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Logic and Probabilistic Operations on the Decision Matrix in a Fuzzy Multi-Criteria Decision-Making Problem

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Abstract

In the framework of (fuzzy) Multi-Criteria Decision-Making, we propose a method that allows the decision maker to subjectively approach the problem by suitably modifying the decision matrix. We consider a decision problem related to a random quantity X with set of values $\{x_1, x_2, \ldots, x_n\}$, and a set of properties $\{C_1, C_2, \ldots, C_m\}$ of X. In this setting, the properties C_j are the criteria of the decision problem, the alternatives represent the events $A_i = (X = x_i)$, for $i = 1, \ldots, n$, and the criteria's weights w_j, for $j = 1, \ldots, m$, are seen as the probabilities of the events "C_j is relevant with respect to the decision problem". For each $i = 1, \ldots, n$ and $j = 1, 2, \ldots, m$, we interpret the scores a_{ij} as membership functions representing "how much alternative A_i satisfies criterion C_j". By adopting the interpretation of membership functions as suitable conditional probabilities, together with the theory of logical operations among conditional events, we allow logical operations among criteria and consistently apply this interpretation to the corresponding scores. In particular, when considering the complement, conjunction, and disjunction of criteria, the resulting scores are the (coherent) previsions of the respective compound conditionals within the framework of conditional random quantities.

Keywords: compound conditionals; conditional events; conditional random quantities; fuzzy sets; MCDM

MSC: 03B48; 03B52; 60A05; 60A86

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1. Introduction

Our daily life is a continuous succession of choices. The more complex these are, the more criteria are needed to make the decision optimal. This is why the study on Multi-Criteria Decision-Making (MCDM) algorithms is an important theoretical problem and a crucial step in many real applications [1–3]. MCDM is a method for structuring and solving decisions involving multiple criteria, with the primary goal of supporting decision makers when faced with an overwhelming number of possible choices for solving a problem. It is widely applied in management, engineering, economics, and environmental science, as it enables decision-making that balances competing objectives. These methods are particularly useful when integrating qualitative and quantitative data under uncertainty, a scenario closely associated with fuzzy logic.

In 1965, Zadeh [4] introduced the concept of fuzzy set, that is, a set characterised by a membership function which assigns to each object a grade of membership. The relationship between probability theory and fuzzy set theory has been widely discussed [5–7]. Coletti and Scozzafava [8–10] proposed an approach to fuzzy set theory based on coherent conditional probability. Agreeing that a classical

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probability approach is not adequate to handle fuzzy logic structures, they considered de Finetti's conditional probability, where conditional events are seen as tri-valued objects that can take values *True*, *False* and *Void*. Through the numerical interpretation of conditional events, probability can model "fuzziness" by expressing partial knowledge in approximate reasoning and hence using conditional probability to interpret the membership functions. In [11], the idea introduced by Coletti and Scozzafava was modified and adapted to conditional random quantities using coherent conditional probability in the betting scheme framework. The authors considered a random quantity X with range of values C_X, the events $A_x = (X = x)$ for each $x \in C_X$, and any property φ related to the random quantity X. Then, an agent randomly selected from a given population, for a given $x \in C_X$, stated whether the property φ holds for X or not. In this context, they considered the event

$$E_\varphi = \text{"They claim that } X \text{ has the property } \varphi\text{"}$$

and the conditional event

$$E_\varphi | A_x = \text{"They claim that } X \text{ has the property } \varphi\text{, knowing that } X = x\text{,"}$$

with conditional probability $P(E_\varphi|A_x)$, seen as a real function $\mu_\varphi(x) = P(E_\varphi|A_x)$ defined on C_X. A fuzzy set was defined as $E^*_\varphi = \{(E_\varphi|A_x, \mu_\varphi(x)) : x \in C_X\}$, where $\mu_\varphi(x) = P(E_\varphi|A_x)$, that plays the role of the membership function of the fuzzy set, is a coherent conditional probability which measures the agent's degree of belief on the conditional event $E_\varphi|A_x$. Moreover, in the framework of conditional random quantities, some "set operations" (see, e.g., [12]), complement and intersection, on fuzzy sets were proposed and the membership functions were interpreted as previsions of compound conditionals.

To go back to MCDM in fuzzy framework, in [13], an algorithm of Multi-Criteria Group Decision-Making (MCGDM) [14–16] was introduced in the context of fuzzy ontologies. In particular, the scores in the decision matrix of each expert were interpreted as values of membership functions expressing how much each alternative satisfies each criterion. The algorithm was used to fuzzify crisp ontologies in a MCGDM framework that treated ontology classes as criteria and configurations as alternatives, enabling a principled assignment of fuzzy membership degrees. A finite panel of experts was considered, each assigned a fuzzy weight $\widetilde{\pi}_k$ reflecting their relative importance with respect to the decision problem. Each expert selected a preferred alternative, and a global geometric compromise A^* was computed via a minimal mean distance operator so that the consensus reflects the experts' views while minimising deviations from individual alternatives.

MCDM methods are intrinsically subjective, because it is the decision maker who, by constructing the decision matrix, characterises the final rank through the scores and weights of the criteria. Then, in a fuzzy MCDM process, it seems natural when interpreting the scores as a membership degrees to look at them in the framework of subjective probability. In this work, within the framework of a fuzzy MCDM, we interpret the scores, represented as degrees of membership, as coherent conditional probabilities. This approach enables the decision maker to subjectively address the problem by suitably modifying the decision matrix. We consider a decision problem related to a random quantity X with set of values $\{x_1, x_2, \ldots, x_n\}$, and a set of properties $\{C_1, C_2, \ldots, C_m\}$ of X. In this setting, the properties C_j are the criteria of the decision problem, the alternatives are given by the events $A_i^X = (X = x_i)$. In case there is no ambiguity on the random quantity considered, to aid readability, we will use A_i to indicate A_i^X, for $i = 1, \ldots, n$. For each $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we interpret the scores a_{ij} as membership functions representing "how much alternative A_i satisfies criterion C_j". Using the conditional events interpretation, we allow logical operations among criteria. More precisely, when considering the complement, conjunction and disjunction of criteria, the scores become the prevision of compound conditionals in the context of conditional random quantities.

In addition, as will become evident at the conclusion of this work, given the introspectable and explainable nature of this framework, it has the potential to be employed in a broad spectrum of decisional applications involving Explainable Artificial Intelligence (XAI) [17–19]. Specifically, we retain every step of this process in order to allow the decision maker to update the criteria by carrying out logical operations on some of them.

The paper is organised as follows. In Section 2, we recall some preliminary notions and results on events, conditional events, compound conditional, and MCDM. In Section 3, we introduce the model and the conditional probability interpretation of the scores in the decision matrix. Moreover, we adapt to this new context the complement and the intersection of fuzzy sets proposed in [11], and we introduced a new definition for the union of fuzzy sets. Then, in Section 4, we introduce logical operations among the criteria of the decision matrix, exploiting the ones for fuzzy sets. In particular, we consider the situations in which the decision maker wants to take the complement of a criterion, a conjunction or disjunction of criteria, or she wants to add or remove a criterion. Finally, in Section 5, we give some conclusions and some ideas for future work.

