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Prosper Rosaire MAMA ASSANDJE * , Joseph Dongho , Thomas Bouetou Bouetou

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Determination of the Integral Curves of the Completely Integrable Gradient System on the Three-Parameter Beta Bivariate Statistical Manifold of the First Kind

Prosper Rosaire Mama Assandje [1,†,‡], Joseph Dongho [2,‡] and Thomas Bouetou Bouetou[3,*]

[1] University of Maroua; www.fs.univ.maroua.cm
[2] University of Maroua; www.fs.univ.maroua.cm
[3] Higher national school of Polytechnic; thomas.bouetou@polytechnique.cm
* Correspondence: mamarosaire@yahoo.fr; Tel.:+237696477562
† Current address: University of Maroua.
‡ These authors contributed equally to this work.

Abstract: In this paper, it is shown that the bivariate beta statistical manifold of the first kind with three parameters has a gradient system Hamiltonian and is completely integrable. It is shown that this system admits a Lax pair representation. The question here is how to construct a gradient system and show that it is completely integrable while determining the integrable curves on the three-parameter bivariate beta variety of the first kind admitting a potential function?To do this, we prove the existence of the potential on the manifold using ovidiu in [9], then using Amari's theorems in [7], we show that this coordinate system admits a dual coordinate system. This will allow us to determine the Riemannian metric on the manifold and to construct the gradient system. We linearise this system using Nakamura's method in [3]. We prove that it is Hamiltonian and completely integrable using the theorem in [11]. It is shown that the Bivariate Beta of the first kind with three parameters, is an exponential function. The gradient system is linearizable. It is proven that the associate gradient system is Hamiltonian with $\mathcal{H}$ which is in involution and satisfying $d\mathcal{H} = 0$. Therefore, the gradient system obtained is a sub-dynamical system of a 4-dimensional system and is a completely integrable system. We show that the gradient system on the statistical model has the following Lax pair representation: $\dot{L} = [L, N]$ where L is a symmetric matrix and N is the diagonal matrix. The gradient system defined by the Bivariate Beta family of the first kind with three parameters odd-dimensional manifold is a completely integrable Hamiltonian system.

Keywords: Hamiltonian function; gradient system; Lax pair representation

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0. Introduction

This paper, it is develops an important example of the main Theorem on complete Integrability of gradient systems on a manifold admitting a potential in odd dimension in [11]. In [10] and [11], the dynamical system describes the passage in time of any object in the space of states S of the physical system. Being given a differentiable manifold S, any section X of the tangent fibred at a point p of S denoted by $TS = \cup_{p\in S} T_pS$, induces a symmetric and positive bilinear form g called a Riemannian metric. In [7], a statistical manifold (S, g, ∇, ∇^*), is a Riemannian manifold (S, g) with a dual connection pair (∇, ∇^*); where g is a Riemannian manifold and ∇ is a affine connection on S and verify

$$g(Y,\ Z) = g\left(\nabla_X Y,\ Z\right) + g\left(Y,\ \nabla^*_X Z\right)$$

where X, Y, Z are the vector fields. In [5], the Riemannian metric g induces an isomorphism between the module of vector fields $\mathfrak{X}(S)$ and the modulus of linear forms $\Omega(S)$ on S, denoted b_g. Let $\theta =$

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$(\theta_1, \dots, \theta_n)$ a coordinate system on S. In [5] and [11], a gradient system on the Riemannian manifold is the negative flow of the gradient field $grad_g(\Phi)$ defined by

$$\overrightarrow{\dot{\theta}} = -[b_g]^{-1}\partial_\theta \Phi(\theta), \tag{1}$$

where $\Phi \in C^\infty(S)$ is the potential function associated with the gradient field $grad_g(\Phi)$ with respect to g. $\partial_\theta \Phi(\theta) = \left(\partial_{\theta_1}\Phi(\theta), \dots, \partial_{\theta_n}\Phi(\theta)\right)^T$, such that $[b_g] = (g_{ij})_{1\le i;j\le n}$, with $(g_{ij})_{1\le i;j\le n}$ is the Fisher information matrix on S. In this paper, the integral curves of the gradient system on the three-parameter bivariate beta manifold of the first kind are determined. Jacobi's integration of the geodesics of a three-axis system of three geodesics on a three-axis ellipsoid, the motion of the vertex under the effect of gravity by Kovalevski in [2] for particular axis ratios. These efforts culminated in the work of Jacobi, who skilfully applied the method of separating variables to partial differential equations, the Hamilton-Jacobi, associated with the mechanical system in order to their integrable nature. In [1], Poincare recognized that integrability is an exceptional phenomenon of the Hamiltonian system. In 1993, Nakamura [3], proves the complete integrability of gradient systems constructed on the Gaussian and multinomial manifold. In the same year, Fujiwara [6], generalized Nakamura's work to even dimensions. In 2023, in [11], it is studied the complete integrability of gradient systems defined on a statistical manifold in odd-dimension. How can we characterize the gradient system on S in order to study the integral curves on the three-parameter bivariate beta manifold of the first kind? Following this question, it is shown on

