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A Proof of the Riemann Reciprocal Sum Formula for All Nontrivial Zeros of the Riemann Zeta Function

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Abstract

The Riemann Reciprocal Sum (formula) for all nontrivial zeros of the Riemann zeta function is a constant, namely, 1/2 $(2 + \gamma - \ln 4\pi)$ [1], and also a supernatural result. Our paper provides a proof of this formula based on research findings since the Riemann Hypothesis was proposed.

Keywords: the Riemann Reciprocal Sum Formula (Constant); the nontrivial zeros; the Riemann zeta function; Riemann Hypothesis

1. Introduction

In his epoch-making paper "On the Number of Primes Less Than a Given Magnitude" [2], Bernhard Riemann proved the only analytic expression, to date, that reveals the essence of prime numbers, which is the following prime counting function [3, p.34] – the number of primes less than any given magnitude x,

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J\left(\sqrt[n]{x}\right), \quad n \in \mathbb{N}^{=} \text{ and } x > 1,$$

where, $\mu(n)$ is the Möbius function, and $J(x)$ is a new step function – the Riemann prime counting function with the evaluation of the terms in the formula [2] [3, p.33] as

$$J(x) = \mathrm{Li}(x) - \sum_{\text{all } \rho} \mathrm{Li}(x^{\rho}) - \ln 2 + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\ln t}, \quad (x > 1),$$

where, $\mathrm{Li}(x)$ and ρ are the logarithmic integral function and the nontrivial zeros of the Riemann zeta function, respectively.

At the same time, Riemann also estimated the number, $N(T)$ [2] [4, proved] [3, p.18] [5, p.17] of nontrivial zeros within the region, $0 < \mathrm{Re}(s) < 1$, $0 < \mathrm{Im}(s) \le T$, which nontrivial zeros should appear as,

$$N(T) \sim \frac{T}{2\pi} \ln \frac{T}{2e\pi};$$

and proposed the renowned Riemann Hypothesis [2] – all nontrivial zeros of the Riemann zeta function lie on the critical line, $\mathrm{Re}(s = \sigma + it) = 1/2$ in the complex plane.

Throughout the mathematical derivation process in Riemann's paper briefly described above, the reciprocal sum (formula) of all nontrivial zeros as

$$\text{(1)} \qquad \sum_{\text{all } \rho} \frac{1}{\rho} = \frac{2 + \gamma - \ln 4\pi}{2}$$

is not necessary, so it did not appear in his paper. Instead, this formula (constant) only exists in the Riemann's nachlass [1] below that conceived his paper.

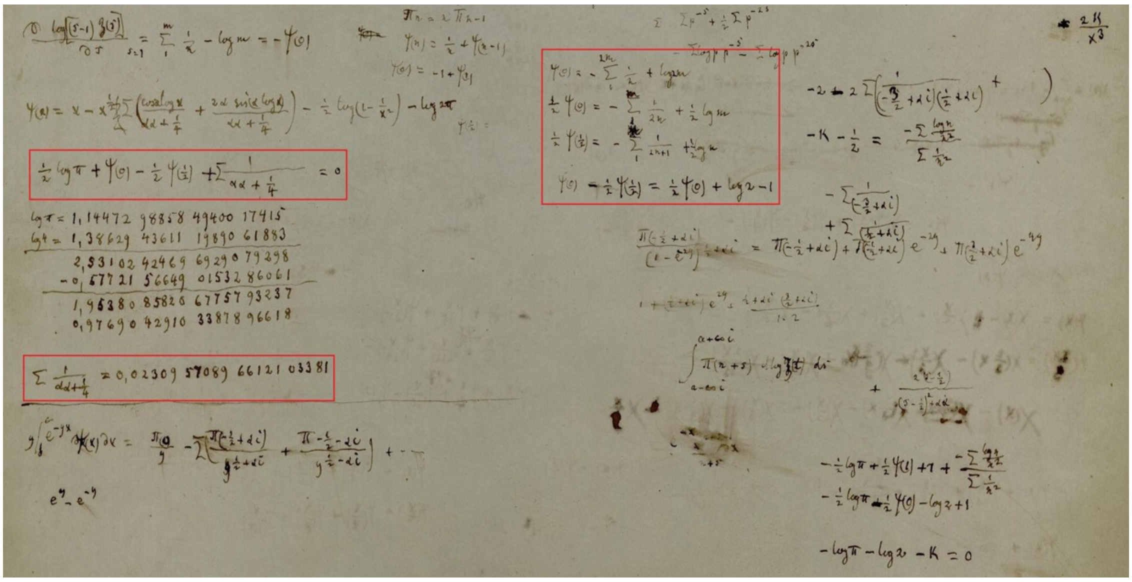

Figure 1. The Riemann's reciprocal sum formula and its numerical value [1, p.72/168].