2. Preliminaries

In this section, we begin by recalling some preliminaries on events and conditional events. Then, we recall the notions of conjunction and disjunction of conditionals in the context of conditional random quantities. Finally, we briefly review some key concepts on MCDM methods.

In the following, we use the same symbol A to refer to an event, a two-valued logical entity which can be *true*, or *false*, and its indicator, which takes the value 1 in the first case and 0 in the second. We use the symbols Ω and $\emptyset$ to refer to the sure event and the impossible event, respectively. The negation of an event A is denoted by $\overline{A}$. Given two events A and B, we denote by $A \wedge B$ (resp., $A \vee B$), or simply by AB, their conjunction (resp., disjunction). When, an event A logically implies an event B, i.e., $A\overline{B} = \emptyset$, we write $A \subseteq B$. We say that n events $E_1, E_2, \dots, E_n$ are logically independent when there are no logical relations among them. Given two events A and H, with $H \neq \emptyset$, the conditional event $A|H$ is a three-valued logical entity which is *true*, or *false*, or *void*, according to whether AH is true, or $\overline{A}H$ is true, or $\overline{H}$ is true, respectively. The negation of a conditional event $A|H$ is defined as $\overline{A|H} = \overline{A}|H$. The logical operations among simple events can be extended to conditional events. This can be done in trivalent logics (see [20–22]), but these definitions do not preserve the logical and probabilistic properties valid for events. Another approach is to define the conjunction and the disjunction of conditionals in the framework of conditional random quantities as follows ([12], see also [23,24]).

Definition 1. *Given two conditional events $A|H$, $B|K$ and a (coherent) probability assessment $P(A|H) = x$, $P(B|K) = y$, the conjunction $(A|H) \wedge (B|K)$ is defined as the following conditional random quantity*

$$(A|H) \wedge (B|K) = (AHBK + x\overline{H}BK + yAH\overline{K})|(H \vee K).$$

Then, the prevision of the conjunction is the following

$$\begin{aligned}\mathbb{P}[(A|H) \wedge (B|K)] = \quad & P(AHBK|(H \vee K)) + P(A|H)P(\overline{H}BK|(H \vee K)) \\ & + P(B|K)P(AH\overline{K}|(H \vee K)).\end{aligned}$$

Within the betting scheme, by starting with a coherent assessment (x, y) on $\{A|H, B|K\}$, if you extend (x, y) (in a coherent way) by adding the assessment $\mathbb{P}[(A|H) \wedge (B|K)] = z$, then you agree to pay z, by receiving the random amount

$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}BK \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

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In other words, you receive:

- 1, if both conditional events $A|H$ and $B|K$ are true;
- 0, if $A|H$ or $B|K$ is false;
- $x = P(A|H)$, if $A|H$ is void and $B|K$ is true;
- $y = P(B|K)$, if $A|H$ is true and $B|K$ is void;
- z, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void.

Notice that, in some particular case, the conjunction $(A|H) \wedge (B|K)$, which is a five-valued object, reduces to a conditional event, that is, a three-valued object.

Theorem 1 ([12] Theorem 7)**.** *Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H, K \neq \emptyset$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if the Fréchet-Hoeffding bounds are satisfied, that is $z \in [z', z'']$, where*

$$z' = \max\{x + y - 1, 0\}, \quad z'' = \min\{x, y\}. \tag{1}$$

In case of some logical dependencies, for the interval $[z', z'']$ of coherent extensions z it holds that $[z', z''] \subseteq [\max\{x + y - 1, 0\}, \min\{x, y\}]$.

Definition 1 can be extended to the case of n conditional events as follows (see [25]).

Definition 2. *Let n conditional events $E_1|H_1, \ldots, E_n|H_n$ be given. For each nonempty strict subset S of $\{1, \ldots, n\}$, let x_S be a prevision assessment on $\bigwedge_{i\in S}(E_i|H_i)$. Then, the conjunction $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ is the conditional random quantity $\mathcal{C}_{1\cdots n}$ defined as*

$$\mathcal{C}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigwedge_{i=1}^n E_i H_i \text{ is true}, \\ 0, & \text{if } \bigvee_{i=1}^n \overline{E}_i H_i \text{ is true}, \\ x_S, & \text{if } (\bigwedge_{i\in S} \overline{H}_i) \wedge (\bigwedge_{i\notin S} E_i H_i) \text{ is true}, \ \emptyset \neq S \subset \{1, \ldots, n\}, \\ x_{1\cdots n}, & \text{if } \bigwedge_{i=1}^n \overline{H}_i \text{ is true}, \end{cases}$$

where $x_{1\cdots n} = x_{\{1,\ldots,n\}} = \mathbb{P}(\mathcal{C}_{1\cdots n})$.

The following theorem says that the prevision of the conjunction $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ satisfies the Fréchet-Hoeffding bounds.

Theorem 2 ([25] Theorem 13)**.** *Let $(E_1|H_1), \ldots, (E_n|H_n)$ be n conditional events with $x_i = P(E_i|H_i)$, $i = 1, ..., n$, and with $\mathbb{P}[(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)] = \mu_n$. Then*

$$\max\{x_1 + \cdots + x_n - (n-1), 0\} \leq \mu_n \leq \min\{x_1, x_2, \ldots, x_n\}.$$

Following the same idea, it is possible to define the disjunction of two conditional events.

Definition 3. *Given two conditional events $A|H$, $B|K$ and a (coherent) probability assessment $P(A|H) = x$, $P(B|K) = y$, the disjunction $(A|H) \vee (B|K)$ is defined as the following conditional random quantity*

$$(A|H) \vee (B|K) = ((AH \vee BK) + x\overline{H}\,\overline{B}K + y\overline{A}H\overline{K})|(H \vee K).$$

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Within the betting scheme, by starting with a coherent assessment (x, y) on $\{A|H, B|K\}$, if you extend (x, y) (in a coherent way) by adding the assessment $\mathbb{P}[(A|H) \vee (B|K)] = w$, then you agree to pay w, by receiving the random amount

$$(A|H) \vee (B|K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \overline{A}H \wedge \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}\,\overline{B}K \text{ is true,} \\ y, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\ w, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

We recall that De Morgan's Laws are satisfied, and hence it holds that

$$\overline{(A|H) \vee (B|K)} = (\overline{A}|H) \wedge (\overline{B}|K) \tag{2}$$

and

$$\overline{(A|H) \wedge (B|K)} = (\overline{A}|H) \vee (\overline{B}|K), \tag{3}$$

where the negations $\overline{(A|H) \vee (B|K)}$ and $\overline{(A|H) \wedge (B|K)}$ are defined as $\overline{(A|H) \vee (B|K)} = 1 - (A|H) \vee (B|K)$ and $\overline{(A|H) \wedge (B|K)} = 1 - (A|H) \wedge (B|K)$, respectively. We also observe that the prevision sum rule is satisfied, that is,

$$A|H + B|K = (A|H) \wedge (B|K) + (A|H) \vee (B|K) \tag{4}$$

and hence

$$P(A|H) + P(B|K) = \mathbb{P}[(A|H) \wedge (B|K)] + \mathbb{P}[(A|H) \vee (B|K)]. \tag{5}$$

From (5), by exploiting the Fréchet-Hoeffding bounds for the conjunction (1), we obtain the Fréchet-Hoeffding bounds for the disjunction :

$$w' = \max\{x, y\}, \; w'' = \min\{x + y, 1\}. \tag{6}$$

Definition 3 can be extended to the case of n conditional events as follows.

