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Not peer-reviewed version

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Posted Date: 16 July 2025

doi: 10.20944/preprints202507.1148.v1

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Article

Generalized Right Weighted Core Inverse in Banach Algebras

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Abstract

In this paper, we introduce the generalized right weighted core inverse, defined via the canonical polar decomposition using a right weighted core invertible element and a quasinilpotent. We present various characterizations of this new generalized inverse with weights and utilize these results to establish new properties of the right pseudo core inverse and the weighted core-EP inverse.

Keywords: core-EP inverse; right core inverse, weighted pseudo core inverse; weighted g-Drazin inverse; generalized weighted core inverse; Banach algebra

MSC: 16U50; 15A09; 16W10

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1. Introduction

A Banach algebra is called a Banach *-algebra if there exists an involution $*: x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda} x^*, (xy)^* = y^*x^*, (x^*)^* = x$. An element a in a Banach *-algebra $\mathcal{A}$ has core inverse if and only if there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by $a^{\#\!\!\!\bigcirc}$. Core inverse is extensively studied by many authors from different views, e.g., [1,6,21,25].

Let $\mathbb{C}^{n\times n}$ be the Banach *-algebra of all $n \times n$ complex matrices with conjugate transpose $*$ and $\mathcal{R}(X)$ represent the range space of a complex matrix X. In 2014, Prasad and Mohana extended core inverse and introduced core-EP inverse for a complex matrix (see [17]). A matrix $A \in C^{n\times n}$ has core-EP inverse X if and only if

$$XAX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

where $k = ind(A)$ is the Drazin index of A. Such X is unique, and we denote it by $A^{\textcircled{\dagger}}$.

In 2020, Gao et al. extended the concept of the core-EP inverse and introduced the notion of weighted core-EP inverse for a complex matrix (see [9]). Let $A, W \in \mathbb{C}^{n\times n}$ and $k = max\{ind(AW), ind(WA)\}$. The weighted core-EP inverse of A is the unique solution to the system:

$$WAWX = (WA)^k[(WA)^k]^\dagger, \mathcal{R}(X) \subseteq \mathcal{R}((AW)^k),$$

and we denote such X by $A^{\textcircled{\dagger},W}$. Then Mosić introduced and studied weighted core-EP inverse for a bounded linear operator between two Hilbert spaces as a generalization of the weighted core-EP inverse of a matrix (see [14]). In 2021, Mosić further extended the weighted core-EP inverse of bounded linear operators on Hilbert spaces to elements of a C^*-algebra and the weighted core-EP inverse in a C^*-algebra was characterized by means of range projections (see [15]).

Wang et al. generalized the core inverse to the right core inverse (see [18]). An element $a \in \mathcal{A}$ has right core inverse if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, axa = a, (ax)^* = ax.$$

If such x exists, we denote it by $a_r^{\#}$.

An element $a \in \mathcal{A}$ has right pseudo core if there exists a $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, a^k = axa^k$$

for some $k \in \mathbb{N}$ (see [18]).

In [5], the authors introduced and studied generalized right core inverse. An element $a \in \mathcal{A}$ has generalized right core inverse if there exists a $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

The preceding x is called generalized right core inverse of a and we denote it by $a_r^{\textcircled{d}}$. We refer the reader more properties of right generalized inverse in [18,19,22,27].

Recently, many authors studied generalized inverse with wights (see [2,8,9,12,16,20,23,24]). In [28], Zhu et al. introduced and studied a weighted generalized inverse as a generalization of core inverse. Let $a, w \in \mathcal{A}$. An element $a \in \mathcal{A}$ is w-core invertible if there exists some $x \in \mathcal{A}$ such that

$$awx^2 = x, (awx)^* = awx, xawa = a.$$

Such an x is called a w-core inverse of a. Many properties of w-core inverse are presented by many authors, e.g., [26,27]. In [29], Zhu et al. extended w-core inverse and introduce right w-core inverse. An element $a \in \mathcal{A}$ has right w-core inverse if there exists $x \in \mathcal{A}$ such that

$$awx^2 = x, (awx)^* = awx, awxa = a.$$

Many properties of right w-core inverse are investigated in [29]. However, the right w-core inverse is unrelated to the weighted core-EP inverse mentioned above. The motivation of this paper is to introduce and study a new class of weighted generalized inverses, such that the class of weighted core-EP inverses can be viewed as a subclass of this new class.

Definition 1.1. *An element $a \in \mathcal{A}$ has generalized right w-weighted core inverse if there exists a $x \in \mathcal{A}$ such that*

$$a(wx)^2 = x, (wawx)^* = wawx, \lim_{n\to\infty} ||(aw)^n - (aw)(xw)(aw)^n||^{\frac{1}{n}} = 0.$$

The preceding x is denoted by $a_r^{\textcircled{d},w}$. It is evident that for any complex matrix, the generalized right weighted core inverse coincides with the weighted core-EP inverse. Consequently, many properties of the weighted core-EP inverse naturally extend to this broader class.

In Section 2, we introduce right w-weighted core inverse for an element in a Banach *-algebra.

Definition 1.2. *An element $a \in \mathcal{A}$ has right w-weighted core inverse if there exists $x \in \mathcal{A}$ such that*

$$a(wx)^2 = x, (wawx)^* = wawx, awxwa = a.$$

If such x exists, we denote it by $a_r^{\#,w}$. The investigation explores several elementary properties of the right w-weighted core inverse, which will be utilized subsequently.

In Section 3, we characterize generalized right weighted core inverse by combining right w-weighted core inverse and quasinilpotent in a Banach *-algebra. We prove that $a \in \mathcal{A}$ has generalized right w-core inverse if and only if there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, (wz)^*(wy) = ywz = 0, z \in \mathcal{A}_r^{\textcircled{\#},w}, y \in \mathcal{A}_w^{qnil}.$$

Here, $\mathcal{A}_w^{qnil} = \{x \in \mathcal{A} \mid \lim\limits_{n\to\infty} ||(xw)^n||^{\frac{1}{n}} = 0$. Surprisingly, we observe that the preceding condition $(wz)^*(wy) = 0$ can be dropped, which add some new properties for the weighted core-EP inverse as well. The polar-like property of the generalized right weighted core inverse is thereby established.

An element $a \in \mathcal{A}$ has right wg-Drazin inverse x if there exists a $x \in \mathcal{A}$ such that

$$a(wx)^2 = x = xwawx, aw - xw(aw)^2 \in \mathcal{A}^{qnil}.$$

We denote such a x by $a_r^{d,w}$. The right wg-Drazin inverse is an one-sided version of weighted generalized Drazin inverse (see [13]). In Section 4, we characterize generalized right weighted core inverse by using the right wg-Drazin inverse. The generalized right weighted core inverse is then constructed based on the right weighted core inverse.

Definition 1.3. *An element $a \in \mathcal{A}$ has right pseudo w-core inverse if there exist $x \in \mathcal{A}$ such that*

$$a(wx)^2 = x, (wawx)^* = wawx, (aw)^k = (aw)(xw)(aw)^k$$

for some $k \in \mathbb{N}$.

Finally, in Section 5, we investigate the right pseudo w-core inverse for elements in a Banach $*$-algebra by using the generalized right w-core inverse. This approach allows us to present many new properties of the weighted core-EP inverse.

Throughout the paper, all Banach *-algebras are complex with an identity. $\mathcal{A}_r^{d,w}, \mathcal{A}_r^{D,w}, \mathcal{A}_r^{\textcircled{\#},w}, \mathcal{A}_r^{\textcircled{d},w}$ and $\mathcal{A}_r^{\textcircled{D},w}$ denote the sets of all right weighted g-Drazin invertible, right weighted Drazin invertible, right weighted core invertible, generalized right weighted core invertible and generalized right pseudo w-core invertible elements in $\mathcal{A}$, respectively.

2. Right w-Weighted Core Inverse

The objective of this section is to elucidate the fundamental properties of the right w-weighted core inverse. If a, x and w satisfy the equations $a = awxwa$ and $(wawx)^* = wawx$, then x is called w-weighted $(1,3)$-inverse of a and is denoted by $a_w^{(1,3)}$. We use $\mathcal{A}_w^{(1,3)}$ to stand for sets of all w-weighted $(1,3)$-invertible elements in $\mathcal{A}$.

Theorem 2.1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{\#},w}$.
(2) There exists some $x \in \mathcal{A}$ such that

$$xwawx = x = a(wx)^2, awxwa = a, (wawx)^* = wawx.$$

(3) $a \in \mathcal{A}_w^{(1,3)}$ and $a\mathcal{A} = awa\mathcal{A}$.

Proof. $(1) \Rightarrow (2)$ By hypothesis, there exists some $z \in \mathcal{A}$ such that

$$a(wz)^2 = z, (wawz)^* = wawz, awzwa = a.$$

Set $x = zwawz$. Then we verify that

$$\begin{aligned} awx &= (awzwa)wz = awz, \\ a(wx)^2 &= [a(wz)^2]wawz = zwawz = x, \\ xwawx &= zwawzw(awz) = zw(awzwa)wz = zwawz = x, \\ awxwa &= (awz)wa = a, \\ (wawx)^* &= (wawz)^* = wawz = wawx. \end{aligned}$$

as desired.

(2) $\Rightarrow$ (1) This is trivial.

(1) $\Rightarrow$ (3) By hypothesis, there exists some $x \in \mathcal{A}$ such that

$$a(wx)^2 = x, (wawx)^* = wawx, awxwa = a.$$

Then $wawxwa = wa$. Hence, $a \in \mathcal{A}_w^{(1,3)}$.

Clearly, $awa \in a\mathcal{A}$. On the other hand, we have $a = (aw)x(wa) = aw[a(wx)^2]wa \in awa\mathcal{A}$. Therefore $a\mathcal{A} = awa\mathcal{A}$.

