Figure 28. The circuit for the inverse 4-qubit QsiHT by the fast path #1.

If we combine these two quantum circuits, we obtain a simple circuit for preparing the state $|x\rangle$ from another state $|y\rangle$. This diagram is shown in Figure 29. Here, in order to not complicate the notations in circuit elements, two parts of the circuits are colored in distinct colors. The angles of rotations in both parts are different. The first part of this circuit is used the set of rotations with 15 angles of the angular representation A_y of the vector y . The second part of the circuit is composed by the set of inverse rotations with the 15 angles of the angular representation A_x of the vector x . This circuit requires 30 angles of these two vectors, or only 15 angles of A_x , if the state $|y\rangle = |0000\rangle$.

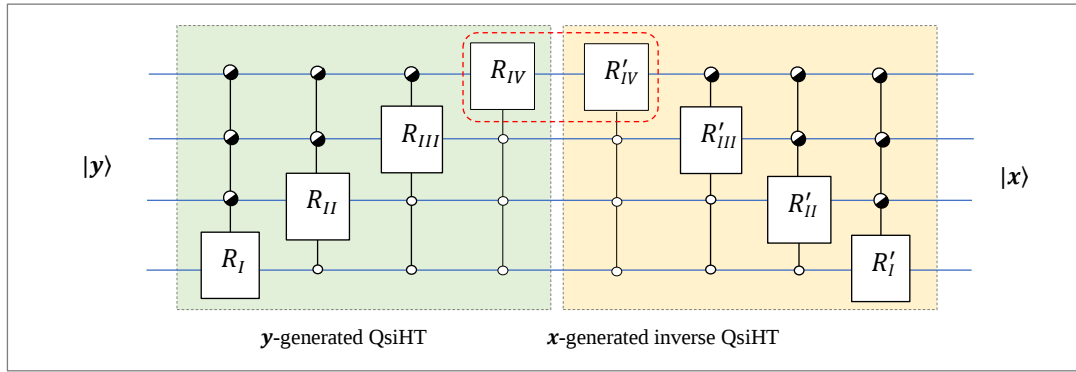


Figure 29. The circuit for transforming one 4-qubit state into another, $|y\rangle \rightarrow |x\rangle$, by using two 4-qubit QsiHTs by the fast path #1.

Then, two controlled gates in the middle of the circuit can be united as one rotation. Therefore, the total number of rotation gates in this circuit with 4 qubit input equals to $g(4) = 2(8 + 4 + 2 + 1) - 1 = 2 \times (2^4 - 1) - 1 = 2 \times 2^4 - 3 = 29$. In general case of r -qubits, these two numbers are equal to $g(r) = 2 \times 2^r - 3$.

It should be noted that the path with partitioning into pairs, which is shown in part (b) in Figure 8, can also be used for the 8-point DsiHT. However, its implementation for the 3-qubit QsiHT is facing difficulties. The first four rotations $R_{\vartheta_k}, k = 1:4$, operate on inputs and outputs on different bit-planes. This stage is described by the matrix

$$H_{8;1} = \begin{bmatrix} c_1 & 0 & 0 & 0 & -s_1 & 0 & 0 & 0 \\ s_1 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & -s_2 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 & -s_3 & 0 \\ 0 & 0 & s_3 & 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 & 0 & 0 & 0 & -s_4 \\ 0 & 0 & 0 & s_4 & 0 & 0 & 0 & c_4 \end{bmatrix},$$

where $c_k = \cos \vartheta_k$ and $s_k = \sin \vartheta_k, k = 1:4$. Also, on stage 2, the 5th rotation uses the inputs number 0 and 4, but outputs number 0 and 2. The 6th rotation uses the inputs number 2 and 6, and outputs number 4 and 6. These rotations are not on the same pair of adjacent bit-planes. Thus, we assume that this path is not suitable for the 3-qubit QsiHT. However, there are other effective paths for the r -qubit QsiHT, which will be shown in the next section. The larger the number of qubits, the more such paths can be found.

7. The DsiHT in the General Form

In this section, we consider the concept of the DsiHT in the general definition [16] and show a way to accomplish the state-to-state transformation in a fast way. Namely, the state transformation will be calculated by using only one QsiHT. This is a one-step process and does not require the use of reverse DsiHT in the method shown above in Figure 29 for 4-qubit superpositions.