Definition 4. *Let n conditional events $E_1|H_1, \ldots, E_n|H_n$ be given. For each non-empty strict subset S of $\{1, \ldots, n\}$, let y_S be a prevision assessment on $\bigvee_{i \in S}(E_i|H_i)$. Then, the disjunction $(E_1|H_1) \vee \cdots \vee (E_n|H_n)$ is the conditional random quantity $\mathcal{D}_{1\cdots n}$ defined as*

$$\mathcal{D}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigvee_{i=1}^{n} E_i H_i \text{ is true,} \\ 0, & \text{if } \bigwedge_{i=1}^{n} \overline{E}_i H_i \text{ is true,} \\ y_S, & \text{if } (\bigwedge_{i \in S} \overline{H}_i) \wedge (\bigwedge_{i \notin S} \overline{E}_i H_i) \text{ is true, } \emptyset \neq S \subset \{1, \ldots, n\}, \\ y_{1\cdots n}, & \text{if } \bigwedge_{i=1}^{n} \overline{H}_i \text{ is true,} \end{cases}$$

where $y_{1\cdots n} = y_{\{1,\ldots,n\}} = \mathbb{P}(\mathcal{D}_{1\cdots n})$.

Likewise observed for the conjunction of n conditional events, the disjunction $(E_1|H_1) \vee \cdots \vee (E_n|H_n)$ satisfies the Fréchet-Hoeffding bounds.

Theorem 3 ([25] Theorem 14). *Let $(E_1|H_1), \ldots, (E_n|H_n)$ be n conditional events with $x_i = P(E_i|H_i)$, $i = 1, ..., n$, and with $\mathbb{P}[(E_1|H_1) \vee \cdots \vee (E_n|H_n)] = \nu_n$. Then*

$$\max\{x_1, x_2, \ldots, x_n\} \leq \nu_n \leq \min\{x_1 + \cdots + x_n, 1\}.$$

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In ([26] Theorem 10 and Theorem 13) it has been shown, under logical independence, the sharpness of the Fréchet-Hoeffding bounds for the prevision of conjunctions and disjunctions of n conditional events.

2.1. Multi-Criteria Decision-Making Problems

We provide a brief introduction to Multi-Criteria Decision-Making (MCDM), a branch of decision-making concerned with comparing several feasible alternatives in order to identify the optimal one, i.e., the option that best aligns with predefined goals, objectives, values, and priorities.

More precisely, for these problems there is a set of n available choices $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ called alternatives, and a set of m criteria (also called goals or attributes) $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$, that are the dimensions along which the alternatives are evaluated.

For every alternative A_i, the decision maker assigns a score a_{ij} indicating how well the criterion C_j is satisfied by A_i. The primary objective of MCDM is to find the optimal alternative A^*. Additionally, the MCDM framework introduces, for each criterion C_j, a weight $w_j \in [0,1]$ which reflects the relative importance of C_j in the decision process. Usually, these weights are normalised so that $\sum_{j=1}^{m} w_j = 1$. The weights may be a representation of the preferences of a single decision maker or a consensus formed by a panel of experts, and extensive literature exists on procedures for weight assignment and elicitation (see, e.g., [27]). Moreover, beyond selecting A^*, the MCDM framework produces a final rank x_i for each alternative A_i, for $i = 1, \ldots, n$, which summarises the overall desirability of each option or the relevance of that option within the decision problem. Such rankings are essential for helping decision makers prioritise alternatives and choose the most suitable option. A decision matrix is often used to organise the relationship between alternatives and criteria. In this type of matrix, the rows correspond to the alternatives A_i and the columns to the criteria C_j. This matrix offers a concise representation of the evaluations a_{ij} and forms the basis for applying MCDM techniques to combine performances and weights into comprehensive rankings and final recommendations.

For example, we show the so-called Weighted Sum Method (WSM), which is one of the most classical methods in MCDM. Formally, for $i = 1, \ldots, m$, let x_i be the rank of the alternative A_i such that $x_i = \sum_{j=1}^{m} a_{ij} w_j$. The alternatives A_i are arranged in descending order according to their final ranking values x_i. The optimal alternative A^* is the one that has the highest ranking value, i.e.,

$$A^* = \arg\max_{A_i} x_i.$$

3. Model and Conditional Probability Interpretation

Let X be a random quantity with set of values $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. Let $\mathcal{E}$ be a meta expert who chooses the relevant properties $\{C_1, C_2, \ldots, C_m\}$ of X, with C_i logically independent. In the framework of a (fuzzy) MCDM, we consider the set of alternatives $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, where $A_i = (X = x_i)$[1] for $i = 1, \ldots, n$, and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ the set of criteria.

To build his decision matrix, the decision maker has to set the criteria's weights w_j and the scores a_{ij} (Table 1).

For each criterion C_j, we consider the event

$$R_j = \text{“}C_j \text{ is relevant with respect to the decision problem.”}$$

Then, the decision maker sets w_j, that is, the weight of criterion C_j with respect to the decision problem, as $w_j = P(R_j)$.

Given a criterion C_j and alternative A_i, $x_i \in \mathcal{X}$, we consider the conditional event

$$E_{C_j}|A_i = \text{“}\mathcal{E} \text{ claims that } X \text{ satisfies property } C_j, \text{ knowing that } X = x_i.\text{”}$$

[1] In case of no ambiguity, $A_i = A_i^X$, for $i = 1, 2, \ldots, n$.

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In the decision-making framework, the decision maker assigns a score a_{ij} to each pair (A_i, C_j) and we interpret this quantity as $a_{ij} = P(E_{C_j}|A_i)$, that is, the conditional probability assigned to the conditional event $E_{C_j}|A_i$.

Table 1. Decision matrix for the decision maker with respect to the alternatives A_i and criteria C_j, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, where $a_{ij} = P(E_{C_j}|A_i)$.

	Criteria			
U Alternatives	w_1 C_1	w_2 C_2	$\cdots$ $\cdots$	w_m C_m
A_1	a_{11}	a_{12}	$\cdots$	a_{1m}
A_2	a_{21}	a_{22}	$\cdots$	a_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

With this interpretation, following the approach given in [28], we have the following definition.