(3) $\Rightarrow$ (1) Since $a \in \mathcal{A}_w^{(1,3)}$, we can find some $x \in \mathcal{A}$ such that $wawxwa = wa$ and $(wawx)^* = wawx$. Write $a = awat$ for some $t \in \mathcal{A}$. Set $x = atwa_w^{(1,3)}$. Then we verify that

$$\begin{aligned} awx &= (awat)wa_w^{(1,3)} = awa_w^{(1,3)}, \\ (wawx)^* &= (wawa_w^{(1,3)})^* = wawa_w^{(1,3)} = wawx, \\ a(wx)^2 &= (awx)(wx) = (awa_w^{(1,3)})w(atwa_w^{(1,3)}) \\ &= (awa_w^{(1,3)}wa)twa_w^{(1,3)} = atwa_w^{(1,3)} = x, \\ awxwa &= (awx)wa = awa_w^{(1,3)}wa = a. \end{aligned}$$

Therefore $a \in \mathcal{A}_r^{\text{\#},w}$. $\square$

Corollary 2.2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\text{\#},w}$.

(2) $\mathcal{A}a = \mathcal{A}(wa)^*(wa)$ and $a\mathcal{A} = awa\mathcal{A}$.

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 2.1, $a\mathcal{A} = awa\mathcal{A}$. By hypothesis, $wa \in \mathcal{A}^{(1,3)}$. In light of [7, Lemma2.3], we have $\mathcal{A}wa = \mathcal{A}(wa)^*(wa)$. As $a = awa_r^{\text{\#},w}wa$, we see that $\mathcal{A}wa = \mathcal{A}a$, and then $\mathcal{A}a = \mathcal{A}(wa)^*(wa)$.

(2) $\Rightarrow$ (1) Write $a = t(wa)^*(wa)$ for a $t \in \mathcal{A}$. Then $wa = (wt)(wa)^*(wa)$, and so $(wa)(wt)^* = (wt)(wa)^*(wa)(wt)^*$. Moreover, $(wa)^* = (wa)^*(wa)(wt)^*$; hence, $(wa)^*(wa) = (wa)^*(wa)(wt)^*(wa)$. Thus, $t(wa)^*(wa) = t(wa)^*(wa)(wt)^*\,(wa)$. Since $a\mathcal{A} = awa\mathcal{A}$, we can find a $z \in \mathcal{A}$ such that $a = awaz$. Then we derive

$$\begin{aligned} a &= t(wa)^*(wa) = t(wa)^*(wa)(wt)^*(wa) \\ &= a(wt)^*(wa) = awaz(wt)^*wa = awxwa, \end{aligned}$$

where $x = az(wt)^*$. Clearly, $wawx = wawaz(wt)^* = wa(wt)^* = (wt)(wa)^*(wa)\,(wt)^*$. This implies that $(wawx)^* = wawx$. Accordingly, $a \in \mathcal{A}_w^{(1,3)}$. This completes the proof by Theorem 2.1. $\square$

Theorem 2.3. *Let $a \in \mathcal{A}_r^{\text{\#},w}$. Then there exists a projection $p \in \mathcal{A}$ such that*

$$pwa = 0 \text{ and } wa + p \in \mathcal{A}_r^{-1}.$$

Proof. Let $x = a_r^{\text{\#},w}$. Then

$$a(wx)^2 = x = xwawx, (wawx)^* = wawx, awxwa = a.$$

Let $p = 1 - wawx$. Then $p^2 = p = p^* \in \mathcal{A}$ and $pwa = 0$. We directly check that

$$\begin{array}{rl} & [wa + 1 - wawx][wx + 1 - wawx] \\ = & (wa)(wx) + (wa)(1 - wawx) + (1 - wawx)(wx) + (1 - wawx)^2 \\ = & (wa)(wx) + (wa)(1 - wawx) + wx - wa(wx)^2 + (1 - wawx) \\ = & 1 + (wa)(1 - wawx). \end{array}$$

By hypothesis, we verify that

$$[1 + wa(1 - wawx)][1 - (wa)(1 - wawx)] = 1.$$

Then

$$[wa + 1 - wawx][wx + 1 - wawx][1 - (wa)(1 - wawx)] = 1.$$

Thus, $wa + 1 - wawx \in \mathcal{A}$ is right invertible, as required. $\square$

Corollary 2.4. *Let $a \in \mathcal{A}, w \in \mathcal{A}^{-1}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\circledast, w}$.

(2) *There exists a unique projection $p \in \mathcal{A}$ such that*

$$pwa = 0 \text{ and } wa + p \in \mathcal{A}_r^{-1}.$$

(3) *There exists a projection $p \in \mathcal{A}$ such that*

$$pwa = 0 \text{ and } wa + p \in \mathcal{A}_r^{-1}.$$

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 2.3, here exists a projection $p \in \mathcal{A}$ such that

$$pwa = 0 \text{ and } wa + p \in \mathcal{A}_r^{-1}.$$

Assume that there exists a projection $q \in \mathcal{A}$ such that

$$qwa = 0 \text{ and } wa + q \in \mathcal{A}_r^{-1}.$$

Then $(1 - p)wa = wa$, and so $\ell(1 - p) \subseteq \ell(wa)$. On the other hand, $wa = (1 - p)(wa + p)$, and then $1 - p = wa(wa + p)_r^{-1}$. This implies that $\ell(wa) = \ell(1 - p)$. Thus, we have $\ell(1 - p) = \ell(wa)$. Likewise, we have $\ell(1 - p) = \ell(wa)$. It follows that $\ell(1 - p) = \ell(1 - q)$. This implies that $p = pq$ and $q = qp$. Accordingly, $p = pq = p^*q^* = (qp)^* = q^* = q$, as required.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) By hypothesis, there exists a projection $p \in \mathcal{A}$ such that

$$pwa = 0, (wa + p)s = 1 \text{ and } ps(1 - p) = 0.$$

Since $(1 - p)(wa + p) = (1 - p)wa = wa$, we see that $1 - p = was = was(1 - p)$. Since $s(1 - p) = (1 - p)s(1 - p) \in w\mathcal{A}$, we may write $s(1 - p) = wx$ for some $x \in \mathcal{A}$. Then $wawx = was(1 - p) = 1 - p$; and so $(wawx)^* = (1 - p)^* = 1 - p = wawx$. Obviously, we have $awxwa = (1 - p)wa = wa$, and so $awxwa = a$.

Since $p(wa + p) = p$, we see that $p = ps$ and $(1 - p)s = s - ps = s - p$. Thus,

$$\begin{array}{rcl} a(wx)^2 & = & awxwx = (1 - p)wx = (1 - p)(wa + p)_r^{-1}(1 - p) \\ & = & [(wa + p)_r^{-1} - p](1 - p) = w[w^{-1}(wa + p)_r^{-1}(1 - p)] = x. \end{array}$$

As $w \in \mathcal{A}^{-1}$, we deduce that $a(wx)^2 = x$. Therefore $a_r^{\circledast} = x$, as asserted. $\square$

Corollary 2.5. *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{\#}}$.

(2) *There exists a projection $p \in \mathcal{A}$ such that*

$$pa = 0 \in \mathcal{A}^{qnil}, \text{ and } a^n + p \in \mathcal{A}_r^{-1}.$$

Proof. Straightforward by choosing $w = 1$ in Theorem 2.3. □

Let $p, q \in \mathcal{A}$ be idempotents. Then for any $x \in \mathcal{A}$, we have $x = pxq + px(1-p) + (1-p)xq + (1-p)x(1-q)$. Thus x can be represented in the matrix form $x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-q) \end{pmatrix}_{(p,q)}$. We have at our disposal all the information necessary to prove the following.

Theorem 2.6. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a_r^{\textcircled{\#},w} = x$.

(2) There exist an idempotent $p \in \mathcal{A}$ and a projection $q \in \mathcal{A}$ such that

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}_{p \times q}, w = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q \times p}$$

and x is represented as

$$x = \begin{pmatrix} (w_1 a_1 w_1)_r^{-1} & x_2 \\ 0 & 0 \end{pmatrix}_{p \times q},$$

where $w_1 a_1 w_1 x_2 = 0, a_1 w_1 (w_1 a_1 w_1)^{-1} w_1 x_2 = x_2$.

Proof. (1) $\Rightarrow$ (2) Let $p = (aw)(a_r^{\textcircled{\#},w} w)$ and $q = (wa)(wa_r^{\textcircled{\#},w})$. Then we verify that

$$\begin{aligned} (1-p)a &= [1 - (aw)(a_r^{\textcircled{\#},w} w)]a \\ &= a - awa_r^{\textcircled{\#},w} wa = 0. \end{aligned}$$

and

$$\begin{aligned} (1-q)wp &= [1 - (wa)(wa_r^{\textcircled{\#},w})]w(aw)(a_r^{\textcircled{\#},w} w) \\ &= w[a - awa_r^{\textcircled{d},w})wa]wa_r^{\textcircled{d},w} w \\ &= 0. \end{aligned}$$

Hence, we may write

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}_{p \times q}, w = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q \times p}.$$

Then we have

$$aw = \begin{pmatrix} a_1 w_1 & a_1 w_2 + a_2 w_3 \\ 0 & 0 \end{pmatrix}_{p \times p}, wa = \begin{pmatrix} w_1 a_1 & w_1 a_2 \\ 0 & 0 \end{pmatrix}_{q \times q}.$$

Since $(1-p)x = [1 - (aw)(a_r^{\textcircled{d},w} w)]a_r^{\textcircled{d},w} = 0$, we may write $x = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}_{p \times q}$.

We compute that

$$\begin{aligned} wawx &= \begin{pmatrix} w_1a_1 & w_1a_2 \\ 0 & 0 \end{pmatrix}_{q\times q} \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q\times p} \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \\ &= \begin{pmatrix} w_1a_1 & w_1a_2 \\ 0 & 0 \end{pmatrix}_{q\times q} \begin{pmatrix} w_1x_1 & w_1x_2 \\ 0 & 0 \end{pmatrix}_{q\times q} \\ &= \begin{pmatrix} w_1a_1w_1x_1 & w_1a_1w_1x_2 \\ 0 & 0 \end{pmatrix}_{q\times p}. \end{aligned}$$

Since $(wawx)^* = wawx$, we have $w_1a_1w_1x_2 = 0$.

