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Anand Aruna Kumar *

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Ermakov Invariants in Stationary Quantum Mechanics: A Bohm–Madelung and Hamilton–Jacobi Perspective

Anand Aruna Kumar [1,2]

[1] Hardware Research Engineer, IBM Reaseach, Albany, NY, USA; anand.aruna.kumar@ibm.com
[2] Private Research

Abstract

The Ermakov–Pinney (EP) equation and its associated invariant are shown to arise naturally in stationary quantum mechanics when the Schrödinger equation is written in Bohm–Madelung (BM) form and the Hamiltonian is diagonal and separable. Under these conditions, the quantum continuity constraint induces a nonlinear amplitude equation of EP type for each degree of freedom, revealing a hidden invariant structure independent of whether the evolution parameter is time or space. This framework is illustrated using the one-dimensional harmonic oscillator, clarifying the role of the second independent solution, which is typically suppressed in standard quantum mechanics. The results establish Ermakov invariants as an intrinsic amplitude-space structure underlying separable stationary problems and motivate further investigation using extended variational formulations of Bohmian dynamics.

Keywords: ermakov-lewis; ermakov-pinney; bohm-madelung; hamilton-jacobi; invariant; separability; continuity and stationary

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1. Introduction

The stationary state description of Quantum Mechanics developed through de Broglie Bohm's method is introduced as a simple application in Peter Holland's, Quantum theory of Motion, Chapter 4, Sec 4.1 [1]. The author describes the general properties of stationary state and its correspondence to WKB's version and concludes with an approximate solution in one dimension, where in reference to Ermakov's differential equation. There had been very little extension of the solution in the rest of the monograph, however, the solution is a first integrable invariant and has interesting applications to description of Bohmian's Hamilton–Jacobi (HJ) equations. The present work generalizes the role of EP invariants in stationary quantum mechanics by showing that they arise naturally within the BM formulation for diagonal and separable Hamiltonians, while preserving the standard probabilistic structure of quantum mechanics.

The EP equation and its associated invariant originate in the nineteenth-century work of Ermakov [2] and were rediscovered and systematized by Lewis and Riesenfeld in the context of time-dependent harmonic oscillators [3]. Subsequent mathematical developments clarified the nonlinear superposition principle and invariant structure of Ermakov systems, notably in the work of Reid and collaborators [4,5], including the concise analysis of Reid and Ray on nonlinear oscillator invariants [6].

Connections between Ermakov-type invariants and quantum mechanics have appeared sporadically, often within trajectory-based or hydrodynamic formulations. In particular, Nassar developed an invariant formulation of quantum systems within the Bohmian and quantum potential framework, emphasizing auxiliary equations closely related to the EP system [7,8]. Despite these contributions, explicit treatments of Ermakov invariants in stationary BM equations and spatially separable Hamiltonians remain limited.

The present work addresses this gap by demonstrating that Ermakov invariants arise naturally in stationary quantum mechanics whenever the Hamiltonian is diagonal and separable, with the EP equation emerging as the amplitude-space counterpart of HJ separability.

The Ermakov–Pinney (EP) equation

$$\rho'' + \Omega^2(x)\rho = \frac{k}{\rho^3} \tag{1}$$

plays a central role in the theory of time-dependent oscillators [9,10] and admits a conserved Ermakov–Lewis invariant (*or* "invariant") [11]. In contrast, stationary quantum mechanics is usually presented as an eigenvalue problem, with little emphasis on hidden invariants beyond the energy.

Ermakov–Lewis invariant structures in quantum mechanics have been discussed in earlier works, particularly by Reinisch [12] and Schuch [13,14] in Bohm–Madelung and Riccati formulations. The present work extends the formalism to separable coordinate systems and demonstrates its application suitable for a wide class of boundary value problems.

In this work we demonstrate that this distinction is largely historical rather than structural. When stationary quantum systems are expressed in the BM representation, the same mathematical structure underlying Ermakov invariants are identical to that seen with time, while the spatial coordinate playing the role of evolution parameter. This occurs whenever the Hamiltonian is diagonal and separable.

Scope of the present work.