Definition 5. *Let X be a random quantity with a set of possible values $\mathcal{X}$ and a property C_j on X. Given a coherent conditional probability P on the family $\mathcal{F}_{C_j} = \{E_{C_j}|A_i, x_i \in \mathcal{X}\}$, we define a* fuzzy set *$E^*_{C_j}$ in $\mathcal{F}_{C_j}$, with membership μ_{C_j}, the set*

$$E^*_{C_j} = \{(E_{C_j}|A_i,\ \mu_{C_j}(x_i)) : x_i \in \mathcal{X}\}, \tag{7}$$

where $\mu_{C_j}(x_i) = P(E_{C_j}|A_i)$.

Following the approach used in [28], we recall the operations of complement and conjunction of fuzzy sets, but using the current notation of Equation (7). In our framework, the negation of the event E_C, denoted by $\overline{E}_C$, is the event

"$\mathcal{E}$ claims that it is not true that X has the property C."

The sentence

"it is not true that X has the property C"

will be simply denoted as

"X has the property $\overline{C}$."

Then,

$$\overline{E}_C = E_{\overline{C}} = \text{"}\mathcal{E} \text{ claims that } X \text{ has the property } \overline{C}.\text{"}$$

Thus, the negation $\overline{E}_C|A_i$ of the conditional event $E_C|A_i$ coincides with the conditional event

$$E_{\overline{C_j}}|A_i = \text{"}\mathcal{E} \text{ claims that } X \text{ has the property } \overline{C}, \text{ knowing that } X = x_i.\text{"}$$

Definition 6. *Given a fuzzy set E^*_C in $\mathcal{F}_X$, the complement of E^*_C, denoted by $\overline{E}^*_C$, is the following fuzzy set in $\mathcal{F}_{\overline{C}}$:*

$$\overline{E}^*_C = E^*_{\overline{C}} = \{(E_{\overline{C}}|A_i, \mu_{\overline{C}}(x_i)) : x_i \in \mathcal{X}\},$$

where $\mu_{\overline{C}}(x) = P(E_{\overline{C}}|A_i)$.

In Definition 6 coherence requires that $\mu_{\overline{C}}(x_i) = P(E_{\overline{C}}|A_i) = P(\overline{E}_C|A_i) = 1 - P(E_C|A_i) = 1 - \mu_C(x_i)$, for every $x_i \in \mathcal{X}$. Therefore,

$$\overline{E}^*_C = \{(\overline{E}_C|A_i, 1 - \mu_C(x_i)) : x_i \in \mathcal{X}\}.$$

Based on Definition 1, given two fuzzy sets E_C^* and E_D^*, we define their intersection $E_{C\wedge D}^*$ as follows.

Definition 7. *Let X, Y be two random quantities and C, D two properties on X and Y, respectively. Moreover, let $E_C^* = \{(E_C|A_i^X, \mu_C(x_i)) : x_i \in \mathcal{X}\}$, with $A_i^X = (X = x_i)$, and $E_D^* = \{(E_D|A_j^Y, \mu_D(y_j)) : y_j \in \mathcal{Y}\}$, with $A_j^Y = (Y = y_j)$, be two fuzzy sets on $\mathcal{F}_C$ and $\mathcal{F}_D$, respectively. The intersection $E_{C\wedge D}^*$ of E_C^* and E_D^* is the following fuzzy set on $\mathcal{F}_{C\wedge D} = \{(E_C|A_i^X) \wedge (E_D|A_j^Y) : (x_i, y_j) \in \mathcal{X} \times \mathcal{Y}\}$:*

$$E_{C\wedge D}^* = E_C^* \wedge E_D^* = \{((E_C|A_i^X) \wedge (E_D|A_j^Y), \mu_{C\wedge D}(x_i, y_j)) : (x_i, y_j) \in \mathcal{X} \times \mathcal{Y}\},$$

where $\mu_{C\wedge D}(x_i, y_j) = \mathbb{P}[(E_C|A_i^X) \wedge (E_D|A_j^Y)]$.

For each pair $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$, the conjunction $(E_C|A_i^X) \wedge (E_D|A_j^Y)$ in Definition 7 can be interpreted as the compound conditional

"They claim that: X has the property C, knowing that $X = x_i$,
and Y has the property D, knowing that $Y = y_j$."

Definition 7 can be extended to the case of n fuzzy sets as follows.

Definition 8. *Let $X_1, \ldots, X_n$ be n random quantities and n associated properties $C_1, \ldots, C_n$, respectively. For each $i = 1, \ldots, n$, we consider the fuzzy set $E_{C_i}^* = \{(E_{C_i}|A_j^i, \mu_{C_i}(x_j^i)) : x_j^i \in \mathcal{X}_i\}$, with $A_j^i = (X_i = x_j^i)$, on $\mathcal{F}_{C_i}$. The fuzzy conjunction $E_{C_1}^* \wedge \cdots \wedge E_{C_n}^*$ is defined as*

$$\begin{aligned} &E_{C_1\wedge\cdots\wedge C_n}^* = E_{C_1}^* \wedge \cdots \wedge E_{C_n}^* = \\ &= \{\Big((E_{C_1}|A_j^1) \wedge \cdots \wedge (E_{C_n}|A_j^n), \mu_{C_1\wedge\cdots\wedge C_n}(x_j^1, \ldots, x_j^n)\Big) : (x_j^1, \ldots, x_j^n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n\}, \end{aligned}$$

where $\mu_{C_1\wedge\cdots\wedge C_n}(x_j^1, \ldots, x_j^n) = \mathbb{P}[(E_{C_1}|A_j^1) \wedge \cdots \wedge (E_{C_n}|A_j^n)]$ and $(E_{C_1}|A_j^1) \wedge \cdots \wedge (E_{C_n}|A_j^n)$ is taken as in Definition 2.

Moreover, to continue with the formalisation of set operations, we consider a new definition of the union of two fuzzy sets.

Definition 9. *Let X, Y be two random quantities and C, D two properties on X and Y, respectively. Moreover, let $E_C^* = \{(E_C|A_i^X, \mu_C(x_i)) : x_i \in \mathcal{X}\}$ and $E_D^* = \{(E_D|A_j^Y, \mu_D(y_j)) : y_j \in \mathcal{Y}\}$ be two fuzzy sets on $\mathcal{F}_C$ and $\mathcal{F}_D$, respectively. We define the union $E_{C\vee D}^*$ of the fuzzy sets E_C^* and E_D^* as the following fuzzy set on $\mathcal{F}_{C\vee D} = \{(E_C|A_i^X) \vee (E_D|A_j^Y), (x_i, y_j) \in \mathcal{X} \times \mathcal{Y}\}$:*

$$E_{C\vee D}^* = \{((E_C|A_i^X) \vee (E_D|A_j^Y), \mu_{C\vee D}(x_i, y_j)) : (x_i, y_j) \in \mathcal{X} \times \mathcal{Y}\},$$

where $\mu_{C\vee D}(x_i, y_j) = \mathbb{P}[(E_C|A_i^X) \vee (E_D|A_j^Y)]$.

For each pair $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$, the disjunction $(E_C|A_i^X) \vee (E_D|A_j^Y)$ in Definition 9 can be interpreted as the compound conditional

"They claim that: X has the property C, knowing that $X = x_i$,
or Y has the property D, knowing that $Y = y_j$."