It is easy to verify that

$$\begin{aligned} & (w_1a_1w_1)x_1 \\ = & [(wawx)w(awxw)][(awxw)a(wawx)][(wawx)w(awxw)] \\ & [(awxw)x(wawx)] = wawx = q. \end{aligned}$$

Therefore

$$x = \begin{pmatrix} (w_1a_1w_1)_r^{-1} & x_2 \\ 0 & 0 \end{pmatrix}_{p\times p},$$

as required.

$(2) \Rightarrow (1)$ By hypothesis, we have an idempotent $p \in \mathcal{A}$ and a projection $q \in \mathcal{A}$ such that

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}_{p\times q}, w = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q\times p}$$

and x is represented as

$$x = \begin{pmatrix} (w_1a_1w_1)_r^{-1} & x_2 \\ 0 & 0 \end{pmatrix}_{p\times q},$$

where $w_1a_1w_1x_2 = 0, a_1w_1(w_1a_1w_1)^{-1}w_1x_2 = x_2$. Then we verify that

$$\begin{aligned} awx &= \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q\times p} \begin{pmatrix} (w_1a_1w_1)_r^{-1} & x_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \\ &= \begin{pmatrix} a_1w_1 & a_1w_2 + a_2w_3 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} (w_1a_1w_1)_r^{-1} & x_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \\ &= \begin{pmatrix} a_1w_1(w_1a_1w_1)_r^{-1} & a_1w_1x_2 \\ 0 & 0 \end{pmatrix}_{p\times q}, \\ a(wx)^2 &= \begin{pmatrix} a_1w_1(w_1a_1w_1)_r^{-1} & a_1w_1x_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1(w_1a_1w_1)_r^{-1} & w_1x_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \\ &= \begin{pmatrix} a_1w_1(w_1a_1w_1)_r^{-1}w_1(w_1a_1w_1)_r^{-1} & a_1w_1(w_1a_1w_1)_r^{-1}w_1x_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \\ &= \begin{pmatrix} (w_1a_1w_1)_r^{-1} & x_2 \\ 0 & 0 \end{pmatrix}_{p\times q}. \end{aligned}$$

 doi:10.20944/preprints202507.1148.v1

Moreover, we have

$$\begin{aligned} wawx &= \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q\times p} \begin{pmatrix} a_1w_1(w_1a_1w_1)_r^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \\ &= \begin{pmatrix} w_1a_1w_1(w_1a_1w_1)_r^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{q\times q} \\ &= q = q^* = (wawx)^*, \\ awxwa &= \begin{pmatrix} a_1w_1(w_1a_1w_1)_r^{-1} & a_1w_1x_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1a_1 & w_1a_2 \\ 0 & 0 \end{pmatrix}_{q\times q} \\ &= \begin{pmatrix} w_1^{-1} & w_1^{-1}(w_1a_1w_1x_2) \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1a_1 & w_1a_2 \\ 0 & 0 \end{pmatrix}_{q\times q} \\ &= a. \end{aligned}$$

Therefore $a \in \mathcal{A}^{\text{\textcircled{\#}},w} = x$, as required. □

Corollary 2.7. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a_r^{\text{\textcircled{\#}}} = x$.

(2) There exist an idempotent $p \in \mathcal{A}$ and a projection $q \in \mathcal{A}$ such that $a = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}_{p\times q}$ and x is represented as

$$x = \begin{pmatrix} (a_1)_r^{-1} & x_2 \\ 0 & 0 \end{pmatrix}_{p\times q},$$

where $a_1x_2 = 0, a_1(a_1)_r^{-1}x_2 = x_2$.

Proof. We obtain the result by choosing $w = 1$ in Theorem 2.6. □

3. Generalized Right w-Weighted Core Decomposition

The purpose of this section is to characterize generalized w-weighted core inverse in a Banach *-algebra by using right weight core inverse and quasinilpotent. The following theorem contains new characterizations for a generalized right w-weighted core inverse.

Theorem 3.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\text{\textcircled{d}},w}$.

(2) There exist $z, y \in \mathcal{A}$ such that

$$a = z + y, (wz)^*(wy) = ywz = 0, z \in \mathcal{A}_r^{\text{\textcircled{\#}},w}, y \in \mathcal{A}_w^{qnil}.$$

(3) There exist $z, y \in \mathcal{A}$ such that

$$a = z + y, ywz = 0, z \in \mathcal{A}_r^{\text{\textcircled{\#}},w}, y \in \mathcal{A}_w^{qnil}.$$

In this case, $a_r^{\text{\textcircled{d}},w} = z_r^{\text{\textcircled{\#}},w}$.

Proof. $(1) \Rightarrow (2)$ By hypotheses, there exists $x \in \mathcal{A}$ such that

$$a(wx)^2 = x, (wawx)^* = wawx,$$
$$\lim_{n\to\infty} ||(aw)^n - (aw)(xw)(aw)^n||^{\frac{1}{n}} = 0.$$

 doi:10.20944/preprints202507.1148.v1

Set $z = awxwa$ and $y = a - awxwa$. Then $a = z + y$. Since $\lim\limits_{n\to\infty} ||(aw)^n - (aw)(xw)(aw)^n||^{\frac{1}{n}} = 0$, we prove that

$$(aw)xwawx = awx, (aw)xw(aw)^2x = (aw)^2x.$$

Step 1. We claim that z has right w-weighted core inverse x. One easily checks that

$$\begin{aligned} z(wx)^2 &= (awxwa)(wx)^2 = awxw[a(wx)^2] = (awxw)x = a(wx)^2 = x, \\ wzwx &= w(awxwa)wx = (wawx)^2, \\ (wzwx)^* &= [(wawx)^*]^2 = (wawz)^2 = wzwx, \\ zwxwz &= (awxwa)wxw(awxwa) = (awxwaw)xwawxwa \\ &= (awxwaw)xwa = awxwa = z. \end{aligned}$$

Thus, $z \in \mathcal{A}_r^{\textcircled{\#},w}$ and $x = z_r^{\textcircled{\#},w}$.

Moreover, we see that

$$[aw - (aw)xw(aw)]awx = 0.$$

Then we verify that

$$\begin{aligned} & ||[aw - (aw)xw(aw))^n||^{\frac{1}{n+1}} \\ = & ||[aw - (aw)xw(aw))^{n-1}[aw - (aw)xw(aw)]||^{\frac{1}{n+1}} \\ = & ||[aw - (aw)xw(aw))^{n-1}aw||^{\frac{1}{n+1}} \\ = & ||[aw - (aw)xw(aw))^{n-2}(aw)^2||^{\frac{1}{n+1}} \\ = & \cdots \\ = & ||[aw - (aw)xw(aw))(aw)^{n-1}||^{\frac{1}{n+1}} \\ = & ||[(aw)^n - (aw)xw(aw)^n||^{\frac{1}{n+1}}. \end{aligned}$$

Accordingly,

$$\lim_{n\to\infty} ||[(aw)^n - (aw)xw(aw)^n)^{n+1}||^{\frac{1}{n+2}} = 0.$$

We infer that $aw - (aw)xw(aw) \in \mathcal{A}^{qnil}$. Therefore $yw = aw - awxwaw \in \mathcal{A}^{qnil}$. That is, $y \in \mathcal{A}_w^{qnil}$.

Moreover, we verify that

$$\begin{aligned} (wz)^*wy &= (wawxwa)^*(wa - wawxwa) = (wa)^*(wawx)^*(1 - wawx)wa \\ &= (wa)^*(wawx)(1 - wawx)wa = (wa)^*(wawxw - wawxwawxw)a \\ &= (wa)^*[wawxw - w(awxwawxw)]a \\ &= (wa)^*[wawxw - w(awxw)]a = 0, \\ ywx &= (a - awxwa)w(awxwa) = awawxwa - [awxw(aw)^2]xwa \\ &= awawxwa - (aw)^2xwa = 0, \end{aligned}$$

as required.

$(2) \Rightarrow (3)$ This is obvious.

$(3) \Rightarrow (1)$ By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, ywz = 0, z \in \mathcal{A}_r^{\textcircled{\#},w}, y \in \mathcal{A}_w^{qnil}.$$

Set $x = z_r^{\textcircled{\#},w}$. Then

$$\begin{aligned} z(wx)^2 &= x, \\ (wzwx)^* &= wzwx, \\ (aw)xw(zw) &= zw. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} a(wx)^2 &= (z+y)(wx)^2 = z(wx)^2 = x, \\ (waw)x &= w(z+y)wx = wzwx, \\ ((waw)x)^* &= (wzwx)^* = wzwx = (waw)x. \end{aligned}$$

We verify that

$$\begin{aligned} & (aw)^2 - (aw)(xw)(aw)^2 \\ = \; & aw(zw+yw) - (aw)(xwzw + xwyw)(zw+yw) \\ = \; & aw(zw+yw) - (aw)[(xwzw)zw + (xwzw)yw + xw(ywz)w + xw(yw)^2] \\ = \; & aw(zw+yw) - [awxw(zw)^2 + (z+y)wxwzwyw + awxw(yw)^2] \\ = \; & awzw + awyw - [awzw + zwxw(zw)yw + awxw(yw)^2] \\ = \; & awzw + awyw - [awzw + zwxw(xw(zw)^2)yw + awxw(yw)^2] \\ = \; & awyw - z(wx)^2w(zw)^2yw - awxw(yw)^2] \\ = \; & awyw - [xw(zw)^2]yw - awxw(yw)^2] \\ = \; & (z+y)wyw - zwyw - (z+y)wxw(yw)^2] \\ = \; & (yw)^2 - zwxw(yw)^2 \\ = \; & (1 - zwxw)(yw)^2. \end{aligned}$$

Since $yw(aw) = yw(zw+yw) = (yw)^2$, we deduce that

$$\begin{aligned} (aw)^n - (aw)(xw)(aw)^n &= [(aw)^2 - (aw)(xw)(aw)^2](aw)^{n-2} \\ &= [(1-zwxw)(yw)^2](aw)^{n-2} \\ &= (1-zwxw)(yw)[yw(aw)^{n-2}] \\ &= (1-zwxw)(yw)(yw)^{n-1}. \end{aligned}$$