The analysis presented here is restricted to stationary quantum systems with diagonal and separable Hamiltonians, for which the BM equations admit a coordinate-wise decomposition and a natural EP structure at the level of the amplitude dynamics. Issues related to angular-variable subtleties, non-separable Hamiltonians, and broken invariance due to non-diagonal (or gauge-coupled) kinetic terms are intentionally not pursued in this manuscript.

Given the broad scope of the present work, spanning coordinate-separable structures, and nonlinear representations of quantum mechanics, it is possible that some related contributions have not been cited. Any such omissions are unintentional, and the authors welcome further references to relevant literature.

2. Stationary BM Formulation and EP Structure

Consider the stationary Schrödinger equation

$$\hat{H}\psi = E\psi, \tag{2}$$

with the wave function ψ in polar decomposition

$$\psi(q_1, \ldots, q_n) = R(q_1, \ldots, q_n)\, e^{iS(q_1,\ldots,q_n)/\hbar}. \tag{3}$$

We assume a diagonal Hamiltonian of the form

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m} + V(q_1, \ldots, q_n), \tag{4}$$

and separability,

$$R = \prod_i R_i(q_i), \qquad S = \sum_i S_i(q_i). \tag{5}$$

Substitution into the Schrödinger equation yields a set of continuity equations and separated energy equations for each degree of freedom.

2.1. Continuity Equation and Ermakov Invariant Connection

The separated continuity equations take the form

$$\frac{d}{dq_i}\left(R_i^2 p_i\right) = 0, \tag{6}$$

which integrate immediately to

$$p_i = \frac{C_i}{R_i^2}, \tag{7}$$

where C_i are constants subject only to global conservation constraints (for example, $\sum_i C_i = 0$ for stationary states).

2.2. The EP Equation and the Ermakov-Lewis Invariant

The separated energy equations are

$$\frac{p_i^2}{2m} + V_i(q_i) + Q_i = E_i, \qquad \sum_i E_i = E, \tag{8}$$

with quantum potential

$$Q_i = -\frac{\hbar^2}{2m}\frac{R_i''}{R_i}. \tag{9}$$

Substituting Eq. (7) and scaling the coefficients yields

$$R_i'' + \Omega_i^2(q_i)R_i = \frac{k_i}{R_i^3}, \tag{10}$$

which is precisely the EP equation for each coordinate q_i.

Thus, *any diagonal and separable stationary quantum system admits EP dynamics in amplitude space.*

The nonlinear term proportional to $1/R^3$ arises directly from the stationary Bohm continuity constraint and distinguishes the EP equation from its linear counterpart. In addition to the differential equation itself, the EP system admits a first integral, commonly referred to as the Ermakov–Lewis invariant.

Specifically, if $y(x)$ is any solution of the associated linear equation

$$y''(x) + \Omega^2(x)\,y(x) = 0, \tag{11}$$

and $R(x)$ satisfies the EP equation,

$$R''(x) + \Omega^2(x)\,R(x) = \frac{k}{R^3(x)}, \tag{12}$$

then the quantity

$$\boxed{I = \frac{1}{2}\left[\left(Ry' - R'y\right)^2 + \frac{k\,y^2}{R^2}\right]} \tag{13}$$

is independent of the coordinate x. Equation (13) therefore provides a first invariant of the stationary amplitude dynamics.

Equation (7) is the key link between stationary quantum mechanics and Ermakov theory.

3. Canonical One-Dimensional Examples

The following examples illustrate how the EP structure manifests for the three canonical one-dimensional stationary quantum systems: the free particle, the harmonic oscillator, and the Coulomb potential.