The definition of the union of two fuzzy sets given above can be extended to the case of n fuzzy sets.

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Definition 10. *Let $X_1, \dots, X_n$ be n random quantities and n associated properties $C_1, \dots, C_n$, respectively. Given n fuzzy sets $E^*_{C_i} = \{(E_{C_i}|A^i_j, \mu_{C_i}(x^i_j)) : x^i_j \in \mathcal{X}_i\}$ with $A^i_j = (X_i = x^i_j)$ for $i = 1, \dots, n$, the fuzzy union $E^*_{C_1} \vee \cdots \vee E^*_{C_n}$ is defined as*

$$E^*_{C_1} \vee \cdots \vee E^*_{C_n} = E^*_{C_1 \vee \cdots \vee C_n} =$$
$$\{((E_{C_1}|A^1_j) \vee \cdots \vee (E_{C_n}|A^n_j), \mu_{C_1 \vee \cdots \vee C_n}(x^1_j, \dots, x^n_j)) : (x^1_j, \dots, x^n_j) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n\},$$

where $\mu_{C_1 \vee \cdots \vee C_n}(x^1_j, \dots, x^n_j) = \mathbb{P}[(E_{C_1}|A^1_j) \vee \cdots \vee (E_{C_n}|A^n_j)]$, for every $(x^1_j, \dots, x^n_j) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$.

4. Operations Among Criteria

In the framework of the decision problem, it is possible that the decision maker would modify the criteria decided by E, that is, she wants to consider the properties of X differently. For instance, if a property C_j is not considered relevant in the context of the decision problem, then it could be interesting to work with its complement $\overline{C_j}$. In the same way, it could be interesting to cluster some properties by considering their conjunction $C_j \wedge C_k$, or their disjunction $C_j \vee C_k$. For all these operations among criteria, the conditional probability interpretation of the scores helps us in finding the new scores of the modified decision matrix.

4.1. Complement

Given a decision problem, the decision maker might want to consider the negation of a property of X rather than the property itself. In this case, looking at the decision matrix, we consider the complement of a criterion C_j, and we modify the scores of the corresponding column. From Definition 6, we find the complement $\overline{E}^*_{C_j}$ of the fuzzy set $E^*_{C_j}$ related to criterion C_j, that is

$$\overline{E}^*_{C_j} = E^*_{\overline{C_j}} = \{(E_{\overline{C_j}}|A_i, \mu_{\overline{C_j}}(x_i)) : x_i \in \mathcal{X}\},$$

where $\mu_{\overline{C_j}}(x_i) = P(E_{\overline{C_j}}|A_i)$.

In the decision matrix, we add the column relative to $\overline{C_j}$ using the scores $\overline{a}_{ij} = P(E_{\overline{C_j}}|A_i) = 1 - a_{ij}$ (Table 2). Moreover, to find the corresponding criterion's weight, we consider the negation of the event R_j, that is,

$$\overline{R_j} = \text{“}C_j \text{ is not relevant with respect to the decision problem”}$$
$$= \text{“}\overline{C}_j \text{ is relevant with respect to the decision problem,”}$$

so that the weight of the new criterion $\overline{C}_j$ becomes

$$w_{\neg j} = P(\overline{R_j}) = 1 - P(R_j) = 1 - w_j. \tag{8}$$

Table 2. Decision matrix for the decision maker where the complement of criterion C_j is considered. The corresponding scores are $\overline{a}_{ij} = 1 - a_{ij}$.

	Criteria					
U	w_1	$\cdots$	w_j	$\cdots$	w_m	$1 - w_j$
Alternatives	C_1	$\cdots$	C_j	$\cdots$	C_m	$\overline{C}_j$
A_1	a_{11}	$\cdots$	a_{1j}	$\cdots$	a_{1m}	$1 - a_{1j}$
A_2	a_{21}	$\cdots$	a_{2j}	$\cdots$	a_{2m}	$1 - a_{2j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_n	a_{n1}	$\cdots$	a_{nj}	$\cdots$	a_{nm}	$1 - a_{nj}$

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4.2. Conjunction

When we consider the conjunction of two properties C_j and C_k, following Definition 7, we have that the correspondent fuzzy set is

$$E^*_{C_j \wedge C_k} = E^*_{C_j} \wedge E^*_{C_k} = \{((E_{C_j}|A_i) \wedge (E_{C_k}|A_t), \mu_{C_j \wedge C_k}(x_i, x_t)) : (x_i, x_t) \in \mathcal{X} \times \mathcal{X}\},$$

where $\mu_{C_j \wedge C_k}(x_i, x_t) = \mathbb{P}[(E_{C_j}|A_i) \wedge (E_{C_k}|A_t)]$. However, when we look at the decision matrix, we do not need to consider the conjunctions where the conditioning events are different, i.e., $A_i \neq A_t$, because different A_i represent different rows of the matrix. Then, we only have to deal with objects where $A_i = A_t$, and exploiting Definition 1 we have that

$$\begin{aligned}(E_{C_j}|A_i) \wedge (E_{C_k}|A_i) &= (E_{C_j} E_{C_k} A_i + a_{ij} \overline{A_i} E_{C_k} A_i + a_{ik} \overline{A_i} E_{C_j} A_i)|A_i \\ &= E_{C_j} E_{C_k} A_i | A_i = (E_{C_j} \wedge E_{C_k})|A_i,\end{aligned}$$

that is, in the decision matrix, we are only interested in the events

$$\begin{aligned}&(E_{C_j} \wedge E_{C_k})|A_i = (E_{C_j \wedge C_k})|A_i \\ &= \text{“}\mathcal{E} \text{ claims that } X \text{ satisfies property } C_j \text{ and } \mathcal{E} \text{ claims that } X \text{ satisfies property } C_k, \\ &\quad \text{knowing that } X = x_i\text{”} \\ &= \text{“}\mathcal{E} \text{ claims that } X \text{ satisfies property } C_j \text{ and property } C_k, \text{ knowing that } X = x_i.\text{”}\end{aligned}$$

The scores a_{ij} and a_{ik} are used to find the new scores $a_{i(j \wedge k)} = P(E_{C_j \wedge C_k}|A_i)$. Using Equation (1) we have that, by coherence, $a_{i(j \wedge k)} \in [\max\{a_{ij} + a_{ik} - 1, 0\}, \min\{a_{ij}, a_{ik}\}]$. In the decision matrix (Table 3), we add the column $C_{j \wedge k}$ relating to the intersection of C_j and C_k, and corresponding to the following fuzzy subset of $E^*_{C_j \wedge C_k}$, that is

$$E^*_{j \wedge k} = \{(E_{C_j \wedge C_k}|A_i, \mu_{C_j \wedge C_k}(x_i)) : x_i \in \mathcal{X}\},$$

where $\mu_{C_j \wedge C_k}(x_i) = P(E_{C_j \wedge C_k}|A_i)$.