Then

$$||(aw)^n - (aw)xw(aw)^n||^{\frac{1}{n}} \quad \leq \quad ||1 - zwxw||||(yw)^n||.$$

Since $y \in \mathcal{A}_w^{qnil}$, we see that

$$\lim_{n\to\infty} ||(yw)^n||^{\frac{1}{n}} = 0.$$

Therefore

$$\lim_{n\to\infty} ||(aw)^n - (aw)(xw)(aw)^n||^{\frac{1}{n}} = 0.$$

Accordingly, $a \in \mathcal{A}_r^{\textcircled{d},w}$, as desired. □

Corollary 3.2. *Let $a \in \mathcal{A}_r^{\textcircled{d},w}$ and $b \in \mathcal{A}^{qnil}$. If $bwa = 0$, then $a + b \in \mathcal{A}_r^{\textcircled{d},w}$. In this case, $(a+b)_r^{\textcircled{d},w} = a_r^{\textcircled{d},w}$.*

Proof. Since $a \in \mathcal{A}_r^{\textcircled{d},w}$, by virtue of Theorem 3.1, there exist $x \in \mathcal{A}_r^{\textcircled{\#},w}$ and $y \in \mathcal{A}_w^{qnil}$ such that $a = x + y, ywx = 0$. As in the proof of Theorem 3.1, $x = awa_r^{\textcircled{d},w}wa$ and $y = a - awa_r^{\textcircled{d},w}wa$. Then $a = x + (y+b)$. As $bwy = bw(a - awa_r^{\textcircled{d},w}wa) = 0$, it follows by [3, Lemma 2.4] that $(y+b)w \in \mathcal{A}^{qnil}$. Hence, $y + b \in \mathcal{A}_w^{qnil}$. Obviously, $(y+b)wx = ywx + bwx = 0$. In light of Theorem 3.1, $a + b \in \mathcal{A}_r^{\textcircled{d},w}$. In this case, $(a+b)_r^{\textcircled{d},w} = x_r^{\textcircled{\#},w} = a_r^{\textcircled{d},w}$. □

Corollary 3.3. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{d}}$.
(2) There exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*y = yz = 0, z \in \mathcal{A}_r^{\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$

(3) There exist $z, y \in \mathcal{A}$ such that

$$a = z + y, ywz = 0, z \in \mathcal{A}_r^{\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$

In this case, $a_r^{\textcircled{d},w} = z^{\textcircled{\#}}$.

Proof. This is obvious by choosing $w = 1$ in Theorem 3.1. $\square$

Corollary 3.4. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{D}}$.
(2) There exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*y = yz = 0, z \in \mathcal{A}_r^{\textcircled{\#}}, y \in \mathcal{A}^{nil}.$$

(3) There exist $z, y \in \mathcal{A}$ such that

$$a = z + y, yz = 0, z \in \mathcal{A}_r^{\textcircled{\#}}, y \in \mathcal{A}^{nil}.$$

In this case, $a^{\textcircled{d}} = z_r^{\textcircled{\#}}$.

Proof. (1) $\Rightarrow$ (2) This is proved in [5, Theorem 5.2].
(2) $\Rightarrow$ (3) This is trivial.
(3) $\Rightarrow$ (1) In view of Theorem 3.1, $a \in \mathcal{A}_r^{\textcircled{d}}$. Since $y, z \in \mathcal{A}_r^D$ and $yz = 0$, we directly verify that $a \in \mathcal{A}_r^D$. Therefore $a \in \mathcal{A}_r^{\textcircled{D}}$ by [5, Lemma 5.1]. $\square$

We present the following example to illustrate Theorem 3.1.

Example 3.5.

The space $\ell^2(\mathbb{N})$ is a Hilbert space consisting of all square-summable infinite sequences of complex numbers. Let $H = \ell^2(\mathbb{N}) \bigoplus \ell^2(\mathbb{N})$ be the Hilbert space of the sum of $\ell^2(\mathbb{N})$ and itself. Let σ be defined on $\ell^2(\mathbb{N})$ by:

$$\sigma(x_1, x_2, x_3, \cdots) = (0, 1^{\frac{1}{1}}x_1, 2^{\frac{1}{2}}x_2, 3^{\frac{1}{3}}x_3, \cdots);$$

τ is defined on $\ell^2(\mathbb{N})$ by:

$$\tau(x_1, x_2, x_3, \cdots) = (2^{\frac{1}{2}}x_2, 3^{\frac{1}{3}}x_3, 4^{\frac{1}{4}}x_4, \cdots)$$

and u is defined on $\ell^2(\mathbb{N})$ by:

$$u(x_1, x_2, x_3, \cdots) = (x_1/1^{\frac{1}{1}}, x_2/2^{\frac{1}{2}}, x_3/3^{\frac{1}{3}}, x_4/4^{\frac{1}{4}}, \cdots).$$

Obviously, we have

$$\begin{aligned}
\lim_{n\to\infty} ||(x_n)^2||/||(x_n/\sqrt[n]{n})^2|| &= \lim_{n\to\infty} (\sqrt[n]{n})^2 = 1, \\
\lim_{n\to\infty} ||(x_n)^2||/||(\sqrt[n]{n}x_n)^2|| &= \lim_{n\to\infty} (\tfrac{1}{\sqrt[n]{n}})^2 = 1.
\end{aligned}$$

Hence $\sum\limits_{n=1}^{\infty} (\sqrt[n]{n}x_n)^2$ and $\sum\limits_{n=1}^{\infty} (x_n/\sqrt[n]{n})^2$ converge. Then σ, τ and u are well defined.

Let γ be the right shift operator defined on $\ell^2(\mathbb{N})$ by:

$$\gamma(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots),$$

and let v is defined on $\ell^2(\mathbb{N})$ by:

$$v(x_1, x_2, x_3, \cdots) = (\frac{1}{3}x_1, \frac{1}{4}x_2, \frac{1}{5}x_3, \cdots).$$

Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, we see that v is well defined.

Let $\delta = v\gamma$. Then it is given by

$$\delta(x_1, x_2, x_3, \cdots) = (\frac{1}{3}x_2, \frac{1}{4}x_3, \frac{1}{5}x_4, \cdots).$$

Let $e_n = (\underbrace{0, 0, \cdots, 0, 1}_{n}, 0, 0, \cdots)$. Then $\delta(e_n) = \frac{1}{n+2}e_{n+1}$. One directly checks that

$$\delta^k(e_n) = \prod_{j=0}^{k-1} \frac{1}{n+j+2} e_{n+k} = \frac{(n+1)!}{(n+k+1)!} e_{n+k}.$$

Then we derive that

$$||\delta^k|| = \frac{1}{(k+1)!}.$$

By virtue of Stirling's formula, we have $(k+1)! \sim \sqrt{2\pi(k+1)}\left(\frac{k+1}{e}\right)^{k+1}$, and then

$$\lim_{k\to\infty} ||\delta^k||^{\frac{1}{k}} = \lim_{k\to\infty} \frac{e^{1+\frac{1}{k}}}{(2\pi)^{\frac{1}{k}}[(k+1)^{\frac{1}{k+1}}]^{1+\frac{1}{k}}(k+1)^{1+\frac{1}{k}}} = 0,$$

i.e., δ is quasinilpotent.

Let $T = \tau \oplus \gamma$ be the direct sum of τ and γ, $W = u \oplus v$ be the direct sum of u and v, $S = \sigma \oplus 0$ be the direct sum of σ and 0. Then T, W and S are operators on H. We see that $T = \alpha + \beta$, where $\alpha = \tau \oplus 0$ and $\beta = 0 \oplus \gamma$.

Claim 1. α has right W-core inverse. We directly check that $\tau u \sigma u = u \tau u \sigma = I$. Then $\alpha WSW = W\alpha WS = I \oplus 0$. Hence,

$$\begin{aligned} \alpha(WS)^2 &= [\alpha WSW]S = S, \\ (W\alpha WS)^* &= W\alpha WS, \\ \alpha WSW\alpha &= \alpha. \end{aligned}$$

Then $S = \alpha_r^{\textcircled{\#},W}$.

Claim 2. $W\beta$ is quasinilpotent. Since $W\beta = 0 \oplus \delta$, we see that $W\beta$ is quasinilpotent.

Claim 3. $\beta W \alpha = 0$. This is obvious.

Therefore T has a generalized right W-core inverse by Theorem 3.1.

As it is well known, an element $a \in \mathcal{A}$ has g-Drazin inverse if and only if it is quasi-polar, i.e., there exists an idempotent $p \in \mathcal{A}$ such that $pa = ap \in \mathcal{A}^{qnil}, a + p \in \mathcal{A}^{-1}$ (see [4, Corollary 15.1.5]). For generalized w-weighted core-EP inverse, we establish the following polar-like characterization.

Theorem 3.6. *Let $a \in \mathcal{A}_r^{\textcircled{d},w}$ and $n \in \mathbb{N}$. Then there exists a projection $p \in \mathcal{A}$ such that*

$$pwa \in \mathcal{A}^{qnil}, (wa)^n + p \in \mathcal{A}_r^{-1} \text{ and } (1-p)\mathcal{A} = wa(1-p)wa\mathcal{A}.$$

Proof. Since $a \in \mathcal{A}_r^{\textcircled{d},w}$, by using Theorem 3.1, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, (wx)^*(wy) = 0, ywx = 0, x \in \mathcal{A}_r^{\textcircled{\#},w}, y \in \mathcal{A}_w^{qnil}.$$

Let $z = x_r^{\circledast,w}$. By virtue of Theorem 2.1, there exist $x \in \mathcal{A}$ such that

$$x(wz)^2 = z = zwxwz, (wxwz)^* = wxwz, xwzwx = x.$$

Let $p = 1 - wxwz$. Then $p^2 = p = p^*, 1 - p \in w\mathcal{A}$ and $pwx = 0$. We directly check that

$$\begin{aligned}
& [(wx)^n + 1 - wxwz][(wz)^n + 1 - wxwz] \\
= \ & (wx)^n(wz)^n + (wx)^n(1 - wxwz) + (1 - wxwz)(wz)^n + (1 - wxwz)^2 \\
= \ & (wx)(wz) + (wx)^n(1 - wxwz) + [wz - wx(wz)^2](wz)^{n-1} + (1 - wxwz) \\
= \ & 1 + (wx)^n(1 - wxwz).
\end{aligned}$$

Then

$$[(wx)^n + 1 - wxwz][(wz)^n + 1 - wxwz][1 - (wx)^n(1 - wxwz)] = 1.$$

Hence, $(wx)^n + 1 - wxwz \in \mathcal{A}_r^{-1}$.