We believe this formula may have helped him in his computations of the first few (at least three) nontrivial zeros on the critical line [1, p.134-137] [3, p.67] [6], especially to calculate and iteratively determine the smallest (first) nontrivial zero [6].

To this day, 166 years have passed, there is still little understanding about the characteristics of the nontrivial zeros for the Riemann zeta function. Even without knowing the value of any one in infinitely many nontrivial zeros, it is magical that the reciprocal sum (infinite series) unexpectedly converges to a constant, 1/2 (2 + γ – ln4π) [1]. Therefore, this formula (constant) is naturally fascinating.

2. The Proof of the Riemann Reciprocal Sum Formula (Constant)

The reciprocal sum [1, p.72/168] of all nontrivial zeros of the Riemann $\zeta(s)$ or $\xi(s)$ function can be expressed as

$$\sum_{\text{all }\rho}\frac{1}{\rho}=\frac{2+\gamma-\ln 4\pi}{2}=\kappa_1\,, \tag{2}$$

where, γ is the Euler- Mascheroni constant.

We notate this as the Riemann reciprocal sum constant and κ_1, separately.

Proof:

2.1. To Verify $\zeta'(0)/\zeta(0)=\ln 2\pi$

We start with the *first* functional equation [2] [3] [7, p.19] obtained by Riemann between $\zeta(s)$ and $\zeta(1-s)$

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\,.$$

Taking the logarithmic differentiation on both sides of the equation above yields

$$\frac{\zeta'}{\zeta}(s)=\ln 2+\ln\pi+\frac{\pi}{2}\frac{\cos\left(\frac{\pi s}{2}\right)}{\sin\left(\frac{\pi s}{2}\right)}-\frac{\Gamma'}{\Gamma}(1-s)-\frac{\zeta'}{\zeta}(1-s)$$

$$=\ln 2\pi+\frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right)-\psi(1-s)-\frac{\zeta'}{\zeta}(1-s),$$

where, $\psi(s)=\Gamma'(s)/\Gamma(s)$ is the Digamma function.

Below is an analysis of the asymptotic behavior of the last three terms in the above equation:

The Taylor expansion of $\cot(\pi s/2)$ [7, p.36] [8, p.118] is shown as

$$\frac{\pi}{2}\cot(x)\Big|_{x=\frac{\pi s}{2}}=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}B_{2n}}{(2n)!}x^{2n-1}\Big|_{x=\frac{\pi s}{2}},\quad \text{for } 0<|x|<\pi$$

$$=\frac{\pi}{2}\left(\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\cdots\right)\Big|_{x=\frac{\pi s}{2}}=\frac{\pi}{2}\left[\frac{2}{\pi s}-\frac{\pi s}{6}+O(s^3)\right]$$

$$=\frac{1}{s}-\frac{\pi^2}{12}s+O(s^3),$$

where B_{2n} are the Bernoulli numbers.

The Taylor expansion of $\psi(x)$ [7, p.13] [9] in the neighborhood of $x = 1$ is written as

$$\psi(x)|_{x=1-s}=-\gamma+\sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)(x-1)^n\Big|_{x=1-s}$$

$$=-\gamma+\sum_{n=1}^{\infty}(-1)^{2n+1}\zeta(n+1)s^n=-\gamma-\zeta(2)s+O(s^2),$$

so,

$$\psi(1-s)=-\gamma-\zeta(2)s+O(s^2),$$

where $\zeta(2)=\pi^2/6$, is Euler's 1735 solution of the Basel Problem.

The Laurent series expansion [7, p.19] [10] of the Riemann ζ function near its pole is given by

$$\zeta(s)=\frac{1}{s-1}+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(s-1)^k=\frac{1}{s-1}+\gamma+O(s-1),$$

where γ_k are the Stieltjes constants and $\gamma_0=\gamma$.