3.1. Free particle: Bohm Continuity ⇒ Ermakov Amplitude and Invariant

For the free particle $V(x) = 0$ and $E = \hbar^2 k_0^2/(2m)$, hence

$$\Omega^2(x) = k_0^2. \tag{14}$$

The Bohmian amplitude therefore satisfies the constant-frequency EP equation

$$R'' + k_0^2 R = \frac{k}{R^3}, \qquad k = \frac{C^2}{\hbar^2}. \tag{15}$$

A particularly transparent solution is the constant-amplitude solution $R(x) = R_0$, for which (15) gives

$$k_0^2 R_0 = \frac{k}{R_0^3} \quad \Rightarrow \quad R_0^4 = \frac{k}{k_0^2} = \frac{C^2}{\hbar^2 k_0^2} \quad \Rightarrow \quad R_0 = \left(\frac{|C|}{\hbar k_0}\right)^{1/2}. \tag{16}$$

The Bohmian momentum $p = S'$ then becomes

$$p(x) = \frac{C}{R_0^2} = \mathrm{sgn}(C)\, \hbar k_0, \tag{17}$$

recovering the constant momentum of the plane wave. The corresponding probability density is $|\psi|^2 = R_0^2 = \text{const}$, consistent with standard quantum mechanics for scattering states.

To connect explicitly with the linear partner equation, note that the associated linear equation $y'' + k_0^2 y = 0$ admits the independent solutions

$$y_1(x) = \cos(k_0 x), \qquad y_2(x) = \sin(k_0 x), \qquad W = y_1 y_2' - y_1' y_2 = k_0. \tag{18}$$

Hence the general Ermakov amplitude can be written as

$$R^2(x) = A\cos^2(k_0 x) + B\sin^2(k_0 x) + 2D\sin(k_0 x)\cos(k_0 x), \qquad AB - D^2 = \frac{k}{k_0^2}. \tag{19}$$

The constant-amplitude solution corresponds to the symmetric choice $A = B = R_0^2$ and $D = 0$, which satisfies $AB - D^2 = R_0^4 = k/k_0^2$.

Finally, for any linear solution y and the Ermakov solution R, the invariant (13) is constant in x.

3.2. Harmonic Oscillator: Weber Basis and Bohmian Ermakov Amplitude

For the one-dimensional harmonic oscillator with $V(x) = \frac{1}{2}m\omega^2 x^2$, the stationary BM reduction yields the EP equation for the amplitude $R(x)$,

$$R''(x) + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right)R(x) = \frac{C^2}{\hbar^2}\frac{1}{R^3(x)}. \tag{20}$$

The associated linear partner equation is obtained by setting $C = 0$:

$$y''(x) + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right)y(x) = 0. \tag{21}$$

Introducing the dimensionless coordinate

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x, \tag{22}$$

Eq. (21) reduces to the Weber equation

$$\frac{d^2y}{d\xi^2} + \left(\nu + \frac{1}{2} - \frac{\xi^2}{4}\right)y = 0, \qquad \nu := \frac{E}{\hbar\omega} - \frac{1}{2}. \tag{23}$$

A natural independent basis is provided by parabolic cylinder functions,

$$y_1(\xi) = D_\nu(\xi), \qquad y_2(\xi) = D_\nu(-\xi). \tag{24}$$

The general Bohmian amplitude solving the nonlinear EP equation (20) can then be written in Pinney form using the same basis:

$$R^2(\xi) = A\, y_1^2(\xi) + B\, y_2^2(\xi) + 2D\, y_1(\xi) y_2(\xi), \qquad AB - D^2 = \frac{k}{W^2}, \tag{25}$$

where $W = y_1 y_2' - y_1' y_2$ is the (constant) Wronskian and $k = C^2/\hbar^2$ as in Eq. (12). The invariant defined in Eq. (13) is therefore constant along x for any choice of y solving the Weber equation and R given by (25).

Why Weber Basis?

The construction of the Ermakov invariant requires a choice of a fundamental pair of independent solutions of the associated linear equation. Naturally, for the harmonic oscillator, this linear partner equation arises in the corresponding separable coordinate system. A convenient representation of its independent solutions is provided by the parabolic cylinder functions $D_\nu(\xi)$ and $D_\nu(-\xi)$. In contrast to the conventional treatment, where the harmonic oscillator is formulated directly in terms of Hermite functions after imposing normalizability, the Ermakov–Pinney framework retains access to the full space of independent solutions prior to the application of boundary conditions. Accordingly, the amplitude admits the invariant quadratic form (25).