Furthermore, we check that

$$\begin{aligned}
yw - ywxwzw &= yw \in \mathcal{A}^{qnil}, \\
(w - wxwzw)y &= (1 - wxwz)wy \in \mathcal{A}^{qnil}, \\
& \quad (1 - wxwz)wx + (w - wxwzw)y \in \mathcal{A}^{qnil}, \\
pwa &= pw(x + y) = (1 - wxwz)wx + pwy = pyw \in \mathcal{A}^{qnil}, \\
wa(1 - p)wa &= w(x + y)(wxwz)wa = wxwxwz(wx + wy) \\
&= wx(wxwz)^*(wx + wy) = wx(wz)^*[(wx)^*wx + (wx)^*wy] \\
&= wx(wz)^*(wx)^*wx = wx(wxwz)wx = (wx)^2 \\
&= w(xwzwx)wx = (wxwz)(wx)^2 = (1 - p)(wx)^2, \\
1 - p &= wxwz = (wx)^2(wz)^3 = [wa(1 - p)wa](wz)^3, \\
(1 - p)\mathcal{A} &= wa(1 - p)wa\mathcal{A}, \\
(wa)^n + p &= (wx + wy)^n + p = (wx)^n + \sum_{i=1}^{n}(wx)^{n-i}(wy)^i + p \\
&= [(wx)^n + p] + \sum_{i=1}^{n}(wx)^{n-i}(wy)^i \\
&= [(wx)^n + p][1 + ((wx)^n + p)_r^{-1}\sum_{i=1}^{n}(wx)^{n-i}(wy)^i].
\end{aligned}$$

as required. □

Corollary 3.7. *Let $a \in \mathcal{A}, w \in \mathcal{A}^{-1}$ and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\circledast,w}$.
(2) *There exists a projection $p \in \mathcal{A}$ such that*

$$pwa \in \mathcal{A}^{qnil}, (wa)^n + p \in \mathcal{A}_r^{-1} \text{ and } (1 - p)\mathcal{A} = wa(1 - p)wa\mathcal{A}.$$

Proof. (1) ⇒ (2) This is obvious by Theorem 3.6.

(2) ⇒ (1) By hypothesis, there exists a projection $p \in \mathcal{A}$ such that

$$pwa \in \mathcal{A}^{qnil}, (wa)^n + p \in \mathcal{A}_r^{-1} \text{ and } (1 - p)\mathcal{A} = wa(1 - p)wa\mathcal{A}.$$

Then $pwa(1 - p)wa = 0$.

Set $x = (1 - p)wa$ and $y = pwa$. Then

$$\begin{aligned}
x^*y &= [(wa)^*(1 - p)^*]pwa = 0, \\
yx &= pwa(1 - p)wa = 0, \\
y &= pwa \in \mathcal{A}^{qnil}.
\end{aligned}$$

 doi:10.20944/preprints202507.1148.v1

Write $(wa+p)s = 1$ and $ps(1-p) = 0$. Then $(1-p)was = (1-p)(wa+p)s = 1-p$, and so $(1-p)was(1-p)wa = (1-p)wa$ and $(1-p)was = (1-p)(wa+p)s = 1-p$. Hence, $x = (1-p)wa \in \mathcal{A}^{(1,3)}$.

Write $x = ws$ and $t = a - s$. Then $a = s + t, (ws)^*(wt) = x^*(wa - x) = x^*[wa - (1-p)wa] = x^*y = 0, tws = w^{-1}(wt)x = w^{-1}[wa - (1-p)wa]x = w^{-1}yx = 0$ and $t \in \mathcal{A}_w^{qnil}$.

We compute that $sws = w^{-1}x^2 = w^{-1}(1-p)wa(1-p)wa = a(1-p)wa$. Since $(1-p)\mathcal{A} = wa(1-p)wa\mathcal{A}$, we can find a $z \in \mathcal{A}$ such that $1 - p = wa(1-p)waz$; hence,

$$s = w^{-1}(1-p)wa = w^{-1}[wa(1-p)waz]wa = a(1-p)waz = (sws)z;$$

hence, $s\mathcal{A} \subseteq sws\mathcal{A}$. Obviously, $sws\mathcal{A} \subseteq s\mathcal{A}$. Accordingly, $s\mathcal{A} = sws\mathcal{A}$. Moreover, we have $\mathcal{A}s = \mathcal{A}ws = \mathcal{A}x = \mathcal{A}x^*x = \mathcal{A}(ws)^*(ws)$. In light of Corollary 2.2, $s \in \mathcal{A}_r^{\text{#},w}$. Therefore $a \in \mathcal{A}_r^{ⓓ,w}$ by Theorem 2.1. □

Corollary 3.8. *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{ⓓ}$.
(2) *There exists a projection $p \in \mathcal{A}$ such that*

$$pa \in \mathcal{A}^{qnil}, a^n + p \in \mathcal{A}_r^{-1} \text{ and } (1-p)\mathcal{A} = a(1-p)a\mathcal{A}.$$

Proof. Straightforward by choosing $w = 1$ in Corollary 3.7. □

4. Connections to Right wg-Drazin Inverses

This section establishes the relationship between the generalized right weighted core inverse and the right wg-Drazin inverse. Let $a \in \mathcal{A}$ and let

$$\{a_r^{d,w}\} = \{x \in \mathcal{A} \mid a(wx)^2 = x = xwawx, aw - xw(aw)^2 \in \mathcal{A}^{qnil}\}.$$

Next, we derive:

Theorem 4.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{ⓓ,w}$.
(2) $\{a_r^{d,w}\} \bigcap \mathcal{A}_r^{\text{#},w} \neq \emptyset$.

In this case, $a_r^{ⓓ,w} = (xw)^2 x_r^{\text{#},w}$ for $x \in \{a_r^{d,w}\} \bigcap \mathcal{A}_r^{\text{#},w}$.

Proof. (1) ⇒ (2) By virtue of Theorem 3.1, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, (wz)^*(wy) = ywz = 0, z \in \mathcal{A}_r^{\text{#},w}, y \in \mathcal{A}_w^{qnil}.$$

Set $x = z_r^{\text{#},w}$. Then

$$a(wx)^2 = (z+y)wxwx = z(wx)^2 = x.$$

Moreover, we see that $xwawx = xw(z+y)wx = xwzwx = x$. Since $awxw = (z+y)wxw = zwxw$, we see that

$$aw - (aw)xw(aw) = aw - (z+y)wxw(aw) = (z+y)w - zwxw(z+y)w.$$

Observing that

$$\begin{aligned} w(z+y) - wzwxw(z+y) &= w(z+y) - (wzwx)^*w(z+y) \\ &= w(z+y) - (wx)^*(wz)^*(wz+wy) \\ &= w(z+y) - (wx)^*(wz)^*wz \\ &= wz + wy - (wzwx)wz = wy \in \mathcal{A}^{qnil}, \end{aligned}$$

by using Cline's formula, we have

$$aw - (aw)xw(aw) \in \mathcal{A}^{qnil}.$$

By Cline's formula again, we see that $aw - xw(aw)^2 \in \mathcal{A}^{qnil}$. Thus, $a \in \mathcal{A}_r^{d,w}$ and $x \in \{a_r^{d,w}\} \bigcap \mathcal{A}_r^{\textcircled{\#},w}$. This implies that $\{a_r^{d,w}\} \bigcap \mathcal{A}_r^{\textcircled{\#},w} \neq \emptyset$.

(2) $\Rightarrow$ (1) Let $z \in \{a_r^{d,w}\} \bigcap \mathcal{A}_r^{\textcircled{\#},w}$. Then

$$a(wz)^2 = z, aw - zw(aw)^2 \in \mathcal{A}^{qnil}.$$

Set $x = (zw)^2 z_r^{\textcircled{\#},w}$. It is easy to verify that

$$\begin{array}{rcl} awx & = & aw(zw)^2 z_r^{\textcircled{\#},w} = [a(wz)^2]wz_r^{\textcircled{\#},w} = zwz_r^{\textcircled{\#},w}, \\ a(wx)^2 & = & (awx)wx = (zwz_r^{\textcircled{\#},w})w(zw)^2 z_r^{\textcircled{\#},w} \\ & = & [zwz_r^{\textcircled{\#},w}wz]wzwz_r^{\textcircled{\#},w} \\ & = & (zw)^2 z_r^{\textcircled{\#},w} = x, \\ wawx & = & w(zwz_r^{\textcircled{\#},w}) = wzwz_r^{\textcircled{\#},w}, \\ (wawx)^* & = & (wzwz_r^{\textcircled{\#},w})^* = wzwz_r^{\textcircled{\#},w} = wawx, \end{array}$$

Hence, we check that

$$\begin{array}{rcl} (zwz_r^{\textcircled{\#},w}wa)wx & = & zwz_r^{\textcircled{\#},w}(wawx) \\ & = & zwz_r^{\textcircled{\#},w}(wzwz_r^{\textcircled{\#},w}) \\ & = & zwz_r^{\textcircled{\#},w}, \\ (w(zwz_r^{\textcircled{\#},w}wa)wx)^* & = & (wzwz_r^{\textcircled{\#},w})^* \\ & = & wzwz_r^{\textcircled{\#},w} = w(zwz_r^{\textcircled{\#},w}wa)wx, \\ (zwz_r^{\textcircled{\#},w}wa)(wx)^2 & = & zwz_r^{\textcircled{\#},w}w(zw)^2 z_r^{\textcircled{\#},w} \\ & = & (zw)^2 z_r^{\textcircled{\#},w} = x, \\ (zwz_r^{\textcircled{\#},w}wa)wxw(zwz_r^{\textcircled{\#},w}wa) & = & zwz_r^{\textcircled{\#},w}w(zwz_r^{\textcircled{\#},w}wa) \\ & = & zwz_r^{\textcircled{\#},w}wa. \end{array}$$

Thus, $zwz_r^{\textcircled{\#},w}wa \in \mathcal{A}_r^{\textcircled{\#},w}$ and $[zwz_r^{\textcircled{\#},w}wa]_r^{\textcircled{\#},w} = x = (zw)^2 z_r^{\textcircled{\#},w}$.