The DisHT by rotations in its original definition consists of the following 2×2 elementary rotations [16,30]

$$T_{\vartheta} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\vartheta) x - \sin(\vartheta) y \\ a \end{bmatrix}. \quad (30)$$

Here, the parameter a is given and the angle is calculated by

$$\vartheta = \arccos\left(\frac{a}{\sqrt{x^2 + y^2}}\right) + \arctan\left(\frac{x}{y}\right), \quad (a^2 \leq x^2 + y^2), \quad (31)$$

$$\left(\vartheta = \arcsin\left(\frac{a}{x}\right), \text{ if } y = 0\right).$$

Thus, the angle of rotation is defined from the angular equation $\sin(\vartheta)x + \cos(\vartheta)y = a$. The N -point DsiHT generated by the vector $\mathbf{x}$ is defined with a given vector of parameters $\mathbf{a} = \{a_1, a_2, \dots, a_{N-1}\}$. It is clear that in the case, when the vector parameter is zero, $\mathbf{a} = \{0, 0, \dots, 0\}$, the DsiHT is the transform described above in Sections 1-6.

As an example, Figure 30 shows the calculation of the 8-point strong DsiHT with the given parameters $\{a_1, a_2, \dots, a_7\}$. The value $x_0^{(7)}$ contains the information of the energy of the generator minus the number $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_7^2}$. It is assumed that $\|\mathbf{a}\| \leq \|\mathbf{x}\|$. Thus, $x_0^{(7)} = \pm\sqrt{\|\mathbf{x}\|^2 - \|\mathbf{a}\|^2}$. When performing the DsiHT, the angles of rotations $T_k = T_{\vartheta_k}, k = 1:7$, are calculated by Eq. 31.

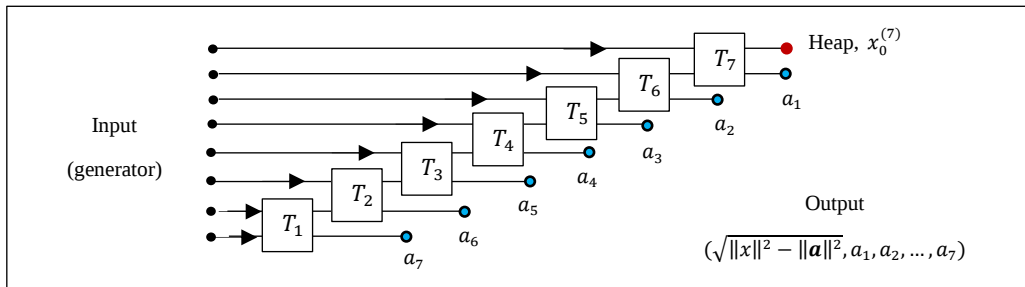


Figure 30. The 8-point strong DsiHT with vector-parameter $\mathbf{a}$.

The DsiHT with the vector-parameter is defined similarly for other paths. The vector-parameter changes the angular representation of the generator, that is, the set of angles of rotations. Figure 31 shows the diagrams for calculating the 4- and 8-point DsiHTs with the fast path in part (a) and (b) respectively.