Here the Wronskian W ensures linear independence, while the invariant guarantees consistency of the amplitude construction. Boundary and normalization conditions then select particular invariant combinations, rather than eliminating solutions at the outset. For discrete values $\nu = n$, the Weber functions reduce to Hermite functions, $D_n(\xi) \propto e^{-\xi^2/4} H_n(\xi/\sqrt{2})$, recovering the standard harmonic-oscillator spectrum.

3.3. Coulomb Potential

For the one-dimensional Coulomb potential $V(x) = -\alpha/x$ on the half-line, assume bound states $E < 0$ and define the decay scale $\beta = \sqrt{-2mE/\hbar^2}$. After the standard rescaling, the stationary Schrödinger equation reduces to Whittaker's differential equation, which provides the natural representation of solutions for Sturm–Liouville problems with a Coulomb-type singularity [15]. We therefore adopt Whittaker functions as the fundamental solution basis. With the Whittaker variable $z = 2\beta x$, a fundamental solution pair is given by $M_{\kappa,1/2}(z)$ and $W_{\kappa,1/2}(z)$, where the Whittaker index is $\kappa = m\alpha/(\hbar^2\beta)$. This leads to the general Ermakov amplitude

$$\rho^2(2\beta x) = A M_{\kappa,1/2}^2(2\beta x) + B W_{\kappa,1/2}^2(2\beta x) + 2C\, M_{\kappa,1/2}(2\beta x)\, W_{\kappa,1/2}(2\beta x), \tag{26}$$

with $AB - C^2 = k/\mathcal{W}^2$, where $\mathcal{W}$ is the Wronskian of the above solution pair. Imposing the bound-state condition $\kappa = n + 1$ yields $E_n = -m\alpha^2/(2\hbar^2(n+1)^2)$ and reduces $M_{\kappa,1/2}$ to the associated Laguerre form,

$$M_{n+1,1/2}(2\beta x) \propto 2\beta x\, e^{-\beta x} L_n^{(1)}(2\beta x).$$

Thus, the textbook Coulomb bound states are recovered as a constrained Ermakov limit.

4. Invariant Structure, Implications and Extensions

4.1. Invariant Structure and Scope

Beyond establishing the existence of an Ermakov–Pinney equation for separable stationary problems, it is instructive to examine the nature and scope of the associated invariant in the examples considered above. The Ermakov–Lewis invariant can be written in the form

$$I = \frac{1}{2}\left[(Ry' - R'y)^2 + k\left(\frac{y}{R}\right)^2 \right], \tag{27}$$

where y is any solution of the associated linear equation and R solves the Ermakov–Pinney equation.

In the linear limit $k = 0$, the invariant reduces to half the squared Wronskian of a fundamental solution pair of the linear equation. This corresponds to a spatial analogue of an adiabatic invariant, which is implicitly present in standard stationary quantum mechanics but is not usually identified as such. From this perspective, the Ermakov–Pinney formulation makes explicit a conserved structure that is otherwise absorbed into basis selection.

For nonzero k, the invariant retains a bounded value even when individual solutions of the linear equation exhibit divergent behavior. This occurs because the invariant depends on a Wronskian-like combination rather than on individual solutions themselves. Consequently, the Ermakov–Pinney framework naturally accommodates general linear combinations of independent solutions and is well suited to boundary-value formulations, without requiring the a priori exclusion of solutions based on asymptotic considerations.

Within this perspective, the Bohm–Madelung equations describe the underlying stationary dynamics, while the Ermakov–Lewis invariant supplies an additional integrability structure that constrains and organizes those dynamics. The invariant acts in amplitude space as a dual to Hamilton–Jacobi separability, providing bounds and consistency conditions without altering the physical content of the theory. Thus, while the Ermakov–Bohm framework reorganizes the description of amplitude dynamics, it preserves the probabilistic predictions of stationary quantum mechanics.

4.2. Implications, Limitations and Extensions

This explicit coordinate-level analysis confirms that the appearance of EP dynamics in stationary quantum mechanics is a structural consequence of separability and not an artifact of a particular coordinate choice. It also clarifies why such invariants fail to exist in systems with non-diagonal kinetic terms or gauge-induced momentum couplings, such as motion in magnetic fields.

The EP framework discussed above captures invariant amplitude structures associated with separable stationary problems. Additional structure may emerge from a variational treatment of the BM equations, resulting in closed relations that regularize the wave function and impose global constraints. These aspects are explored further in [16].