And so

$$\zeta'(s)=\frac{d}{ds}\left[\frac{1}{s-1}+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(s-1)^k\right]=\frac{-1}{(s-1)^2}+\gamma_1+O(s-1),.$$

hence

$$\frac{\zeta'(s)}{\zeta(s)}=\frac{\frac{-1}{(s-1)^2}+\gamma_1+O(s-1)}{\frac{1}{s-1}+\gamma+O(s-1)}=\frac{-1-\gamma(s-1)+O[(s-1)^2]}{(s-1)},$$

also namely,

$$\frac{\zeta'(1-s)}{\zeta(1-s)}=\frac{1}{-s}[-1-\gamma s+O(s^2)]=\frac{1}{s}[1+\gamma s+O(s^2)].$$

So that,

$$\begin{aligned}\left.\frac{\zeta'}{\zeta}(s)\right|_{s=0} &= \lim_{s\to 0}\left[\ln 2+\ln\pi+\frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right)-\frac{\Gamma'}{\Gamma}(1-s)-\frac{\zeta'}{\zeta}(1-s)\right]\\ &= \ln 2\pi+\lim_{s\to 0}\left\{\left(\frac{1}{s}-\frac{1}{s}\right)+(\gamma-\gamma)+\left[\zeta(2)-\frac{\pi^2}{12}\right]s+\frac{O(s^2)}{s}+O(s^2)\right\}\\ &= \ln 2\pi+\lim_{s\to 0}\left\{\left[\zeta(2)-\frac{\pi^2}{12}\right]s+O(s)+O(s^2)\right\}=\ln 2\pi.\end{aligned}$$

For alternative proofs about this subsection see [3, p,66-67] and [3, p.134-135].

2.2. *To Confirm* $\Gamma'(1)/\Gamma(1)=-\gamma$

By the product formula with the form of the Weierstrass factor theorem [7, p.1] [8, p.132] of the Gamma function,

$$\Gamma(s)=\frac{e^{-\gamma s}}{s}\prod_{k=1}^{\infty}\left(1+\frac{s}{k}\right)^{-1}e^{\frac{s}{k}},$$

taking logarithmic differentiation on both sides of the above equation yields

$$\frac{\Gamma'(s)}{\Gamma(s)}=-\gamma-\frac{1}{s}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+s}\right)=-\gamma+\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+s}\right).$$

Notice: the formula above is the series form of the Digamma Function.

So,

$$\frac{\Gamma'(1)}{\Gamma(1)}=\left.\frac{\Gamma'(s)}{\Gamma(s)}\right|_{s=1}=\lim_{s\to 1}\left[-\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+s}\right)\right]=-\gamma.$$

2.3. *To Prove* $\kappa_1=\sum_{\text{all }\rho}\frac{1}{\rho}=\frac{2+\gamma-\ln 4\pi}{2}$

By the definition of the Riemann $\xi(s)$ function [2] [3] which Riemann constructed

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

taking logarithmic differentiation on both sides of the above equation yields

$$\frac{\xi'}{\xi}(s)=\frac{1}{s-1}-\frac{1}{2}\ln\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)+\frac{\zeta'}{\zeta}(s).$$

By the Product formula [2] [3] of the Riemann $\xi(s)$ function,

$$\xi(s)=\xi(0)\prod_{\text{all }\rho}\left(1-\frac{s}{\rho}\right),$$

taking logarithmic differentiation on both sides of the above equation yields

$$\frac{\xi'}{\xi}(s)=-\sum_{\text{all }\rho}\frac{1}{\rho-s}.$$

Combining the two $\xi'(s)/\xi(s)$ equations above, and limiting them on both sides, gives us

$$\lim_{s\to 0}\sum_{\text{all }\rho}\frac{1}{\rho-s}=\lim_{s\to 0}\left[-\frac{\xi'}{\xi}(s)\right]=\lim_{s\to 0}\left[\frac{-1}{s-1}+\frac{1}{2}\ln\pi-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)-\frac{\zeta'}{\zeta}(s)\right].$$

Therefore, we finally obtain (2) as

$$\kappa_1 = \sum_{\text{all }\rho}\frac{1}{\rho} = \sum_{n=1}^{\infty}\frac{1}{\rho_n} = \frac{-1}{0-1} + \frac{1}{2}\ln\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{0}{2}+1\right) - \frac{\zeta'}{\zeta}(0) = \frac{2+\gamma-\ln 4\pi}{2}.$$

Regarding the absolute convergence of the Riemann reciprocal sum formula, $\kappa_1 = \sum 1/\rho$, it is better to refer to Edwards's brilliant proof [3, p.42-43]. □

3. Remarks

(a) Besides helping Riemann calculate the first few nontrivial zeros, the determination of the Riemann reciprocal sum constant, κ_1 for all nontrivial zeros may also have been one of the key factors motivating the Riemann Hypothesis;

(b) This constant (formula) is also needed in deriving the Riemann $\xi(s)$ product formula;

(c) This constant (formula) may have even more important potential applications.

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