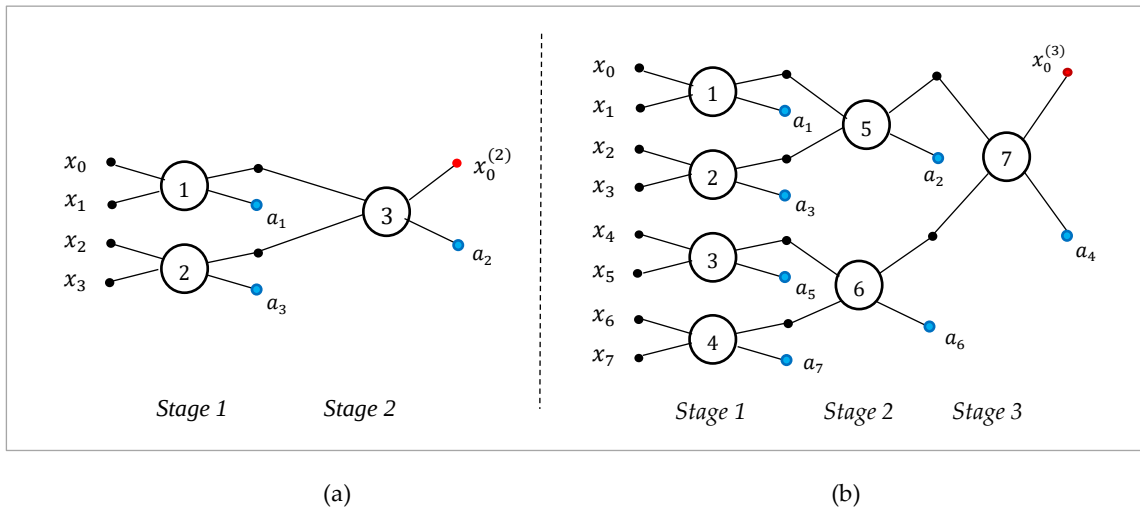


Figure 31. (a) The 4-point DsiHT and (b) the 8-point DsiHT with the paths with splitting into pairs.

Example 7 (2-qubit transformation)

Consider two 4-D vectors $\mathbf{x} = (1, 2, -1, 3)$ and $\mathbf{y} = (4, -1, 1, 2)$ after normalization by $\|\mathbf{x}\| = \sqrt{15}$ and $\|\mathbf{y}\| = \sqrt{22}$. The corresponding 2-qubit superpositions are

$$|\mathbf{x}\rangle = \frac{1}{\sqrt{15}}(|0\rangle + 2|1\rangle - |2\rangle + 3|3\rangle)$$

and

$$|\mathbf{y}\rangle = \frac{1}{\sqrt{22}}(4|0\rangle - |1\rangle + |2\rangle + 2|3\rangle).$$

Both 2-qubits are entangled. The vector parameter for the DsiHT is taken to be equal to

$$\mathbf{a} = (y(1), y(2), y(3)) = \frac{1}{\sqrt{22}}(-1, 1, 2).$$

The norm of this vector equals to $\|\mathbf{a}\| = \sqrt{1 + 1 + 4}/\sqrt{22} = \sqrt{6}/\sqrt{22} = \sqrt{3/11}$. The DsiHT signal-flow graph with the generator $\mathbf{x}$ and this vector parameter is shown in Figure 32.

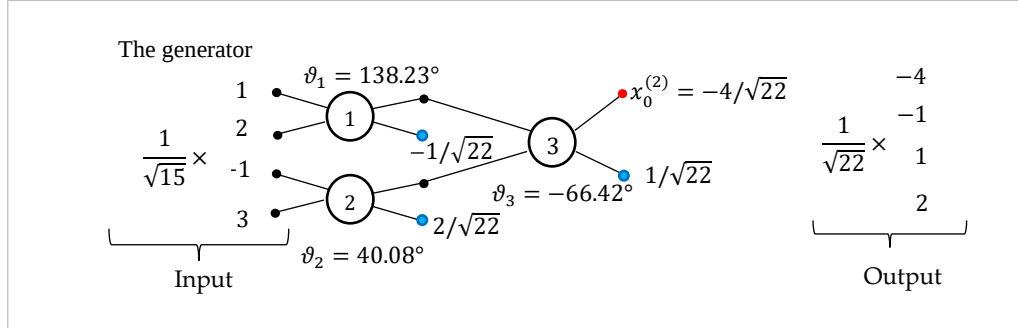


Figure 32. The signal-flow graph of the 4-point DsiHT with the fast path.