5. Conclusions

We have shown that Ermakov–Pinney equations and their associated invariants arise naturally in stationary quantum mechanics whenever the Hamiltonian is diagonal and separable, within the BM formulation. In this setting, the spatial coordinate plays the role of an evolution parameter, and the Ermakov invariant encodes a hidden geometric structure of the amplitude dynamics.

The framework unifies the free particle, harmonic oscillator, and Coulomb potential within a common invariant-based description and preserves the standard probabilistic interpretation of quantum mechanics. While additional structures emerge for non-separable systems and angular variables, these are beyond the scope of the present work and are addressed in an extended Hamilton–Jacobi formulation.

The results presented here emphasize that Ermakov invariants are not restricted to time-dependent problems but are intrinsic to the structure of stationary quantum mechanics under suitable separability conditions.

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Conflicts of Interest: "The author declare no conflict of interest." The research is a privately developed idea of the author, and has no bearings to affiliated employer, IBM Research, USA.

Appendix A

Coordinate Separation of the Bohm Quantum Potential

This appendix provides the technical details underlying the separation of the Bohm quantum potential in stationary quantum mechanics when the Hamiltonian is diagonal and separable. The purpose is to demonstrate explicitly that the Ermakov–Pinney structure derived in the main text is not an artifact of a particular coordinate choice, but follows generally for orthogonal coordinate systems after suitable amplitude rescalings.

Appendix A.1. Relation to Hamilton–Jacobi Separability

In classical mechanics, Hamilton–Jacobi separability guarantees integrability of the phase function S. In Bohmian mechanics, this structure is mirrored:

- the phase S_i satisfies Hamilton–Jacobi–type equations,
- the amplitude R_i satisfies EP equations.

Appendix A.2. General form of the Quantum Potential

The Bohm quantum potential is defined as

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{A1}$$

where R is the amplitude in the polar decomposition $\psi = Re^{iS/\hbar}$.

Assume separability of the amplitude,

$$R(q_1, \dots, q_n) = \prod_{i=1}^{n} R_i(q_i), \tag{A2}$$

in an orthogonal coordinate system $\{q_i\}$ with diagonal metric coefficients $h_i(q_i)$.

Substituting the product form of R yields

$$\frac{\nabla^2 R}{R} = \frac{1}{\prod_{j=1}^{n} h_j}\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\frac{\prod_{j=1}^{n} h_j}{h_i^2}\,\frac{1}{R}\frac{\partial R}{\partial q_i}\right) = \frac{1}{\prod_{j=1}^{n} h_j}\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\frac{\prod_{j=1}^{n} h_j}{h_i^2}\,\frac{R_i'}{R_i}\right). \tag{A3}$$

demonstrating that the quantum potential separates additively,

$$Q = \sum_{i=1}^{n} Q_i(q_i). \tag{A4}$$

Equivalently, expanding the derivative gives a form that isolates the usual second-derivative term and a “measure” contribution from the coordinate system:

$$\frac{\nabla^2 R}{R} = \sum_{i=1}^{n}\frac{1}{h_i^2}\frac{R_i''}{R_i} + \sum_{i=1}^{n}\frac{1}{h_i^2}\left[\frac{\partial}{\partial q_i}\ln\!\left(\frac{\prod_{j=1}^{n} h_j}{h_i^2}\right)\right]\frac{R_i'}{R_i}. \tag{A5}$$

This additive structure is essential for the emergence of independent EP equations for each degree of freedom.

Additive structure and emergence of EP equations.

For diagonal and separable Hamiltonians in orthogonal coordinates, the Bohm quantum potential admits an additive decomposition,

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = \sum_i Q_i(q_i), \tag{A6}$$

provided the amplitude factorizes as $R = \prod_i R_i(q_i)$ and the coordinate operators are in Sturm–Liouville form. This additive structure is essential, as it allows each coordinate contribution to enter the quantum Hamilton–Jacobi equation independently.