$$S = \left\{ p_\theta(x) = \frac{1}{B(a,b,c)} x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1}, \begin{array}{l} \theta = (a,\ b,\ c) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x = (x_1,\ x_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x_1 + x_2 < 1 \\ B(a,b,c) = \frac{\Gamma(a).\Gamma(b).\Gamma(c)}{\Gamma(a+b+c)} \end{array} \right\}$$

where p_θ is the Bivariate Beta family of the first kind with three parameters, that p_θ is an exponential function. Therefore, according to Ovidiu [9], $\Phi = -\log \frac{1}{B(a,b,c)}$ is its related potential function. It is concluded that S admits a dual-coordinate pair $(\theta,\ \eta)$. The gradient system is linearizable and we have to take the following form $\dot{\eta}_i = -\eta_i$. It is proven that the associate gradient system is Hamiltonian with $\mathcal{H}$ which is in involution and satisfying $d\mathcal{H} = 0$. Therefore, applying the fundamental theorem and prove that On a statistical manifold S of three-dimensional defined by the Bivariate Beta family of the first kind with three parameters, the gradient system obtained is a sub-dynamical system of a 4-dimensional system and is a completely integrable system. From Stirling's formula [4], it is shown that the potential function is given by: $\Phi(\theta) = (a+b+c-\frac{1}{2})\log(a+b+c-1) + (\frac{1}{2}-a)\log(a-1) + (\frac{1}{2}-b)\log(b-1) + (\frac{1}{2}-c)\log(c-1) - \log(2\pi) - 2$. and for all $\Upsilon = \left\{(a,b,c) \in \mathbb{R}^3 \,|\, 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1\right\}$ the cube of edges $1cm$. We show that the dual potential function associated on S is defined for all $(a,b,c) \in \mathbb{R}^3 - \{\Upsilon\}$ is given by, $\Psi(\eta) = -\frac{a}{2(a-1)} - \frac{b}{2(b-1)} - \frac{c}{2(c-1)} + \frac{1}{2}\log(a+b+c-1) - \frac{1}{2}\log(a-1) - \frac{1}{2}\log(b-1) - \frac{1}{2}\log(c-1) - k$. So, we prove that for all Υ the cube such that $\Upsilon = \left\{(a,b,c) \in \mathbb{R}^3 \,|\, 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1\right\}$, let $(a,b,c) \in \mathbb{R}^3 - \{\Upsilon\}$, let $D =< A(0, \frac{3}{2}, \frac{3}{2});\ \vec{e}\begin{pmatrix}1\\0\\0\end{pmatrix} >$ the vector line, let $\mathbb{V} =< \left(\frac{8bc-15b-15c+27}{4cb-8b-8c+15}, b, c\right) >$

- If $(a,b,c) \in \mathbb{V} \cup \{D\}$, then S then is algebraic manifold.
- If $(a,b,c) \in \mathbb{R}^3 - \{\Upsilon; \mathbb{V} \cup \{D\}\}$ then G is a Pseudo- Riemannian metric.

We also, prove that the gradient system on S obtained by using Stirling's formula has the following linear representation $\dot{\eta}_i = -\eta_i$. Using [11], we prove that the gradient system is Hamiltonian and completely integrable. We show that the gradient system on S has the following Lax pair representation: $\dot{L} = [L, N]$ where L is a symmetric matrix and N is the diagonal matrix. After the introduction, in section 2, recall the preliminaries motion on theory of statistical manifold, in section 3 we determine the Riemannian structure on Bivariate Beta family of the first kind with three parameters distribution,

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in section 4, linearization and integrability of Gradient Systems, in section 5, we determine the Riemannian structure related to Stirling's formula. In section 6, we have the integrability of Gradient Systems. At the end, we show that this gradient system admits a Lax pair representation.

1. Preliminaries

Let $S = \left\{ p_\theta(x), \begin{array}{c} \theta \in \Theta \\ x \in \mathcal{X} \end{array} \right\}$ be the set of probabilities p_θ, parametrized by Θ, open a subset of $\mathbb{R}^n$; on the sample space $\mathcal{X} \subseteq \mathbb{R}$. Let $\mathcal{F}(\mathcal{X}, \mathbb{R})$ be the space of real-valued smooth functions on $\mathcal{X}$. According to Ovidiu [9], the log-likelihood function is a mapping defined by

$$\begin{array}{rcl} l : S & \longrightarrow & \mathcal{F}(\mathcal{X}, \mathbb{R}) \\ p_\theta & \longmapsto & l\left(p_\theta\right)(x) = \log p_\theta(x) \end{array}$$

Sometimes, for convenient reasons, this will be denoted by $l(x, \theta) = l\left(p_\theta\right)(x)$.
In [3] and [7], the Fisher information defined by

$$(g_{ij})_{1 \le i;j \le n} = \left(-\mathbb{E}[\partial_{\theta_i} \partial_{\theta_j} l(x, \theta)] \right) \tag{2}$$

Denote $G = (g_{ij})_{1 \le i;j \le n}$ the Fisher information matrix, the gradient system is given by

$$\vec{\dot{\theta}} = -G^{-1} \partial_\theta \Phi(\theta). \tag{3}$$

The complete integrability of gradient system (3) is proven if the Theorem 1 in [11] is verify.