Write $a = a_1 + a_2$, where $a_1 = zwz_r^{\textcircled{\#},w}wa$ and $a_2 = a - zwz_r^{\textcircled{\#},w}wa$. We verify that

$$\begin{array}{rcl} a_2 wa_1 & = & [a - zwz_r^{\textcircled{\#},w}wa]w[zwz_r^{\textcircled{\#},w}wa] \\ & = & awzwz_r^{\textcircled{\#},w}wa - zwz_r^{\textcircled{\#},w}wawzwz_r^{\textcircled{\#},w}wa \\ & = & awzwz_r^{\textcircled{\#},w}wa - zwz_r^{\textcircled{\#},w}wz_r^{\textcircled{\#},w}wa \\ & = & z_r^{\textcircled{\#},w}wa - z(wz_r^{\textcircled{\#},w})^2 wa = 0. \end{array}$$

Moreover, we check that

$$\begin{array}{rl} & (aw - zw(aw)^2)[1 - zwz_r^{\textcircled{\#},w}w] \\ = & aw - zw(aw)^2 - (1 - zwaw)awzwz_r^{\textcircled{\#},w}w \\ = & aw - zw(aw)^2 - (1 - zwaw)awzwz(wz_r^{\textcircled{\#},w})^2 w \\ = & aw - zw(aw)^2 - (1 - zwaw)[a(wz)^2](wz_r^{\textcircled{\#},w})^2 w \\ = & aw - zw(aw)^2 - (1 - zwaw)z(wz_r^{\textcircled{\#},w})^2 w \\ = & aw - zw(aw)^2 \in \mathcal{A}^{qnil}. \end{array}$$

By using Cline's formula (see [11, Theorem 2.1]),

$$a_2 = a - zwz_r^{\textcircled{\#},w}wa = [1 - zwz_r^{\textcircled{\#},w}w](a - za^2) \in \mathcal{A}^{qnil}.$$

Thus $a = a_1 + a_2$ is the generalized right weighted core decomposition of a. Accordingly,

$$a_r^{e,\textcircled{d}} = (a_1)_r^{\textcircled{\#},w} = (zw)^2 z_r^{\textcircled{\#},w},$$

as asserted. $\square$

As an immediate consequence, we provide formulas of the weighted core-EP inverse of a complex matrix.

Corollary 4.2. *Let $A, W \in \mathbb{C}^{n\times n}$. Then*

$$A^{\textcircled{\dagger},W} = [A^{D,W}W]^2(A^{D,W})^{\textcircled{\#},W},$$

where $k = max\{ind(AW), ind(WA)\}$.

Proof. This is obvious by Theorem 4.1. $\square$

Lemma 4.3. *Let $z \in \{a_r^{d,w}\}$. Then*

$$\lim_{n\to\infty} ||((aw)^n - awzw(aw)^n)^*||^{\frac{1}{n}} = \lim_{n\to\infty} ||(aw)^n - awzw(aw)^n||^{\frac{1}{n}} = 0.$$

Proof. Let $x = a - awzwa$. Then $xw \in \mathcal{A}^{qnil}$. For any $\lambda \in \mathbb{C}$, we have $1 - \overline{\lambda}xw \in \mathcal{A}^{-1}$, and so $1 - \lambda(xw)^* \in \mathcal{A}^{-1}$. Thus, $(xw)^* \in \mathcal{A}^{qnil}$. Obviously, $(1 - awzw)awzw = 0$, we see that $(aw - awzw)^n = (1 - awzw)(aw)^n$. Hence, we derive that

$$\begin{aligned} ||((aw)^n - awzw(aw)^n)^*||^{\frac{1}{n}} &= ||((aw)^n)^*(1 - awzw)^*|| \\ &= ||((aw)^n)^*[(1 - awzw)^n]^*|| \\ &= ||[(aw - awzwaw)^n]^*|| = ||((xw)^*)^n||. \end{aligned}$$

Since $(xw)^* \in \mathcal{A}^{qnil}$, we have

$$\lim_{n\to\infty} ||((aw)^n - awzw(aw)^n)^*||^{\frac{1}{n}} = 0.$$

Analogously, we prove that $\lim_{n\to\infty} ||(aw)^n - awzw(aw)^n||^{\frac{1}{n}} = 0.$, as asserted. $\square$

Lemma 4.4. *Let $a \in \mathcal{A}_r^{\textcircled{d},w}$. Then*

$$\lim_{n\to\infty} ||((aw)^n - awa_r^{\textcircled{d},w}w(aw)^n)^*||^{\frac{1}{n}} = 0.$$

Proof. Construct x, y as in the proof of Theorem 3.1. Since $(1 - awxw)awxwaw = 0$, we deduce that $(aw - awxwaw)^n = (1 - awxw)(aw)^n$. Then

$$\begin{aligned} & ||((aw)^n - axww(aw)^n)^*|| \\ = & ||[(1 - axww)(aw)^n]^*|| \\ = & ||((yw)^*)^n||^{\frac{1}{n}} = 0. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} ||(a^n - a^n(a_r^{e,\textcircled{d}})^n a^n)^*||^{\frac{1}{n}} = 0,$$

as asserted. $\square$

We are ready to prove:

Theorem 4.5. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_r^{\textcircled{\#},w}$ if and only if there exist $x \in \mathcal{A}$ and $z \in \{a_r^{d,w}\}$ such that*

$$a(wx)^2 = x = xwawx, x\mathcal{A} = z\mathcal{A}, \mathcal{A}x = \mathcal{A}(wawz)^*.$$

In this case, $a_r^{\textcircled{\#},w} = x$.

Proof. $\Longrightarrow$ Choose $x = a_r^{\textcircled{\#},w}$. As in the proof of Theorem 4.1, $x = xwawx = a(wx)^2$. By using Theorem 4.1, we can find $z \in \{a_r^{d,w}\} \bigcap \mathcal{A}_r^{\#,w}$ such that

$$x = (zw)^2 z_r^{\#,w}.$$

Then we have

$$a(wz)^2 = z = zwawz, aw - (zw)(aw)^2 \in \mathcal{A}^{qnil}.$$

Obviously, $z = zwz_r^{\#,w}wz = (zw)^2(z_r^{\#,w}w)^2 z = xwz_r^{\#,w}wz$. Accordingly, we have $x\mathcal{A} = z\mathcal{A}$. Since $a(wx)^2 = x$, we have $(wa)(wx) = (wa)^n(wx)^n$, and then

$$\begin{array}{rcl} x & = & xwawx = x(wawx)^* = x((wa)^n(wx)^n)^* = x((wx)^n)^*((wa)^n)^* \\ & = & x((wx)^n)^*((wa)^n - wawz(wa)^n)^* + x((wx)^n)^*((wa)^n)^*(wawz)^* \\ & = & x((wx)^n)^*((wa)^n - wawz(wa)^n)^* + x(wawx)^*(wawz)^*. \end{array}$$

Hence,

$$\begin{array}{rcl} ||x - x(wawx)^*(wawz)^*||^{\frac{1}{n}} & = & ||x((wx)^n)^*((wa)^n - wawz(wa)^n)^*||^{\frac{1}{n}} \\ & \leq & ||x((wx)^n)^*||^{\frac{1}{n}}||((wa)^n - wawz(wa)^n)^*||^{\frac{1}{n}}. \end{array}$$

In view of Lemma 4.3,

$$\lim_{n\to\infty} ||((wa)^n - wawz(wa)^n)^*||^{\frac{1}{n}} = 0,$$

we derive that

$$\lim_{n\to\infty} ||x - x(wawx)^*(wawz)^*||^{\frac{1}{n}} = 0;$$

hence, $x = x(wawx)^*(wawz)^*$. Then $\mathcal{A}x \subseteq \mathcal{A}(wawz)^*$.

Since $a(wz)^2 = z$, we have $(wa)^n(wz)^n = wawz$, and then we derive that

$$\begin{array}{rl} & ||(wawz)^* - (wawz)^*wawx||^{\frac{1}{n}} \\ = & ||((wz)^n)^*((wa)^n)^* - ((wz)^n)^*(wawx(wa)^n)^*||^{\frac{1}{n}} \\ = & ||((wz)^n)^*((wa)^n - wawx(wa)^n)^*||^{\frac{1}{n}} \\ \leq & ||((wz)^n)^*||^{\frac{1}{n}}||((wa)^n - wawx(wa)^n)^*||^{\frac{1}{n}}. \end{array}$$

In light of Lemma 4.4, we see that

$$\lim_{n\to\infty} ||((wa)^n - wawx(wa)^n)^*||^{\frac{1}{n}} = 0.$$

Then

$$\lim_{n\to\infty} ||(wawz)^* - (wawz)^*wawx||^{\frac{1}{n}} = 0,$$

and so $(wawz)^* = (wawz)^*wawx$. Hence $\mathcal{A}(wawz)^* \subseteq \mathcal{A}x$. Therefore $\mathcal{A}x = \mathcal{A}(wawz)^*$, as required.

$\Longleftarrow$ By hypothesis, there exist $x \in \mathcal{A}$ and $z \in \{a_r^{d,w}\}$ such that

$$a(wx)^2 = x = xwawx, x\mathcal{A} = z\mathcal{A}, \mathcal{A}x = \mathcal{A}(wawz)^*.$$

We claim that $a_r^{\textcircled{\#},w} = x$.