The matrix of the DsiHT is equal to

$$H_4 = \begin{bmatrix} -0.2984 & -0.2664 & 0.7012 & -0.5901 \\ 0.6661 & -0.7459 & 0 & 0 \\ 0.6836 & 0.6105 & 0.3061 & -0.2576 \\ 0 & 0 & 0.6439 & 0.7651 \end{bmatrix}, \det(H_4) = 1. \quad (32)$$

The angular representation of the generator is equal to $A_x = \{\vartheta_1, \vartheta_2, \vartheta_3\} = \{138.2359^\circ, 40.0829^\circ, -66.4201^\circ\}$. The transform of the generator equals to

$$H_4 \mathbf{x}' = \frac{1}{\sqrt{22}}(-4, -1, 1, 2)'$$

The amplitude of the first state $|0\rangle$ is $-4/\sqrt{22}$ not $4/\sqrt{22}$, as in the state $|y\rangle$. Therefore, with the additional 0-controlled phase shift gate

$$P_-(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (33)$$

we obtain the following matrix equation:

$$(P_-(\pi) \oplus I_2) H_4 \mathbf{x}' = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} H_4 \mathbf{x}' = \frac{1}{\sqrt{22}} \begin{bmatrix} 4 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \mathbf{y}'.$$

Thus, with three rotations $\{R_{\vartheta_1, (0,1)}, R_{\vartheta_2, (2,3)}, R_{\vartheta_3, (0,2)}\}$ plus the 0-controlled phase shift gate, the following 2-qubit transformation of states holds:

$$|\mathbf{x}\rangle = \frac{1}{\sqrt{15}}(|0\rangle + 2|1\rangle - |2\rangle + 3|3\rangle) \rightarrow |\mathbf{y}\rangle = \frac{1}{\sqrt{22}}(4|0\rangle - |1\rangle + |2\rangle + 2|3\rangle).$$

The corresponding circuit for this 2-qubit transform is shown in Figure 33.

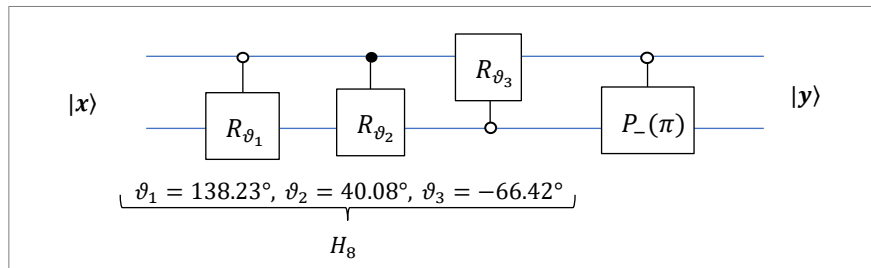


Figure 33. The circuit for transforming the 2-qubit $|\mathbf{x}\rangle$ into the 2-qubit $|\mathbf{y}\rangle$.

Table 5 shows the results of measurement of the 2-qubit $|y\rangle$ by implementing this circuit in Qiskit.

Table 5. Measured magnitudes of the 2-qubit state $|y\rangle$ after transferring the state $|x\rangle$.

Basis States	Magnitudes of $ y\rangle$			
	Theoretical	1,000 shots	10,000 shots	100,000 shots
00	0.8528	0.8555	0.8518	0.8531
01	0.2132	0.2000	0.2128	0.2128
10	0.2132	0.2121	0.2145	0.2149
11	0.4264	0.4278	0.4278	0.4252
MSRE		7.56×10^{-3}	1.42×10^{-3}	6.42×10^{-4}

The same circuit can be used for obtaining the 2-qubit $|y\rangle$ from other 2-qubits. Only the angles of three rotation gates will be changed. For example, the transformation of 2-qubits

$$|x_1\rangle = \frac{1}{\sqrt{46}}(|0\rangle + 5|1\rangle + 2|2\rangle + 4|3\rangle) \rightarrow |y\rangle = \frac{1}{\sqrt{22}}(4|0\rangle - |1\rangle + |2\rangle + 2|3\rangle)$$

requires a new set of angles, $A_{x_1} = \{\vartheta_1, \vartheta_2, \vartheta_3\} = \{52.3485^\circ, 33.8996^\circ, -21.7498^\circ\}$ in the circuit in Figure 33.

We also can consider another path for such 2-qubit QsiHT, which is illustrated in Figure 34. The order of the last two rotations changes, and this results in a change in the rotation angles.