2. Riemannian structure on Bivariate Beta family of the first kind with three parameters distribution

In this section, we present the particular properties of beta bivariate Beta family of the first kind with three parameters and associated information metric and associated gradient system.

2.1. Particular properties of Beta Bivariate Beta family of the first kind with three parameters.

Let

$$S = \left\{ p_\theta(x) = \frac{1}{B(a,b,c)} x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1}, \begin{array}{l} \theta = (a,\ b,\ c) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x = (x_1,\ x_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x_1 + x_2 < 1 \\ B(a,b,c) = \frac{\Gamma(a).\Gamma(b).\Gamma(c)}{\Gamma(a+b+c)} \end{array} \right\}$$

a statistical manifold.

Proposition 1. *The Bivariate Beta family of the first kind with three parameters distribution*

$$p_\theta(x) = \frac{1}{B(a,b,c)} x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1} \tag{4}$$

is an exponential family for all $B(a,b,c) > 0$.

Proof. We have,

$$p_\theta(x) = \exp\left[(a-1)\log x_1 + (b-1)\log x_2 + (c-1)\log(1 - x_1 - x_2) + \log \frac{1}{B(a,b,c)} \right]$$

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Whose, we have

$$\begin{aligned} p_\theta(x) &= \exp\Big[-\log x_1 - \log x_2 - \log(1-x_1-x_2) + a\log x_1 + b\log x_2 \\ &\quad + c\log(1-x_1-x_2) + \log\frac{1}{B(a,b,c)}\Big] \end{aligned}$$

whose,

$$p_\theta(x) = \exp\left[C(x) + a f_1(x_1) + b\, f_2(x_2) + c f_3(x_1,x_2) - \Phi(\theta)\right]$$

with:

$$f_1(x_1) = \log x_1,\ f_2(x_2) = \log x_2;\ f_3(x_1,x_2) = \log(1-x_1-x_2),$$
$$\Phi(\theta) = -\log\frac{1}{B(a,b,c)}$$

More explicitly the potential function given by

$$\Phi(\theta) = \log\Gamma(a) + \log\Gamma(b) + \log\Gamma(c) - \log\Gamma(a+b+c) \tag{5}$$

and

$$C(x) = -\log x_1 - \log x_2 - \log(1-x_1-x_2).$$

According to Ovidiu Scalin in his book [9] this family is an exponential family. □

2.2. Associated information metric and associated gradient system.

Lemma 1. *The Fisher information matrix for the Bivariate Beta family of the first kind with three parameters is given by*

$$G = \begin{pmatrix} -\phi_{aa}(a)+\phi_{aa}(a+b+c) & \phi_{ab}(a+b+c) & \phi_{ac}(a+b+c) \\ \phi_{ab}(a+b+c) & -\phi_{bb}(b)+\phi_{bb}(a+b+c) & \phi_{bc}(a+b+c) \\ \phi_{ac}(a+b+c) & \phi_{bc}(a+b+c) & -\phi_{cc}(c)+\phi_{cc}(a+b+c) \end{pmatrix} \tag{6}$$

with

$$\begin{aligned}
\phi_{aa}(a) &= -\phi_a^2(a) + \phi_a^2(a+b+c) \\
\phi_{aa}(a+b+c) &= -\frac{\frac{\partial^2\Gamma(a)}{\partial a^2}}{\Gamma(a)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial a^2}}{\Gamma(a+b+c)} \\
\phi_{bb}(b) &= -\phi_b^2(b) + \phi_b^2(a+b+c) \\
\phi_{bb}(a+b+c) &= -\frac{\frac{\partial^2\Gamma(b)}{\partial b^2}}{\Gamma(b)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial b^2}}{\Gamma(a+b+c)} \\
\phi_{cc}(a) &= -\phi_c^2(c) + \phi_c^2(a+b+c) \\
\phi_{cc}(a+b+c) &= -\frac{\frac{\partial^2\Gamma(c)}{\partial c^2}}{\Gamma(c)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial c^2}}{\Gamma(a+b+c)} \\
\phi_{ab}(a+b+c) &= \frac{\partial_a\partial_b\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_a\Gamma(a+b+c).\partial_b\Gamma(a+b+c)}{\Gamma^2(a+b+c)} \\
\phi_{ac}(a+b+c) &= \frac{\partial_a\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_a\Gamma(a+b+c).\partial_c\Gamma(a+b+c)}{\Gamma^2(a+b+c)} \\
\phi_{bc}(a+b+c) &= \frac{\partial_b\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_b\Gamma(a+b+c).\partial_c\Gamma(a+b+c)}{\Gamma^2(a+b+c)}
\end{aligned}$$

and $\phi_a(a) = \frac{\partial_a\Gamma(a)}{\Gamma(a)}$, $\phi_b(b) = \frac{\partial_b\Gamma(b)}{\Gamma(b)}$, $\phi_c(c) = \frac{\partial_c\Gamma(c)}{\Gamma(c)}$, $\phi_a(a+b+c) = \frac{\partial_a\Gamma(a+b+c)}{\Gamma(a+b+c)}$, $\phi_b(a+b+c) = \frac{\partial_b\Gamma(a+b+c)}{\Gamma(a+b+c)}$, $\phi_c(a+b+c) = \frac{\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)}$, *where* $\phi_{aa}(a) = \partial_a\phi_a(a)$,
$\partial_a = \frac{\partial}{\partial_a}$