For what concerns the weight of the new criterion $C_{j \wedge k}$, we consider the conjunction of the events R_j and R_k,

$$R_j \wedge R_k = R_{j \wedge k} = \text{“}C_j \wedge C_k \text{ is relevant with respect to the decision problem.”}$$

Then, using the Fréchet-Hoeffding bounds for simple events, we have that

$$w_{j \wedge k} = P(R_{j \wedge k}) = P(R_j \wedge R_k) \in [\max\{w_j + w_k - 1, 0\}, \min\{w_j, w_k\}]. \tag{9}$$

Table 3. Decision matrix for the decision maker where the conjunction $C_{j \wedge k}$ of the columns C_j and C_k is also considered. The corresponding scores are $a_{i(j \wedge k)} = P(E_{C_j \wedge C_k}|A_i)$.

	Criteria							
	w_1	$\cdots$	w_j	$\cdots$	w_k	$\cdots$	w_m	$w_{j \wedge k}$
Alternatives	C_1	$\cdots$	C_j	$\cdots$	C_k	$\cdots$	C_m	$C_{j \wedge k}$
A_1	a_{11}	$\cdots$	a_{1j}	$\cdots$	a_{1k}	$\cdots$	a_{1m}	$a_{1(j \wedge k)}$
A_2	a_{21}	$\cdots$	a_{2j}	$\cdots$	a_{2k}	$\cdots$	a_{2m}	$a_{2(j \wedge k)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_n	a_{n1}	$\cdots$	$a_{n(j \wedge k)}$	$\cdots$	$a_{n(j \wedge k)}$	$\cdots$	a_{nm}	$a_{n(j \wedge k)}$

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The previous conjunction of two criteria can be extended to the conjunction of N criteria using Definition 8. Indeed, if we consider the conjunction $C_{j_1} \wedge C_{j_2} \wedge \cdots \wedge C_{j_N}$, we have the fuzzy set

$$E^*_{C_{j_1}} \wedge \cdots \wedge E^*_{C_{j_N}} = E^*_{C_{j_1} \wedge \cdots \wedge C_{j_N}} =$$
$$\{((E_{C_{j_1}}|A_1) \wedge \cdots \wedge (E_{C_{j_N}}|A_n), \mu_{C_{j_1} \wedge \cdots \wedge C_{j_N}}(x_1, \ldots, x_n)) : (x_1, \ldots, x_n) \in \mathcal{X} \times \cdots \times \mathcal{X}\},$$

where $\mu_{C_{j_1} \wedge \cdots \wedge C_{j_N}}(x_1, \ldots, x_n) = \mathbb{P}[(E_{C_{j_1}}|A_1) \wedge \cdots \wedge (E_{C_{j_N}}|A_n)]$, for every $(x_1, \ldots, x_n) \in \mathcal{X} \times \cdots \times \mathcal{X}$. Then, for the new class $C_{j_1 \wedge j_2 \wedge \cdots \wedge j_N}$ in the decision matrix, we consider the fuzzy subset $E^*_{j_1 \wedge j_2 \wedge \cdots \wedge j_N}$ of $E^*_{C_{j_1}} \wedge \cdots \wedge E^*_{C_{j_N}}$, that is

$$E^*_{j_1 \wedge j_2 \wedge \cdots \wedge j_N} = \{((E_{C_{j_1}}|A_i) \wedge \cdots \wedge (E_{C_{j_N}}|A_i), \mu_{C_{j_1} \wedge \cdots \wedge C_{j_N}}(x_i)) : x_i \in \mathcal{X}\},$$

where $\mu_{C_{j_1} \wedge \cdots \wedge C_{j_N}}(x_i) = \mathbb{P}[(E_{C_{j_1} \wedge \cdots \wedge C_{j_N}}|A_i)]$.

In the decision matrix, for each A_i with $i = 1, \ldots, n$, the scores $a_{ij_1}, \ldots, a_{ij_N}$ corresponding to the columns $C_{j_1}, \ldots, C_{j_N}$ are used to find the new scores $a_{i(j_1 \wedge j_2 \wedge \cdots \wedge j_N)} = \mathbb{P}[(E_{C_{j_1} \wedge \cdots \wedge C_{j_N}}|A_i)]$. By coherence, using Theorem 2, it holds that

$$a_{i(j_1 \wedge j_2 \wedge \cdots \wedge j_N)} \in [\max\{a_{ij_1} + \cdots + a_{ij_N} - (N-1), 0\}, \min\{a_{ij_1}, \ldots, a_{ij_N}\}].$$

The new criterion's weight $w_{j_1 \wedge \cdots \wedge j_N}$, corresponding to the probability of the event

$$R_{j_1 \wedge \cdots \wedge j_N} = \text{“}C_{j_1} \wedge \cdots \wedge C_{j_N} \text{ is relevant with respect to the decision problem,”}$$

which, by Theorem 2, has coherent probability assessments in the interval (see [25])

$$[\max\{w_{j_1} + \cdots + w_{j_N} - (N-1), 0\}, \min\{w_{j_1}, \ldots, w_{j_N}\}].$$

4.3. Disjunction

To deal with the disjunction of two criteria C_j and C_k, following Definition 9, we have that the correspondent fuzzy set is

$$E^*_{C_j \vee C_k} = E^*_{C_j} \vee E^*_{C_k} = \{((E_{C_j}|A_i) \vee (E_{C_k}|A_t), \mu_{C_j \vee C_k}(x_i, x_t)) : (x_i, x_t) \in \mathcal{X} \times \mathcal{X}\},$$

where $\mu_{C_j \vee C_k}(x_i, x_t) = \mathbb{P}[(E_{C_j}|A_i) \vee (E_{C_k}|A_t)]$. However, on the same line as what we did for the conjunction, we do not need to consider the disjunctions where $A_i \neq A_t$, because different A_i represent different rows of the matrix. Then, we only have to deal with objects where $A_i = A_t$, and exploiting Definition 1 we have that

$$\begin{aligned}(E_{C_j}|A_i) \vee (E_{C_k}|A_i) &= ((E_{C_j}A_i \vee E_{C_k}A_i) + a_{ij}\overline{A_i}\,\overline{E}_{C_k}A_i + a_{ik}\overline{A_i}\,\overline{E}_{C_j}A_i)|A_i \\ &= (E_{C_j}A_i \vee E_{C_k}A_i)|A_i \\ &= (E_{C_j} \vee E_{C_k})|A_i,\end{aligned}$$

that is, in the decision matrix, we are only interested in the events

$$\begin{aligned}&(E_{C_j} \vee E_{C_k})|A_i = (E_{C_j \vee C_k})|A_i \\ &= \text{“}\mathcal{E} \text{ claims that } X \text{ satisfies property } C_j \text{ or } \mathcal{E} \text{ claims that } X \text{ satisfies property } C_k, \\ &\quad \text{knowing that } X = x_i\text{”} \\ &= \text{“}\mathcal{E} \text{ claims that } X \text{ satisfies property } C_j \text{ or property } C_k, \text{ knowing that } X = x_i.\text{”}\end{aligned}$$

The new scores $a_{i(j \vee k)} = P(E_{C_j \vee C_k}|A_i)$ are obtained using a_{ij}, a_{ik}, and Equation (6). By coherence, $a_{i(j \vee k)} \in [\max\{a_{ij}, a_{ik}\}, \min\{a_{ij} + a_{ik}, 1\}]$. In the decision matrix, we add the column $C_{j \vee k}$ relating to

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the union of the columns C_j and C_k (Table 4), and corresponding to the following fuzzy subset of $E^*_{C_j \vee C_k}$, that is

$$E^*_{j\vee k} = \{(E_{C_j \vee C_k}|A_i, \mu_{C_j \vee C_k}(x_i)) : x_i \in \mathcal{X}\},$$

where $\mu_{C_j \vee C_k}(x_i) = P(E_{C_j \vee C_k}|A_i)$.