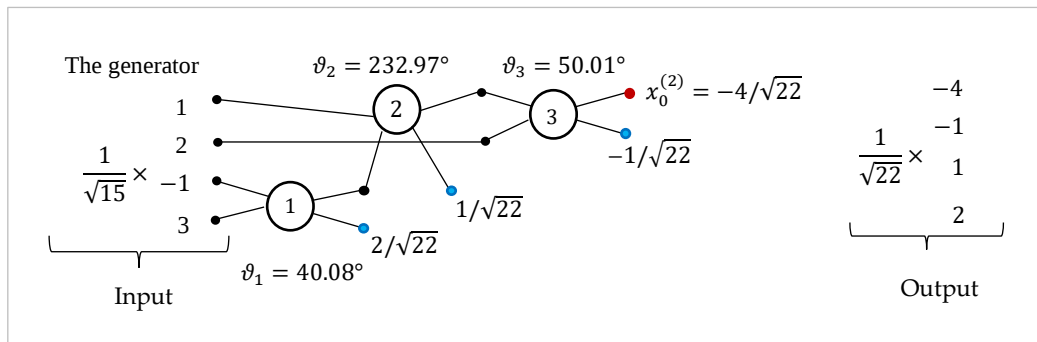


Figure 34. The second signal-flow graph of the 4-point DsiHT.

In this case, the matrix of the 2-qubit QsiHT transform is equal to

$$H_4 = \begin{bmatrix} -0.3870 & -0.7662 & 0.3925 & -0.3303 \\ -0.4614 & 0.6426 & 0.4680 & -0.3939 \\ -0.7983 & 0 & -0.4608 & 0.3878 \\ 0 & 0 & 0.6439 & 0.7651 \end{bmatrix}, \det(H_4) = 1. \quad (34)$$

The angular representation of the generator for this transform is equal to $A_x = \{\vartheta_1, \vartheta_2, \vartheta_3\} = \{40.0829^\circ, 233.9712^\circ, 50.0125^\circ\}$. The corresponding circuit is shown in Figure 35.

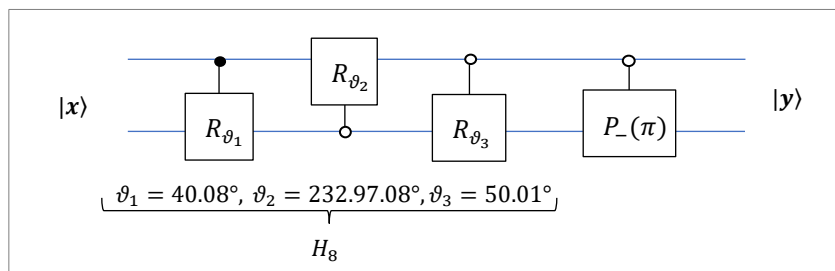


Figure 35. The circuit for transforming the 2-qubit $|x\rangle$ into the 2-qubit $|y\rangle$.

It should be noted for comparison, that the existent estimation of rotation gates in number of $\mu_1(r) = 2^{r+1} - 2$ [14,15,37] gives us the number 6, for the $r = 2$ case. However, some operations, such as global phase and normalization, were not considered, and these publications do not contain a single illustrative example of implementing the schemes $|\mathbf{x}\rangle \rightarrow |\mathbf{y}\rangle$. The circuits in Figures 33 and 35 use one phase shift gate and only three controlled elementary rotations, that is, $\mu(r) = 2^r - 1$ and this estimation is valid for integers $r \geq 2$.

Example 8 (3-qubit preparation)

Consider two 8-D vectors $\mathbf{x} = (1, 2, -1, 3, 1, 2, -3, -1)$ and $\mathbf{y} = (8, -1, 1, 2, 1, -2, 3, 1)$ after normalization them by $\|\mathbf{x}\| = \sqrt{30}$ and $\|\mathbf{y}\| = \sqrt{85}$, respectively. These two vectors represent the 3-qubit superpositions $|\mathbf{x}\rangle$ and $|\mathbf{y}\rangle$,

$$|\mathbf{x}\rangle = \frac{1}{\sqrt{30}}(|0\rangle + 2|1\rangle - |2\rangle + 3|3\rangle + |4\rangle + 2|5\rangle - 3|6\rangle - |7\rangle)$$

and

$$|\mathbf{y}\rangle = \frac{1}{\sqrt{85}}(8|0\rangle - |1\rangle + |2\rangle + 2|3\rangle + |4\rangle - 2|5\rangle + 3|6\rangle + |7\rangle).$$