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Proof. we have

$$l(x,\theta) = \log p_\theta(x) = C(x) + af_1(x_1) + b\, f_2(x_2) + cf_3(x_1,x_2) - \Phi(\theta). \tag{7}$$

using the expression (7) we have the following relation;

$$\begin{aligned}
\frac{\partial l(x,\theta)}{\partial_a} &= f_1(x_1) + \frac{\partial_a \Gamma(a)}{\Gamma(a)} - \frac{\partial_a \Gamma(a+b+c)}{\Gamma(a+b+c)} \\
\frac{\partial l(x,\theta)}{\partial_b} &= f_2(x_2) + \frac{\partial_b \Gamma(b)}{\Gamma(b)} - \frac{\partial_b \Gamma(a+b+c)}{\Gamma(a+b+c)} \\
\frac{\partial l(x,\theta)}{\partial_c} &= f_3(x_1,x_2) + \frac{\partial_c \Gamma(c)}{\Gamma(c)} - \frac{\partial_c \Gamma(a+b+c)}{\Gamma(a+b+c)}
\end{aligned}$$

So we obtain the following expression

$$\begin{aligned}
\frac{\partial^2 l(x,\theta)}{\partial_{a^2}} &= -\phi_a^2(a) + \phi_a^2(a+b+c) + \frac{\frac{\partial^2\Gamma(a)}{\partial a^2}}{\Gamma(a)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial a^2}}{\Gamma(a+b+c)} \\
\frac{\partial^2 l(x,\theta)}{\partial_{b^2}} &= -\phi_b^2(b) + \phi_b^2(a+b+c) + \frac{\frac{\partial^2\Gamma(b)}{\partial b^2}}{\Gamma(b)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial b^2}}{\Gamma(a+b+c)} \\
\frac{\partial^2 l(x,\theta)}{\partial_{c^2}} &= -\phi_c^2(c) + \phi_c^2(a+b+c) + \frac{\frac{\partial^2\Gamma(c)}{\partial c^2}}{\Gamma(c)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial c^2}}{\Gamma(a+b+c)} \\
\frac{\partial^2 l(x,\theta)}{\partial_a \partial_b} &= \frac{\partial_a\partial_b\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_a\Gamma(a+b+c).\partial_b\Gamma(a+b+c)}{\Gamma^2(a+b+c)} \\
\frac{\partial^2 l(x,\theta)}{\partial_a \partial_c} &= \frac{\partial_a\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_a\Gamma(a+b+c).\partial_c\Gamma(a+b+c)}{\Gamma^2(a+b+c)} \\
\frac{\partial^2 l(x,\theta)}{\partial_b \partial_c} &= \frac{\partial_b\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_b\Gamma(a+b+c).\partial_c\Gamma(a+b+c)}{\Gamma^2(a+b+c)}
\end{aligned}$$

by setting

$$\begin{aligned}
\phi_{aa}(a) &= -\phi_a^2(a) + \phi_a^2(a+b+c) \text{ and } \phi_{aa}(a+b+c) = -\frac{\frac{\partial^2\Gamma(a)}{\partial a^2}}{\Gamma(a)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial a^2}}{\Gamma(a+b+c)} \\
\phi_{bb}(b) &= -\phi_b^2(b) + \phi_b^2(a+b+c) \text{ and } \phi_{bb}(a+b+c) = -\frac{\frac{\partial^2\Gamma(b)}{\partial b^2}}{\Gamma(b)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial b^2}}{\Gamma(a+b+c)} \\
\phi_{cc}(a) &= -\phi_c^2(c) + \phi_c^2(a+b+c) \text{ and } \phi_{cc}(a+b+c) = -\frac{\frac{\partial^2\Gamma(c)}{\partial c^2}}{\Gamma(c)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial c^2}}{\Gamma(a+b+c)} \\
\phi_{ab}(a+b+c) &= \frac{\partial_a\partial_b\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_a\Gamma(a+b+c).\partial_b\Gamma(a+b+c)}{\Gamma^2(a+b+c)} \\
\phi_{ac}(a+b+c) &= \frac{\partial_a\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_a\Gamma(a+b+c).\partial_c\Gamma(a+b+c)}{\Gamma^2(a+b+c)} \\
\phi_{bc}(a+b+c) &= \frac{\partial_b\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_b\Gamma(a+b+c).\partial_c\Gamma(a+b+c)}{\Gamma^2(a+b+c)}
\end{aligned}$$