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Claim 1.
$$\lim_{n\to\infty} ||(aw)^n - awxw(aw)^n||^{\frac{1}{n}} = 0.$$
Write $z = xy$ for some $y \in \mathcal{A}$. For any $n \in \mathbb{N}$, we have
$$\begin{aligned} (aw)^n &= [(aw)^n - (aw)(zw)(aw)^n] + (aw)(zw)(aw)^n, \\ awxw(aw)^n &= awxw[(aw)^n - awzw(aw)^n] + awxwawzw(aw)^n \\ &= awxw[(aw)^n - awzw(aw)^n] + aw(xwawx)yw(aw)^n \\ &= awxw[(aw)^n - awzw(aw)^n] + awxyw(aw)^n \\ &= awxw[(aw)^n - (aw)(zw)(aw)^n] + awzw(aw)^n. \end{aligned}$$
Hence,
$$(aw)^n - awxw(aw)^n = (1 - awxw)[(aw)^n - (aw)(zw)(aw)^n],$$
and so
$$||(aw)^n - awxw(aw)^n||^{\frac{1}{n}} \leq ||1 - awxw||^{\frac{1}{n}} ||(aw)^n - (aw)(zw)(aw)^n||^{\frac{1}{n}},$$
we have
$$\lim_{n\to\infty} ||(aw)^n - awxw(aw)^n||^{\frac{1}{n}} = 0.$$
Claim 2. $(wawx)^* = wawx$.

Since $\mathcal{A}x = \mathcal{A}(wawz)^*$, we have $x^*\mathcal{A} = wawz\mathcal{A}$. Write $wawz = x^*s$ for some $s \in \mathcal{A}$. Since $xwawx = x$, we have $x^*w^*a^*w^*x^* = x^*$, and so $(wawx)^*x^* = x^*$. We infer that $(wawx)^*wawz = (wawx)^*(x^*s) = [(wawx)^*x^*]s = (xwawx)^*s = x^*s = wawz$. Since $x\mathcal{A} = z\mathcal{A}$, we can find $t \in \mathcal{A}$ such that $x = zt$. Then $(wawx)^*(wawx) = (wawx)^*waw(zt) = [(wawx)^*wawz]t = (wawz)t = waw(zt) = wawx$. Hence $(wawx)^* = [(wawx)^*(wawx)]^* = (wawx)^*(wawx) = wawx$.

Therefore $a_r^{ⓓ,w} = x$, as asserted. □

Corollary 4.6. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_r^{ⓓ}$ if and only if there exist $x \in \mathcal{A}$ and $z \in \{a_r^d\}$ such that*
$$ax^2 = x = xax, x\mathcal{A} = z\mathcal{A}, \mathcal{A}x = \mathcal{A}(az)^*.$$
In this case, $a_r^{ⓓ} = x$.

Proof. Straightforward from Theorem 4.5. □

5. Right Pesudo Weighted Core Inverses

In this section, we are concerned with right pesudo weighted core inverse in a Banach *-algebra. Let $a \in \mathcal{A}$, and let
$$\{a_r^{D,w}\} = \{x \in \mathcal{A} \mid a(wx)^2 = x = xwawx, (aw)^k = awxw(aw)^k \text{ for some } k \in \mathbb{N}\}.$$
Evidently, $a \in \mathcal{A}$ has the right w-Drazin inverse if and only if $\{a_r^{D,w}\} \neq \emptyset$.

Lemma 5.1. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_r^{Ⓓ,w}$ if and only if*

(1) $a \in \mathcal{A}_r^{ⓓ,w}$;
(2) $\{a_r^{D,w}\} \neq \emptyset$.

In this case, $a_r^{Ⓓ,w} = a_r^{ⓓ,w}$.

Proof. $\Longrightarrow$ This implication is obvious.

$\Longleftarrow$ Let $x = a_r^{ⓓ,w}$. Then
$$a(wx)^2 = x = xwawx, (wawx)^* = wawx, \lim_{n\to\infty} ||(aw)^n - awxw(aw)^n||^{\frac{1}{n}} = 0.$$

Let $y \in \{a_r^{D,w}\}$. Then $a(wy)^2 = y = ywawy, (aw)^k = awyw(aw)^k$ for some $k \in \mathbb{N}$. Set $z = ywawx$. Then we verify that

$$\begin{aligned} wawz &= waw(ywawx) = wawy(wa)^k(wx)^k = w[awyw(aw)^k]x(wx)^{k-1} \\ &= w(aw)^k x(wx)^{k-1} = waw(aw)^{k-1}(xw)^{k-1}x \\ &= waw(aw)(xw)x = (wa)^2(wx)^2 = wawx. \end{aligned}$$

Hence $(wawz)^* = (wawx)^* = wawx = wawz$. We observe that

$$\begin{aligned} awzw &= awywawxw = awyw(aw)^k(xw)^k = awxw, \\ a(wz)^2 &= awzwz = awxwz = awxwywawx = awxw[a(wy)^2]wawx \\ &= [a(wy)^2]wawx = ywawx = z, \\ zwawz &= ywawx(wawz) = ywaw(xwawx) = ywawx = z. \end{aligned}$$

Moreover, we have

$$\begin{aligned} (aw)^k - awzw(aw)^k &= (aw)^k - awxw(aw)^k \\ &= (aw)(ay)(aw)^k - awxw[(aw)(ay)(aw)^k] \\ &= (aw)^n(ay)^n(aw)^k - awxw[(aw)^n(ay)^n(aw)^k] \\ &= [(aw)^n - awxw(aw)^n](ay)^n(aw)^k. \end{aligned}$$

Hence,

$$||(aw)^k - awzw(aw)^k|| \leq ||(aw)^n - awxw(aw)^n||||(ay)^n(aw)^k||.$$

Since $\lim\limits_{n\to\infty} ||(aw)^n - awxw(aw)^n||^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n\to\infty} ||(aw)^k - awzw(aw)^k||^{\frac{1}{n}} = 0;$$

and then $(aw)^k = awzw(aw)^k$. Therefore $a \in \mathcal{A}_r^{\mathbb{D},w}$. $\square$

We are ready to prove:

Theorem 5.2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\mathbb{D},w}$.
(2) There exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_r^{\#,w},\ yw \in \mathcal{A}^{nil}.$$

(3) There exist $x, y \in \mathcal{A}$ such that

$$a = x + y, ywx = 0, x \in \mathcal{A}_r^{\#,w},\ yw \in \mathcal{A}^{nil}.$$

Proof. $(1) \Rightarrow (2)$ By virtue of Lemma 5.1, $a \in \mathcal{A}_r^{\textcircled{d},w}$ and $a_r^{\textcircled{d},w} = a_r^{\mathbb{D},w}$. In view of Theorem 3.1, we can find $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_r^{\#}, \ y \in \mathcal{A}_w^{qnil}.$$

Explicitly, we have $y = a - awa_r^{\textcircled{d},w}wa = a - awa_r^{\mathbb{D},w}wa$. Set $z = a_r^{\mathbb{D},w}$. Then $a(wz)^2 = z = zwawz, (wawz)^* = wawz$ and $awaz(aw)^k = (aw)^k$ for some $k \in \mathbb{N}$. It is easy to see that

$$\begin{aligned} (1 - awzw)aw(awz) &= (1 - awzw)aw((aw)^{k-1}z(wz)^{k-2}) \\ &= (1 - awzw)(aw)^k z(wz)^{k-2} = 0. \end{aligned}$$

Moreover, we verify that

$$\begin{aligned}
(aw - zw(aw)^2)^{k+2} &= (1 - zwaw)aw[aw - zw(aw)^2]^k[aw - zw(aw)^2] \\
&= (1 - zwaw)aw[aw - zw(aw)^2]^{k-1}[aw - zw(aw)^2]aw \\
&= (1 - zwaw)aw(aw - zw(aw)^2)^{k-1}(aw)^2 \\
&\vdots \\
&= (1 - zwaw)aw(aw - zw(aw)^2)(aw)^k \\
&= (1 - zwaw)((aw)^k - awzw(aw)^k)(aw) \\
&= 0.
\end{aligned}$$

This implies that $aw - zw(aw)^2 \in \mathcal{A}^{nil}$. Therefore $yw = aw - awzwaw \in \mathcal{A}^{nil}$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_r^{\#,w},\ yw \in \mathcal{A}^{nil}.$$

Since $\mathcal{A}^{nil} \subseteq \mathcal{A}^{qnil}$. In view of Theorem 3.1, $a \in \mathcal{A}_r^{\textcircled{D},w}$ and $z := a_r^{\textcircled{D},w} = x_r^{\#,w}$. Hence, $a(wz)^2 = z$ and $(wawz)^* = wawz$. Write $(yw)^k = 0 (k \in \mathbb{N})$. Then

$$\begin{aligned}
(aw)^k &= \sum_{i=0}^{k}(xw)^i(yw)^{n-i} = (yw)^k + xw(yw)^{k-1} + \cdots + (xw)^{k-1}yw + (xw)^k \\
&= xw(yw)^{k-1} + \cdots + (xw)^{k-1}yw + (xw)^k.
\end{aligned}$$

Then

$$\begin{aligned}
awzw(aw)^k &= (xw + yw)x_r^{\#}wx[(yw)^{k-1} + \cdots + (xw)^{k-2}yw + (xw)^{k-1}] \\
&= xw[(yw)^{k-1} + \cdots + (xw)^{k-2}y + (xw)^{k-1}] = (aw)^k.
\end{aligned}$$

Therefore $a \in \mathcal{A}_r^{\textcircled{D},w}$, as asserted. $\square$

Recall that a has w-Drazin inverse x provided that the following identities are satisfied:

$$xwawx = x, awx = xwa, xw(aw)^k = (aw)^{k+1}$$

for some $k \in \mathbb{N}$.

Corollary 5.3. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{D},w}$ if and only if*

(1) $a \in \mathcal{A}^{D,w}$;
(2) there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_r^{\#},\ y \in \mathcal{A}^{nil}.$$

(3) there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, ywx = 0, x \in \mathcal{A}_r^{\#},\ y \in \mathcal{A}^{nil}.$$

Proof. $\Longrightarrow$ This is obvious by Theorem 5.2 and [5, Lemma 2].