At the same time, separation of the stationary continuity equation implies that each Bohmian momentum component satisfies

$$p_i(q_i) = \frac{C_i}{R_i^2(q_i)}, \tag{A7}$$

where C_i are separation constants subject only to a global constraint. Substitution into the quantum Hamilton–Jacobi equation then yields, for each degree of freedom, a nonlinear second-order equation of the EP type,

$$R_i''(q_i) + \Omega_i^2(q_i)\, R_i(q_i) = \frac{1}{R_i^3(q_i)} \frac{C_i^2}{\hbar^2}. \tag{A8}$$

Thus, the EP structure arises directly from the combination of additive separability and the Bohm continuity constraint.

General Orthogonal-Coordinate Statement

From a broader perspective, the additive separation of the quantum Hamilton–Jacobi and amplitude equations mirrors the classical separability of the Helmholtz equation in orthogonal coordinate systems. The systematic classification of such separable Helmholtz problems, including the eleven canonical coordinate systems, is developed in detail by Morse and Feshbach [17]. Within this framework, the emergence of Ermakov–Pinney equations reflects the invariant structure associated with separated Sturm–Liouville operators.

In any orthogonal coordinate system with diagonal metric coefficients, a suitable rescaling of the form

$$R_i(q_i) = \frac{\rho_i(q_i)}{\sqrt{h_i(q_i)}} \tag{A9}$$

removes first-derivative terms originating from the coordinate measure and yields a second-order equation for ρ_i of the form

$$\rho_i'' + \Omega_i^2(q_i)\rho_i = \frac{k_i}{\rho_i^3}. \tag{A10}$$

Thus, for diagonal and separable Hamiltonians, the Bohm quantum potential always admits a decomposition leading to EP equations for the rescaled amplitudes.

References

1. Peter R. Holland, Quantum theory of Motion, First Edition., Cambridge University Press, 1993.
2. V. P. Ermakov, "Second-order differential equations. Conditions of complete integrability," Univ. Izv. Kiev Series III, **9**, 1–25 (1880).
3. H. R. Lewis, "Class of Exact Invariants for Classical and Quantum Time-Dependent Harmonic Oscillators," J. Math. Phys. **9**, 1976 (1968).
4. J. L. Reid, "An exact solution of the nonlinear second-order differential equation $y'' + p(x)y + cy^{-3} = 0$," Phys. Lett. A **34**, 409–410 (1971).
5. J. L. Reid and J. R. Ray, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM) **64** , 365-6 (1984).
6. P.E. Ried and J. Ray, "On canonical invariants and nonlinear oscillator systems," Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM) **55**(6), 321 (2012).
7. A. B. Nassar, Ermakov and non-Ermakov systems in quantum dissipative models. J. Math. Phys. 27, 755–758 (1986).
8. A.B. Nassar and S. Miret-Artés, *Bohmian Mechanics, Open Quantum Systems and Continuous Measurements* Springer International Publishing, 2017.
9. A. B. Nassar, "Time-dependent Harmonic Oscillator: An Ermakov-Nelson Process." Phys. Rev. A **32**, 1862 (1985).
10. A. B. Nassar, "Ermakov and Non-Ermakov Systems in Quantum Dissipative Models." "J. Phys. A: Math. Gen. **18**, L509 (1986)."
11. S. C. Mancas and H. C. Rosu, "Ermakov–Lewis invariants and Reid systems," Phys. Lett. A **378**, 1443–1449 (2014).

12. G. Reinisch, Hamiltonian formulation of quantum mechanics with Ermakov invariants, Physica A, **206**, 229–252, (1994).
13. D. Schuch, Quantum theory from a nonlinear perspective, SIGMA, **4**, 043, 0805.1667, (2008).
14. D. Schuch, *Quantum Theory from a Nonlinear Perspective: Riccati Equations in Fundamental Physics*, Fundamental Theories of Physics, 101, Springer, (2018).
15. Arfken, Weber and Harris, *Mathematical Methods for Physicists*, Seventh Edition., Academic Press, 2013.
16. Anand Aruna Kumar, S. K. Srivatsa and Rajesh Tengli, "A regularisation method to obtain analytical solutions to de Broglie-Bohm wave equations" arxiv preprint 2512.18555.
17. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw–Hill, New York, 1953.