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we obtain the following expression

$$
\begin{aligned}
\frac{\partial^2 l(x,\theta)}{\partial_{a^2}} &= \phi_{aa}(a) - \phi_{aa}(a+b+c) \\
\frac{\partial^2 l(x,\theta)}{\partial_{b^2}} &= \phi_{bb}(b) - \phi_{bb}(a+b+c) \\
\frac{\partial^2 l(x,\theta)}{\partial_{c^2}} &= \phi_{cc}(a) - \phi_{cc}(a+b+c) \\
\frac{\partial^2 l(x,\theta)}{\partial_{ab}} &= -\phi_{ab}(a+b+c) \\
\frac{\partial^2 l(x,\theta)}{\partial_{ac}} &= -\phi_{ac}(a+b+c) \\
\frac{\partial^2 l(x,\theta)}{\partial_{bc}} &= -\phi_{bc}(a+b+c).
\end{aligned}
$$

Using (2), we have

$$
\begin{aligned}
g_{aa}(\theta) &= -\phi_{aa}(a) + \phi_{aa}(a+b+c) \\
g_{bb}(\theta) &= -\phi_{bb}(b) + \phi_{bb}(a+b+c) \\
g_{cc}(\theta) &= -\phi_{cc}(c) + \phi_{cc}(a+b+c) \\
g_{ab}(\theta) &= \phi_{ab}(a+b+c) \\
g_{ac}(\theta) &= \phi_{ac}(a+b+c) \\
g_{bc}(\theta) &= \phi_{bc}(a+b+c).
\end{aligned}
$$

□

Since our coordinate system $\theta = (a, b, c)$ admits a dual pair $\eta = (\eta_1, \eta_2, \eta_3)$ such that

$$
\begin{cases}
\eta_1 = \phi_a(a) - \phi_a(a+b+c) \\
\eta_2 = \phi_b(b) - \phi_b(a+b+c) \\
\eta_3 = \phi_c(c) - \phi_c(a+b+c)
\end{cases} \tag{8}
$$

According to Amari [7] there exists a dual potential function Ψ which verifies the Legendre equation. We have

$$
\begin{aligned}
\Psi(\eta) &= a\phi_a(a) - a\phi_a(a+b+c) + b\phi_a(a) - b\phi_b(a+b+c) + c\phi_c(c) - c\phi_c(a+b+c) \\
&\quad - \log\Gamma(a) - \log\Gamma(b) - \log\Gamma(c) + \log\Gamma(a+b+c)
\end{aligned}
$$

Theorem 1. *Let*

$$
S = \left\{ p_\theta(x) = \frac{1}{B(a,b,c)} x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1}, \quad \begin{array}{l} \theta = (a,\ b,\ c) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x = (x_1,\ x_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x_1 + x_2 < 1 \\ B(a,b,c) = \frac{\Gamma(a).\Gamma(b).\Gamma(c)}{\Gamma(a+b+c)} \end{array} \right\}
$$

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be the statistical manifold, where $p_\theta(x)$ is the density function of the Bivariate Beta family of the first kind with three parameters family. The gradient system defined on S is given by

$$\begin{cases} \dot{a} = -\zeta_1(a+b+c)\left(\phi_a(a) - \phi_a(a+b+c)\right) - \zeta_4(a+b+c)\left(\phi_b(b)\right) \\ -\phi_b(a+b+c) - \zeta_5(a+b+c)\left(\phi_c(c) + \phi_c(a+b+c)\right) \\ \dot{b} = -\zeta_4(a+b+c)\left(\phi_a(a) - \phi_a(a+b+c)\right) - \zeta_2(a+b+c)\left(\phi_b(b)\right) \\ -\phi_b(a+b+c) - \zeta_6(a+b+c)\left(\phi_c(c) + \phi_c(a+b+c)\right) \\ \dot{c} = -\zeta_5(a+b+c)\left(\phi_a(a) - \phi_a(a+b+c)\right) - \zeta_6(a+b+c)\left(\phi_b(b)\right) \\ -\phi_b(a+b+c) - \zeta_3(a+b+c)\left(\phi_c(c) + \phi_c(a+b+c)\right) \end{cases} \tag{9}$$

where

$$\begin{aligned}
\zeta_1(a+b+c) &= \frac{\phi_{bb}(b)\phi_{cc}(c) - \phi_{cc}(c)\phi_{cc}(a+b+c) - \phi_{cc}(c)\phi_{bb}(a+b+c)}{m(a,b,c)} \\
&\quad \frac{+\phi_{bb}(a+b+c)\phi_{cc}(a+b+c) - \phi_{bc}^2(a+b+c)}{m(a,b,c)} \\
\zeta_2(a+b+c) &= \frac{\phi_{aa}(a)\phi_{cc}(c) - \phi_{aa}(a)\phi_{cc}(a+b+c) - \phi_{cc}(c)\phi_{aa}(a+b+c)}{m(a,b,c)} \\
&\quad \frac{+\phi_{aa}(a+b+c)\phi_{cc}(a+b+c) - \phi_{ac}^2(a+b+c)}{m(a,b,c)} \\
\zeta_3(a+b+c) &= \frac{\phi_{aa}(a)\phi_{bb}(b) - \phi_{aa}(a)\phi_{bb}(a+b+c) - \phi_{bb}(b)\phi_{aa}(a+b+c)}{m(a,b,c)} \\
&\quad \frac{+\phi_{aa}(a+b+c)\phi_{bb}(a+b+c) - \phi_{ab}^2(a+b+c)}{m(a,b,c)} \\
\zeta_4(a+b+c) &= \frac{\phi_{bc}(a+b+c)\phi_{ac}(a+b+c) + \phi_{ab}(a+b+c)\phi_{cc}(c)}{m(a,b,c)} \\
&\quad \frac{-\phi_{ab}(a+b+c)\phi_{cc}(a+b+c)}{m(a,b,c)} \\
\zeta_5(a+b+c) &= \frac{\phi_{ab}(a+b+c)\phi_{bc}(a+b+c) + \phi_{ac}(a+b+c)\phi_{bb}(b)}{m(a,b,c)} \\
&\quad \frac{-\phi_{ac}(a+b+c)\phi_{bb}(a+b+c)}{m(a,b,c)} \\
\zeta_6(a+b+c) &= \frac{\phi_{bc}(a+b+c)\phi_{aa}(a) - \phi_{bc}(a+b+c)\phi_{aa}(a+b+c)}{m(a,b,c)} \\
&\quad \frac{+\phi_{ab}(a+b+c)\phi_{ac}(a+b+c)}{m(a,b,c)}
\end{aligned}$$