For what concerns the weight of the new criterion $C_{j\vee k}$, we consider the disjunction of the events R_j and R_k,

$$R_j \vee R_k = R_{j\vee k} = \text{“}C_j \vee C_k \text{ is relevant with respect to the decision problem.”}$$

Then, using the Fréchet-Hoeffding bounds for simple events, we have that

$$w_{j\vee k} = P(R_{j\vee k}) = P(R_j \vee R_k) \in [\max\{w_j, w_k\}, \min\{w_j + w_k, 1\}]. \quad (10)$$

Table 4. Decision matrix for the decision maker, where the disjunction $C_{j\vee k}$ of the columns C_j and C_k is added. The corresponding scores are $a_{i(j\vee k)} = P(E_{C_j \vee C_k}|A_i)$.

	Criteria							
	w_1	$\cdots$	w_j	$\cdots$	w_k	$\cdots$	w_m	$w_{j\vee k}$
Alternatives	C_1	$\cdots$	C_j	$\cdots$	C_k	$\cdots$	C_m	$C_{j\vee k}$
A_1	a_{11}	$\cdots$	a_{1j}	$\cdots$	a_{1k}	$\cdots$	a_{1m}	$a_{1(j\vee k)}$
A_2	a_{21}	$\cdots$	a_{2j}	$\cdots$	a_{2k}	$\cdots$	a_{2m}	$a_{2(j\vee k)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_n	a_{n1}	$\cdots$	a_{nj}	$\cdots$	a_{nk}	$\cdots$	a_{nm}	$a_{n(j\vee k)}$

The disjunction of two criteria can be extended to the disjunction of N criteria using Definition 10. Indeed, if we consider the disjunction $C_{j_1} \vee C_{j_2} \vee \cdots \vee C_{j_N}$, we have the fuzzy set

$$E^*_{C_{j_1}} \vee \cdots \vee E^*_{C_{j_N}} = E^*_{C_{j_1} \vee \cdots \vee C_{j_N}} =$$
$$\{((E_{C_{j_1}}|A_1) \vee \cdots \vee (E_{C_{j_N}}|A_n), \mu_{C_{j_1} \vee \cdots \vee C_{j_N}}(x_1, \ldots, x_n)) : (x_1, \ldots, x_n) \in \mathcal{X} \times \cdots \times \mathcal{X}\},$$

where $\mu_{C_{j_1} \vee \cdots \vee C_{j_N}}(x_1, \ldots, x_n) = \mathbb{P}[(E_{C_{j_1}}|A_1) \vee \cdots \vee (E_{C_{j_N}}|A_n)]$, for every $(x_1, \ldots, x_n) \in \mathcal{X} \times \cdots \times \mathcal{X}$. Then, for the new class $C_{j_1 \vee j_2 \vee \cdots \vee j_N}$ in the decision matrix, we consider the fuzzy subset $E^*_{j_1 \vee j_2 \vee \cdots \vee j_N}$ of $E^*_{C_{j_1}} \vee \cdots \vee E^*_{C_{j_N}}$, that is

$$E^*_{j_1 \vee j_2 \vee \cdots \vee j_N} = \{((E_{C_{j_1}}|A_i) \vee \cdots \vee (E_{C_{j_N}}|A_i), \mu_{C_{j_1} \vee \cdots \vee C_{j_N}}(x_i)) : x_i \in \mathcal{X}\},$$

where $\mu_{C_{j_1} \vee \ldots \vee C_{j_N}}(x_i) = \mathbb{P}[(E_{C_{j_1} \vee \cdots \vee C_{j_N}}|A_i)]$.

For each $i = 1, \ldots, n$, the new score $a_{i(j_1 \vee j_2 \vee \cdots \vee j_N)} = \mathbb{P}[(E_{C_{j_1} \vee \cdots \vee C_{j_N}}|A_i)]$ is built using the scores $a_{ij_1}, a_{ij_2}, \ldots, a_{ij_N}$ of the original decision matrix, and by Theorem 3, the coherent assessment are given by the Fréchet-Hoeffdings bounds, i.e.,

$$a_{i(j_1 \vee j_2 \vee \cdots \vee j_N)} \in [\max\{a_{ij_1}, \ldots, a_{ij_N}\}, \min\{a_{ij_1} + \cdots + a_{ij_N}, 1\}].$$

The new criterium's weight $w_{j_1 \vee \cdots \vee j_N}$ will correspond to the probability of the event

$$R_{j_1 \vee \cdots \vee j_N} = \text{“}C_{j_1} \vee \cdots \vee C_{j_N} \text{ is relevant with respect to the decision problem.”}$$

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Then by Theorem 3, the coherent probability assessments on $w_{j_1 \vee \cdots \vee j_N}$ are in the interval

$$[\max\{w_{j_1}, \ldots, w_{j_N}\}, \min\{w_{j_1} + \cdots + w_{j_N}, 1\}].$$

4.4. De Morgan's Laws and Sum Rule

As recalled in Section 2, for the conjunction and the disjunction of conditionals in Definition 1 and Definition 3, respectively, both De Morgan's laws and the prevision sum rule are satisfied. Then, these relations are also valid for the conjunction and disjunction of criteria described in Section 4.2 and Section 4.3. Exploiting Equation (2), it holds that

$$\overline{(E^*_{C_j} \vee E^*_{C_k})} = \overline{E}^*_{C_j} \wedge \overline{E}^*_{C_k}.$$

Then, the fuzzy subset $\overline{E}^*_{j\vee k}$ of $\overline{(E^*_{C_j} \vee E^*_{C_k})}$ we need for the decision matrix is

$$\overline{E}^*_{j\vee k} = \{(E_{\overline{C_j \vee C_k}} | A_i, \mu_{\overline{C_j \vee C_k}}(x_i)) : x_i \in \mathcal{X}\}$$

and it corresponds to the new criterion $\overline{C_j \vee C_k} = \overline{C}_{j\vee k}$.