Our goal is to obtain the vector $\mathbf{y}$ from $\mathbf{x}$ using only one 8-point DsiHT, or 3-qubit QsiHT. Therefore, the vector-parameter for the 8-point DsiHT with the generator $|\mathbf{x}\rangle$ is taken to be equal to

$$\mathbf{a} = (y(1), y(2), y(3), \dots, y(7)) = \frac{1}{\sqrt{85}}(-1, 1, 2, 1, -2, 3, 1).$$

The norm of this vector is equal to $\|\mathbf{a}\| = \sqrt{85 - 64}/\sqrt{85} = \sqrt{21/85} = 0.4971$. The angular representation of the vector $\mathbf{x}$ equals to

$$A_x = \{\vartheta_k; k = 1:7\} = \{131.9727^\circ, 49.4953^\circ, 148.6630^\circ, -29.2633^\circ, -63.0605^\circ, -87.9558^\circ, -48.5902^\circ\}.$$

The matrix of the 8-point DsiHT has 32 zero coefficients and is equal to

$$H_8 = \begin{bmatrix} -0.2004 & -0.2228 & 0.3830 & -0.4484 & -0.0229 & -0.0139 & 0.6539 & 0.3664 \\ 0.7435 & -0.6688 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5962 & 0.6628 & 0.2943 & -0.3445 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7604 & 0.6495 & 0 & 0 & 0 & 0 \\ 0.2272 & 0.2526 & -0.4343 & 0.5084 & -0.0202 & -0.0123 & 0.5767 & 0.3231 \\ 0 & 0 & 0 & 0 & 0.5201 & -0.8541 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8536 & 0.5197 & 0.0311 & 0.0174 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.4888 & 0.8724 \end{bmatrix}.$$

The transform of the generator $\mathbf{x}$ equals to

$$H_8 \mathbf{x}' = \frac{1}{\sqrt{85}}(-8, -1, 1, 2, 1, -2, 3, 1)'.$$

Here, the sign of the first component should be changed, because $y(0) = 8$. As in Example 5, we can use the phase shift gate $P_-(\pi)$ with two control qubits, that is $P_-(\pi) \oplus I_6$, and reconstruct the sign of $y(0)$. We obtain the given vector $\mathbf{y}$,

$$(P_-(\pi) \oplus I_6) H_8 \mathbf{x}' = \begin{bmatrix} -1 & & & & & & & \\ & 1 & & & & & & \\ & & I_2 & & & & & \\ & & & I_4 & & & & \end{bmatrix} H_8 \mathbf{x}' = \frac{1}{\sqrt{85}} \begin{bmatrix} 8 \\ -1 \\ 1 \\ 2 \\ 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} = \mathbf{y}'.$$

Thus, with seven rotations plus the 0-controlled phase shift gate, the following 3-qubit transformation holds: $|\mathbf{x}\rangle \rightarrow |\mathbf{y}\rangle$. The circuit for this transformation is shown in Figure 36. In this figure, the rotations in three stages are defined by

$$R_I = \{R_{\vartheta_1;(0,1)}, R_{\vartheta_2;(2,3)}, R_{\vartheta_3;(4,5)}, R_{\vartheta_4;(6,7)}\}, \quad R_{II} = \{R_{\vartheta_5;(0,2)}, R_{\vartheta_6;(4,6)}\}, \quad R_{III} = \{R_{\vartheta_7;(0,4)}\}.$$

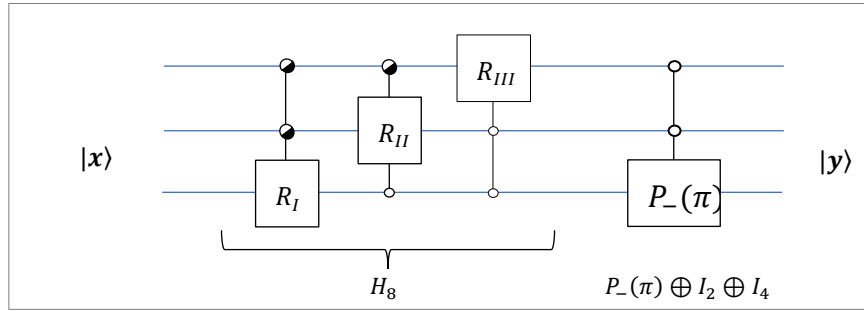


Figure 36. The circuit for the 3-qubit transformation of states, $|x\rangle \rightarrow |y\rangle$.