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and

$$
\begin{aligned}
\phi_{aa}(a) &= -\phi_a^2(a) + \phi_a^2(a+b+c) \\
\phi_{aa}(a+b+c) &= -\frac{\frac{\partial^2\Gamma(a)}{\partial a^2}}{\Gamma(a)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial a^2}}{\Gamma(a+b+c)} \\
\phi_{bb}(b) &= -\phi_b^2(b) + \phi_b^2(a+b+c) \\
\phi_{bb}(a+b+c) &= -\frac{\frac{\partial^2\Gamma(b)}{\partial b^2}}{\Gamma(b)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial b^2}}{\Gamma(a+b+c)} \\
\phi_{cc}(a) &= -\phi_c^2(c) + \phi_c^2(a+b+c) \\
\phi_{cc}(a+b+c) &= -\frac{\frac{\partial^2\Gamma(c)}{\partial c^2}}{\Gamma(c)} - \frac{\frac{\partial^2\Gamma(a+b+c)}{\partial c^2}}{\Gamma(a+b+c)} \\
\phi_{ab}(a+b+c) &= \frac{\partial_a\partial_b\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_a\Gamma(a+b+c).\partial_b\Gamma(a+b+c)}{\Gamma^2(a+b+c)} \\
\phi_{ac}(a+b+c) &= \frac{\partial_a\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_a\Gamma(a+b+c).\partial_c\Gamma(a+b+c)}{\Gamma^2(a+b+c)} \\
\phi_{bc}(a+b+c) &= \frac{\partial_b\partial_c\Gamma(a+b+c)}{\Gamma(a+b+c)} - \frac{\partial_b\Gamma(a+b+c).\partial_c\Gamma(a+b+c)}{\Gamma^2(a+b+c)}
\end{aligned}
$$

where

$$
\begin{aligned}
m(a,b,c) &= -\phi_{aa}(a)\phi_{bb}(b)\phi_{cc}(c) + \phi_{aa}(a)\phi_{bb}(b)\phi_{cc}(a+b+c) \\
&\quad -\phi_{aa}(a)\phi_{bb}(a+b+c)\phi_{cc}(a+b+c)\phi_{aa}(a)\phi_{bc}^2(a+b+c) \\
&\quad +\phi_{aa}(a+b+c)\phi_{bb}(b)\phi_{cc}(c) - \phi_{aa}(a+b+c)\phi_{bb}(b)\phi_{cc}(a+b+c) \\
&\quad -\phi_{aa}(a+b+c)\phi_{bb}(a+b+c)\phi_{cc}(a+b+c) - \phi_{aa}(a+b+c)\phi_{bc}^2(a+b+c) \\
&\quad +2\phi_{ab}(a+b+c)\phi_{bc}(a+b+c)\phi_{ac}(a+b+c)\phi_{ab}^2(a+b+c)\phi_{cc}(c) \\
&\quad -\phi_{ab}^2(a+b+c)\phi_{cc}(a+b+c) + \phi_{ac}^2(a+b+c)\phi_{bb}(b) \\
&\quad -\phi_{ac}^2(a+b+c)\phi_{bb}(a+b+c)
\end{aligned}
$$

is the determinant of the matrix G.