For $i = 1, 2, \ldots, n$, exploiting the relations and the lower-upper bounds found in the previous Section 4.1, Section 4.2, and Section 4.3, the scores of $\overline{C}_{j\vee k}$ can be determined as $\overline{a}_{i(j\vee k)} = 1 - a_{i(j\vee k)} \in [\overline{a}'_{i(j\vee k)}, \overline{a}''_{i(j\vee k)}]$, where, by coherence,

$$\begin{aligned} \overline{a}'_{i(j\vee k)} &= 1 - \min\{a_{ij} + a_{ik}, 1\} = \max\{1 - a_{ij} - a_{ik}, 0\}, \\ \overline{a}''_{i(j\vee k)} &= 1 - \max\{a_{ij}, a_{ik}\} = \min\{1 - a_{ij}, 1 - a_{ik}\}. \end{aligned}$$

Applying the same argument for the weights, using Equation (8) and Equation (10), the new weight will be $\overline{w}_{j\vee w} = 1 - w_{j\vee w} \in [\overline{w}'_{j\vee k}, \overline{w}''_{j\vee k}]$, where

$$\begin{aligned} \overline{w}'_{j\vee k} &= 1 - \min\{w_j + w_k, 1\} = \max\{1 - w_j - w_k, 0\}, \\ \overline{w}''_{j\vee k} &= 1 - \max\{w_j, w_k\} = \min\{1 - w_j, 1 - w_k\}. \end{aligned}$$

In the same way, for the other De Morgan's relation in Equation (3), it holds that

$$\overline{(E^*_{C_j} \wedge E^*_{C_k})} = \overline{E}^*_{C_j} \vee \overline{E}^*_{C_k}.$$

Then, the fuzzy subset $\overline{E}^*_{j\wedge k}$ of $\overline{(E^*_{C_j} \wedge E^*_{C_k})}$ we need for the decision matrix is

$$\overline{E}^*_{j\wedge k} = \{(E_{\overline{C_j \wedge C_k}} | A_i, \mu_{\overline{C_j \wedge C_k}}(x_i)) : x_i \in \mathcal{X}\}$$

and it corresponds to the new class $\overline{C_j \wedge C_k} = \overline{C}_{j\wedge k}$.

For $i = 1, \ldots, n$, the new scores of $\overline{C}_{j\vee k}$ can be determined as follows $\overline{a}_{i(j\wedge k)} = 1 - a_{i(j\wedge k)} \in [\overline{a}'_{i(j\wedge k)}, \overline{a}''_{i(j\wedge k)}]$, where

$$\begin{aligned} \overline{a}'_{i(j\wedge k)} &= 1 - \min\{a_{ij}, a_{ik}\} = \max\{1 - a_{ij}, 1 - a_{ik}\}, \\ \overline{a}''_{i(j\wedge k)} &= 1 - \max\{a_{ij} + a_{ik} - 1\} = \min\{2 - a_{ij} - a_{ik}, 1\}. \end{aligned}$$

Exploiting Equation (8) and Equation (9), the new weight will be $\overline{w}_{j\wedge k} = 1 - w_{j\wedge k} \in [\overline{w}'_{j\wedge k}, \overline{w}''_{j\wedge k}]$, where

$$\begin{aligned} \overline{w}'_{j\wedge k} &= 1 - \min\{w_j, w_k\} = \max\{1 - w_j, 1 - w_k\}, \\ \overline{w}''_{j\wedge k} &= 1 - \max\{w_j + w_k - 1, 0\} = \min\{2 - w_j - w_k, 1\}. \end{aligned}$$

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Moreover, the sum rule (Equation (4)) is satisfied. That is, given two criteria C_j and C_k, between the correspondent fuzzy sets, $E^*_{C_j}$ and $E^*_{C_k}$, it holds that

$$E^*_{C_j} + E^*_{C_k} = E^*_{C_j} \wedge E^*_{C_k} + E^*_{C_j} \vee E^*_{C_k}$$

and the same relation holds for taking the fuzzy subsets needed for the decision matrix

$$E^*_{C_j} + E^*_{C_k} = E^*_{j\wedge k} + E^*_{j\vee k}.$$

For every i, we have the following relation among the scores

$$a_{ij} + a_{ik} = a_{i(j\wedge k)} + a_{i(j\vee k)}$$

and the weights

$$w_j + w_k = w_{j\wedge k} + w_{j\vee k}.$$

4.5. Adding or Removing a Criterion

In the case the decision maker decides that a certain criterion C_j is not relevant for the decision problem, then this criterion can be removed. Removing a criterion from the decision matrix corresponds to simply deleting a column, and it has no other impacts on the decision matrix.
If the decision maker decides that it is necessary to consider properties of X that are not present in the set $\mathcal{C}$, it is possible to add a criterion. Let us suppose to consider a new property C_{m+1} of X, with no logical relation with $C_1, C_2, \ldots, C_m$. Using this criterion in the decision problem corresponds to adding a new column. The decision maker must assess the new scores $a_{i(m+1)}$, for $i = 1, \ldots, n$, that is, she must give a coherent probability evaluation on the events

$$\text{"}\mathcal{E} \text{ claims that } X \text{ satisfies property } C_{m+1}, \text{ knowing that } X = x_i,\text{"} \quad \text{for } i = 1, \ldots, n.$$

Moreover, she has to assign a weight w_{m+1} to the new criterion C_{m+1}, by assess a probability on the event $R_{m+1} =$"C_{m+1} is relevant with respect to the decision problem".

5. Conclusions and Future Works

In this work, we adopted an MCDM framework in which a single expert is responsible for selecting both the criteria and the alternatives associated with a given decision problem. More specifically, we focused on a fuzzy MCDM setting, where the criteria are modelled as fuzzy sets, and the entries of the decision matrix represent the degree of membership of an alternative A_i with respect to a criterion C_j. This modelling choice (interpreting the scores as membership degrees) was originally introduced in [13,29] in the context of aggregating fuzzy ontologies.

Furthermore, following [11], we adapted the fuzzy interpretation of the scores in the decision matrix, viewing them as coherent conditional probabilities. This interpretation allowed us to model operations among criteria exploiting the complements, intersections, and unions of the fuzzy sets representing the corresponding criteria. Thus, a decision maker can approach the decision problem iteratively by suitably modifying the initial decision matrix. Therefore, the final decision matrix will result from the decision maker applying logical operations to the initial list of criteria. This final decision matrix retains all the original criteria, as well as those that have been combined and/or complemented by the decision maker, thus the number of columns is augmented accordingly.

This methodological approach enables decision makers to refine an initial set of criteria in a manner that is both introspectable and explainable. Indeed, since each criterion is retained in the whole process, every step (union, intersection, and complement) is introspectable and fully explainable. This feature makes it possible to apply this framework to applications in the context of Explainable Artificial Intelligences (XAIs), such as [30].

In future works, we will consider operations on the rows of the decision matrix, which corresponds to operations on the alternatives, e.g., aggregating two or more alternatives by using logical operations. Moreover, the operations described in this work could be extended to the case of an MCGDM, where multiple experts are considered, and applied to the algorithm proposed in [13].

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Abbreviations

The following abbreviations are used in this manuscript:

MCDM	Multi-Criteria Decision-Making
MCGDM	Multi-Criteria Group Decision-Making
WSM	Weighted Sum Method
XAI	Explainable Artificial Intelligences

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