The results of simulation of this circuit in Qiskit are shown in Table 6.

Table 6. Measured magnitudes of the 3-qubit after state transformation $|x\rangle \rightarrow |y\rangle$.

Basis States	Magnitudes of $ y\rangle$			
	Theoretical	1,000 shots	10,000 shots	100,000 shots
000	0.8667	0.8672	0.8682	0.8678
001	0.1085	0.1000	0.1105	0.1101
...	...	...	...	...
110	0.3254	0.3271	0.3239	0.3262
111	0.1085	0.1048	0.1086	0.1086
MSRE		1.33×10^{-3}	3.44×10^{-4}	1.60×10^{-4}

When comparing with the known estimation $\mu_1(r) = 2^{r+1} - 2$ for the $r = 3$ case, we obtain the number 14, respectively. The circuit of $|x\rangle \rightarrow |y\rangle$ transformation in Figure 34 includes seven elementary rotations, that is, $\mu(3) = 7$.

The quantum circuits for the 4-qubit state-to-state transformation, which are given in Figures 25-28, only angles $\vartheta_k, k = 1:7$, should be changed according to the vector parameter $\mathbf{a} = (y(1), y(2), \dots, y(7))$. This vector changes the angular representation of A_x of the generator. Similarly, with the complete set of k -fold uniformly controlled 1-qubit gates on n qubits, the quantum circuit can be described for the r -qubit state-to-state transformation, when $r \geq 2$.

8. Conclusion

In this work, we describe in detail a new method of state-to-state transformation with only elementary rotations, by using the concept of the DsiHT in its general form. This transformation is generated by the given superposition and the set of angular equations with the given parameters. The analogue of this transform, namely the QsiHT, is described, and the quantum circuits of its implementation were considered. Amplitudes of quantum superpositions are considered real. On the examples with the DsiHT with strong and fast paths, the importance of this characteristic for the transform is shown. The QsiHT with the fast path allows us to build the circuit for r -qubit state-to-state transformation with only $(2^r - 1)$ elementary rotations, each with only one angle. The number of such possible paths increases with the number r of qubits and they can be found, and the best ones can be selected for a specific implementation of the proposed method. The problem of state preparation with complex amplitudes can also be considered and solved by the complex QsiHT [32,36].

It is important to mention the following. We present a “permutation-free method” for transforming one quantum superposition into another. This is our vision for building quantum circuits for multi-qubit gates. If there is a desire (or need) to modify the proposed quantum circuits

to circuits with (not controlled) rotations, then the following should be considered. As shown in [14], the $(r - 1)$ -fold uniformly controlled 1-qubit gates on n -qubits can be implemented by 2^{r-1} CNOTs controlled by one qubit. Therefore, no more than $2^{r-1} + 2^{r-2} + \dots + 2 + 1 = 2^r - 1$ one-qubit CNOTs are required, or seven CNOTs, for the realization of the 3-qubit QsiHT in the circuit in Figure 36. One controlled 1-qubit phase shift gate is needed only to change the sign of the first amplitude in the transformed superposition $|y\rangle$. This is the trivial operation of the sign change, $y_0 \rightarrow -y_0$, on the first bit. Note that the probability $p_0 = |y_0|^2$ of observing the first state $|0\rangle$ in $|y\rangle$ does not require this operation. The best-known number of CNOTs is $2(2^r - r - 1)$ [14,37], or eight for the 3-qubit case, and 22 and 52 CNOTs for the 4- and 5-qubit cases, respectively. For 4- and 5-qubit state-to-state transformations, the modified circuits will require no more than $2^4 - 1 = 15$ and $2^5 - 1 = 31$ CNOTs, respectively.

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Abbreviations

The following abbreviations are used in this manuscript:

DsiHT	Discrete signal-induced Heap transform
QsiHT	Quantum signal-induced Heap transform
MSRE	Mean square root error
BP	Bit plane

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