Proof. Using the lemma 1 we have

$$
\begin{aligned}
m(a,b,c) &= detG = -\phi_{aa}(a)\phi_{bb}(b)\phi_{cc}(c) + \phi_{aa}(a)\phi_{bb}(b)\phi_{cc}(a+b+c) \\
&\quad -\phi_{aa}(a)\phi_{bb}(a+b+c)\phi_{cc}(a+b+c)\phi_{aa}(a)\phi_{bc}^2(a+b+c) \\
&\quad +\phi_{aa}(a+b+c)\phi_{bb}(b)\phi_{cc}(c) - \phi_{aa}(a+b+c)\phi_{bb}(b)\phi_{cc}(a+b+c) \\
&\quad -\phi_{aa}(a+b+c)\phi_{bb}(a+b+c)\phi_{cc}(a+b+c) - \phi_{aa}(a+b+c)\phi_{bc}^2(a+b+c) \\
&\quad +2\phi_{ab}(a+b+c)\phi_{bc}(a+b+c)\phi_{ac}(a+b+c)\phi_{ab}^2(a+b+c)\phi_{cc}(c) \\
&\quad -\phi_{ab}^2(a+b+c)\phi_{cc}(a+b+c) + \phi_{ac}^2(a+b+c)\phi_{bb}(b) \\
&\quad -\phi_{ac}^2(a+b+c)\phi_{bb}(a+b+c)
\end{aligned}
$$

we obtain:

$$
G^{-1} = \begin{pmatrix} \zeta_1(a+b+c) & \zeta_4(a+b+c) & \zeta_5(a+b+c) \\ \zeta_4(a+b+c) & \zeta_2(a+b+c) & \zeta_6(a+b+c) \\ \zeta_5(a+b+c) & \zeta_6(a+b+c) & \zeta_3(a+b+c) \end{pmatrix} \qquad (10)
$$

Using the relation (6) and (10) in (3) we have the result. □

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3. Linearization and integrability of Gradient Systems.

Thereafter we show that this equation can be reduced to the following linear system

$$\frac{d\eta}{dt} = -\eta. \tag{11}$$

Indeed, since

$$\dot{\vec{\eta}} = G\vec{\theta} \tag{12}$$

so, using (**??**) and (9) in (12) we obtain

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{pmatrix} = \begin{pmatrix} -\phi_a(a) + \phi_a(a+b+c) \\ -\phi_b(b) + \phi_b(a+b+c) \\ -\phi_c(c) + \phi_c(a+b+c) \end{pmatrix}$$

which clearly shows that system (18) is linearized. We have the following lemma.

Lemma 2. *Let*

$$S = \left\{ p_\theta(x) = \frac{1}{B(a,b,c)} x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1}, \begin{array}{l} \theta = (a,\ b,\ c) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x = (x_1,\ x_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x_1 + x_2 < 1 \\ B(a,b,c) = \frac{\Gamma(a).\Gamma(b).\Gamma(c)}{\Gamma(a+b+c)} \end{array} \right\}$$

be the statistical manifold, where p_θ is the density function of the Bivariate Beta family of the first kind with three parameters family. The Hamiltonian of the gradient system (9) on S is given by $\mathcal{H} = Q_1P_1 + Q_1'P_1'$ such that:

$$\mathcal{H}(a,b,c) = \frac{\phi_b(b) - \phi_b(a+b+c)}{\phi_a(a) - \phi_a(a+b+c)} + \frac{\phi_c(c) - \phi_c(a+b+c)}{\phi_b(b) - \phi_b(a+b+c)} \tag{13}$$

Proof. Using the main Theorem in [11], and using (8), by setting

$$P_1 = -\frac{1}{\phi_a(a) - \phi_a(a+b+c)},\ Q_1 = \phi_b(b) - \phi_b(a+b+c),$$

$$P_1' = -\frac{1}{\phi_b(b) - \phi_b(a+b+c)},\ Q_1' = \phi_c(c) - \phi_c(a+b+c).$$

We have

$$\mathcal{H}(a,b,c) = \frac{\phi_b(b) - \phi_b(a+b+c)}{\phi_a(a) - \phi_a(a+b+c)} + \frac{\phi_c(c) - \phi_c(a+b+c)}{\phi_b(b) - \phi_b(a+b+c)}.$$

□

We have the following theorem

Theorem 2. *Let*

$$S = \left\{ p_\theta(x) = \frac{1}{B(a,b,c)} x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1}, \begin{array}{l} \theta = (a,\ b,\ c) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x = (x_1,\ x_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \\ x_1 + x_2 < 1 \\ B(a,b,c) = \frac{\Gamma(a).\Gamma(b).\Gamma(c)}{\Gamma(a+b+c)} \end{array} \right\}$$

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the statistical manifold. The gradient system obtained (9) is a sub-dynamical system of 4-dimensional system and is a completely integrable Hamiltonian system with the Hamiltonian $\mathcal{H}$, such that:

$$\begin{pmatrix} \dot{P}_1 \\ \dot{Q}_1 \\ \dot{P'}_1 \\ \dot{Q'}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial P_1} \\ \frac{\partial \mathcal{H}}{\partial Q_1} \\ \frac{\partial \mathcal{H}}{\partial P'_1} \\ \frac{\partial \mathcal{H}}{\partial Q'_1} \end{pmatrix} \tag{14}$$

where $\overline{\wedge} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ *is a Poisson tensor.*

Proof. Using the main Theorem in [11], and using (8), by setting

$$P_1 = -\frac{1}{\phi_a(a) - \phi_a(a+b+c)}, \; Q_1 = \phi_b(b) - \phi_b(a+b+c),$$

$$P'_1 = -\frac{1}{\phi_b(b) - \phi_b(a+b+c)}, \; Q'_1 = \phi_c(c) - \phi_c(a+b+c).$$