$\Longleftarrow$ In view of Theorem 5.2, $a \in \mathcal{A}_r^{\textcircled{D},w}$. Therefore we complete the proof by [5, Lemma 2]. $\square$

Theorem 5.4. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{D},w}$.
(2) There exists $m \in \mathbb{N}$ such that $a(wa)^{k-1} \in \mathcal{A}_r^{\#,w}$ for any $k \geq m$.
(3) $a(wa)^{k-1} \in \mathcal{A}_r^{\#,w}$ for some $k \in \mathbb{N}$.

 doi:10.20944/preprints202507.1148.v1

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $z \in \mathcal{A}$ and $m \in \mathbb{N}$ such that

$$a(wz)^2 = z, (wawz)^* = wawz, (aw)^{m-1} = (aw)(zw)(aw)^{m-1}.$$

Let $x = (zw)^{k-1}z(k \geq m)$. Then $(aw)^{k-1} = (aw)(zw)(aw)^{k-1}$, and so

$$\begin{array}{rcl}
(wa)^k wx & = & (wa)^{k-1}waw(zw)^{k-1}z \\
& = & (wa)^{k-1}w[a(wz)^2](wz)^{k-2} \\
& = & (wa)^{k-1}wz(wz)^{k-2} \\
& = & (wa)^{k-1}(wz)^{k-1} = wawz, \\
\big(a(wa)^{k-1}\big)(wx)^2 & = & w^{-1}(wawz)(wx) = awzw(zw)^{k-1}z \\
& = & [a(wz)^2]w(zw)^{k-2}z \\
& = & zw(zw)^{k-2}z = (zw)^{k-1}z = x, \\
w\big(a(wa)^{k-1}\big)wx & = & (wa)^k wx = wawz, \\
\big(w[a(wa)^{k-1}]wx\big) & = & w[a(wa)^{k-1}]wx, \\
\big(a(wa)^{k-1}\big)wxw\big(a(wa)^{k-1}\big) & = & \big(a(wa)^{k-1}\big)w(zw)^{k-1}zw\big(a(wa)^{k-1}\big) \\
& = & (aw)^k(zw)^k[a(wa)^{k-1}] = awzwa(wa)^{k-1} \\
& = & awzw[a(wa)^{k-1}] = [awzw(aw)^{k-1}] \\
& = & a(wa)^{k-1}.
\end{array}$$

Therefore $a(wa)^{k-1} \in \mathcal{A}_r^{\oplus,w}$, as required.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Set $x = a(wa)^{k-2}w[a(wa)^{k-1}]_r^{\oplus,w}$. Then we verify that

$$\begin{array}{rcl}
awx & = & awa(wa)^{k-2}w[a(wa)^{k-1}]_r^{\oplus,w} = a(wa)^{k-1}w[a(wa)^{k-1}]_r^{\oplus,w}, \\
(wawx)^* & = & \big(w[a(wa)^{k-1}]w[a(wa)^{k-1}]_r^{\oplus,w}\big)^* \\
& = & w[a(wa)^{k-1}]w[a(wa)^{k-1}]_r^{\oplus,w} = wawx, \\
a(wx)^2 & = & a(wa)^{k-1}w[a(wa)^{k-1}]_r^{\oplus,w}(wa)^{k-1}w[a(wa)^{k-1}]_r^{\oplus,w} \\
& = & a(wa)^{k-1}w[a(wa)^{k-1}]_r^{\oplus,w}wa(wa)^{k-1}(wa)^{k-1}\big([a(wa)^{k-1}]_r^{\oplus,w}\big)^2 \\
& = & [a(wa)^{k-1}w[a(wa)^{k-1}]_r^{\oplus,w}wa(wa)^{k-1}](wa)^{k-1}\big([a(wa)^{k-1}]_r^{\oplus,w}\big)^2 \\
& = & a(wa)^{k-1}(wa)^{k-1}\big([a(wa)^{k-1}]_r^{\oplus,w}\big)^2 \\
& = & a(wa)^{k-2}w[a(wa)^{k-1}]\big([a(wa)^{k-1}]_r^{\oplus,w}\big)^2 \\
& = & a(wa)^{k-2}w[a(wa)^{k-1}]_r^{\oplus,w} = x, \\
awxw(aw)^k & = & [a(wa)^{k-1}w[a(wa)^{k-1}]_r^{\oplus,w}wa(wa)^{k-1}]w \\
& = & a(wa)^{k-1}w = (aw)^k.
\end{array}$$

This completes the proof. $\square$

Corollary 5.5. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{\scriptsize D},w}$.

(2) $a(wa)^{k-1} \in \mathcal{A}_w^{(1,3)}$ and $(aw)^k\mathcal{A} = (aw)^{k+1}\mathcal{A}$ for some $k \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 5.4, $a(wa)^{k-1} \in \mathcal{A}_w^{(1,3)}$. By hypothesis, there exist $x \in \mathcal{A}$ such that

$$a(wx)^2 = x, (wawx)^* = wawx, (aw)^k = (aw)(xw)(aw)^k$$

for some $k \in \mathbb{N}$. Then $(aw)^k = (aw)^{k+1}(xw)^{k+1}(aw)^k$; hence, $(aw)^k\mathcal{A} = (aw)^{k+1}\mathcal{A}$, as required.

(2) $\Rightarrow$ (1) In view of Theorem 5.4, we will suffice to prove $a(wa)^{k-1} \in \mathcal{A}_r^{\oplus,w}$.

We claim that $a(wa)^{k-1}\mathcal{A} = a(wa)^{k-1}wa(wa)^{k-1}\mathcal{A}$. Obviously, we have $a(wa)^{k-1}wa(wa)^{k-1}\mathcal{A} \subseteq a(wa)^{k-1}\mathcal{A}$. Since $a(wa)^{k-1} \in \mathcal{A}_w^{(1,3)}$, we can find some $z \in \mathcal{A}$ such that $a(wa)^{k-1} = a(wa)^{k-1}wzwa(wa)^{k-1}$.

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As $(aw)^k\mathcal{A}=(aw)^{k+1}\mathcal{A}$, by induction, we have $(aw)^k\mathcal{A}=(aw)^{k+m}\mathcal{A}$ for any $m\in\mathbb{N}$. Thus, we derive that

$$\begin{array}{rcl} a(wa)^{k-1} & = & a(wa)^{k-1}wzwa(wa)^{k-1}\\ & = & (aw)^kzwa(wa)^{k-1}\in(aw)^{2k}zwa(wa)^{k-1}\mathcal{A}\\ & \subseteq & (aw)^k(aw)^{k-1}aw\mathcal{A}\\ & = & a(wa)^{k-1}wa(wa)^{k-1}\mathcal{A}, \end{array}$$

and then $a(wa)^{k-1}\mathcal{A}=a(wa)^{k-1}wa(wa)^{k-1}\mathcal{A}$. By virtue of Theorem 2.1, $a(wa)^{k-1}\in\mathcal{A}_r^{\textcircled{\#},w}$. Therefore $a\in\mathcal{A}_r^{\textcircled{D},w}$ by Theorem 5.4. $\square$

We proceed to prove:

Theorem 5.6. *Let $a\in\mathcal{A}$. Then $a\in\mathcal{A}_r^{\textcircled{D},w}$ if and only if there exist a projection $p\in\mathcal{A}$ and $k\in\mathbb{N}$ such that*

$$wa+p\in\mathcal{A}_r^{-1},(1-p)\mathcal{A}=wa(1-p)wa\mathcal{A},(pwa)^k=0,a(wa)^{k-1}\in\mathcal{A}^{(1,3)}.$$

Proof. $\Longrightarrow$ In light of Theorem 3.6, there exists a projection $p\in\mathcal{A}$ such that

$$wa+p\in\mathcal{A}_r^{-1},pwa\in\mathcal{A}^{qnil},(1-p)\mathcal{A}=wa(1-p)wa\mathcal{A}.$$

Explicitly, $p=1-wawa_r^{\textcircled{D},w}$. As in the proof of Theorem 3.1, we check that $pwa=(1-wawa_r^{\textcircled{D},w})wa\in\mathcal{A}^{nil}$. Write $(pwa)^m=0$ for some $m\in\mathbb{N}$. By using Theorem 5.4, we can find a $k\geq m$ such that $a(wa)^{k-1}\in\mathcal{A}^{(1,3)}$, as required.

$\Longleftarrow$ By hypothesis, there exist a projection $p\in\mathcal{A}$ and $k\geq 2$ such that

$$wa+p\in\mathcal{A}_r^{-1},(1-p)\mathcal{A}=wa(1-p)wa\mathcal{A},(pwa)^{k-1}=0,a(wa)^{k-1}\in\mathcal{A}_w^{(1,3)}.$$

Then $pwa(1-p)wa=0$, and so $pwa=pwap$. Hence, $p(wa)^2=(pwa)wa=(pwap)wa=(pwa)^2$. By induction, we have $p(wa)^{k-1}=(pwa)^{k-1}=0$. This implies that $(wa)^{k-1}=(1-p)(wa)^{k-1}$. Since $1-p\in wa(1-p)\mathcal{A}$, by induction, $1-p\in(wa)^m$ for any $m\geq 2$. Then $(wa)^{k-1}\in(wa)^{k+1}\mathcal{A}$. Therefore $(aw)^k=a(wa)^{k-1}w\in a(wa)^{k+1}\mathcal{A}\subseteq(aw)^{k+1}\mathcal{A}$. This implies that $(aw)^k\mathcal{A}=(aw)^{k+1}\mathcal{A}$. According to Corollary 5.5, $a\in\mathcal{A}_r^{\textcircled{D},w}$. $\square$

As an immediate consequence, we derive

Corollary 5.7. *Let $a\in\mathcal{A}$. Then $a\in\mathcal{A}^{\textcircled{D},w}$ if and only if $a\in\mathcal{A}^{D,w}$ and there exist a projection $p\in\mathcal{A}$ and $k\in\mathbb{N}$ such that*

$$wa+p\in\mathcal{A}_r^{-1},(1-p)\mathcal{A}=wa(1-p)wa\mathcal{A},(pwa)^k=0,a(wa)^{k-1}\in\mathcal{A}^{(1,3)}.$$

Remark 5.8. *Generalized weighted left core inverse can be defined dually. The corresponding results for the generalized weighted left core inverse can be established in a similar way.*

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