We have

$$\dot{P}_1 = P_1, \; \dot{Q}_1 = -Q_1, \; \dot{P'}_1 = P'_1, \; \dot{Q'}_1 = -Q'_1.$$

Using the Lemma 1 we have

$$\frac{\partial \mathcal{H}}{\partial Q_1} = P_1, \; \frac{\partial \mathcal{H}}{\partial P_1} = Q_1, \; \frac{\partial \mathcal{H}}{\partial Q_2} = P_2, \; \frac{\partial \mathcal{H}}{\partial P'_1} = Q_1.$$

We have the following system

$$\begin{cases} \dot{P}_1 = \frac{\partial \mathcal{H}}{\partial Q_1} \\ \dot{Q}_1 = -\frac{\partial \mathcal{H}}{\partial P_1} \\ \dot{P'}_1 = \frac{\partial \mathcal{H}}{\partial Q'_1} \\ \dot{Q'}_1 = -\frac{\partial \mathcal{H}}{\partial P'_1} \end{cases}. \tag{15}$$

the system (15) take the form:

$$\begin{pmatrix} \dot{P}_1 \\ \dot{Q}_1 \\ \dot{P'}_1 \\ \dot{Q'}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial P_1} \\ \frac{\partial \mathcal{H}}{\partial Q_1} \\ \frac{\partial \mathcal{H}}{\partial P'_1} \\ \frac{\partial \mathcal{H}}{\partial Q'_2} \end{pmatrix}.$$

We see that (9) is Hamiltonian system. Since the system (9) is completely integrable on S. □

4. Riemannian structure related to Stirling's formula.

In this part, we study the geometric structure on the first species beta manifold using the approximation related to the Stirling's formula [4].

4.1. Associated information metric and gradient system.

Using James Stirling's formula which was recalled by Mortici, Cristinel [4] we have the following lemma:

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Lemma 3. *Let $(a,b,c) \in]1;+\infty[\times]1;+\infty[\times]1;+\infty[$. The potential function Φ is defined on the space of three dimensional deprived of the cube of edges 1cm and of which one of the vertices are the origin of the frame and the three positive axes define the supports of these edges, its expression is given by*

$$\begin{aligned}\Phi(\theta) &= (a+b+c-\frac{1}{2})\log(a+b+c-1)+(\frac{1}{2}-a)\log(a-1)+(\frac{1}{2}-b)\log(b-1)\\ &+(\frac{1}{2}-c)\log(c-1)-\log(2\pi)-2.\end{aligned}$$

Proof.
Using (5) and Stirling [4] we have

$$\begin{aligned}\Phi(\theta) &= (a+b+c-\frac{1}{2})\log(a+b+c-1)+(\frac{1}{2}-a)\log(a-1)+(\frac{1}{2}-b)\log(b-1)\\ &+(\frac{1}{2}-c)\log(c-1)-\log(2\pi)-2. \end{aligned} \quad (16)$$

Relation (16) is defined only for the values of a,b ,and c such that:

$$(a,b,c) \in]1;+\infty[\times]1;+\infty[\times]1;+\infty[.$$

By setting

$$\mathrm{Y} = \left\{(a,b,c) \in \mathbb{R}^3 \mid 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1\right\}.$$

Y defines the cube of edges $1cm$. Then $(a,b,c) \in \mathbb{R}^3 - \{\mathrm{Y}\}$. □

So, we have the following Lemma

Lemma 4. *Let $\mathrm{Y} = \left\{(a,b,c) \in \mathbb{R}^3 \mid 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1\right\}$ the cube of edges 1cm. The information metric defined for all $(a,b,c) \in \mathbb{R}^3 - \{\mathrm{Y}\}$ is given by*

$$G = \begin{pmatrix} \frac{1}{a+b+c-1} - \frac{a-\frac{3}{2}}{(a-1)^2} & \frac{1}{a+b+c-1} & \frac{1}{a+b+c-1} \\ \frac{1}{a+b+c-1} & \frac{1}{a+b+c-1} - \frac{b-\frac{3}{2}}{(b-1)^2} & \frac{1}{a+b+c-1} \\ \frac{1}{a+b+c-1} & \frac{1}{a+b+c-1} & \frac{1}{a+b+c-1} - \frac{c-\frac{3}{2}}{(c-1)^2} \end{pmatrix} \quad (17)$$

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Proof. Using the relation (1), and the potential function (16) we obtain the following coefficients,

$$
\begin{aligned}
g_{11}(\theta) &= \frac{1}{a+b+c-1} - \frac{a-\frac{3}{2}}{(a-1)^2}.\\
g_{22}(\theta) &= \frac{1}{a+b+c-1} - \frac{b-\frac{3}{2}}{(b-1)^2}.\\
g_{12}(\theta) &= \frac{1}{a+b+c-1}.\\
g_{21}(\theta) &= \frac{1}{a+b+c-1}\\
g_{31}(\theta) &= \frac{1}{a+b+c-1}.\\
g_{32}(\theta) &= \frac{1}{a+b+c-1}.\\
g_{13}(\theta) &= \frac{1}{a+b+c-1}.\\
g_{23}(\theta) &= \frac{1}{a+b+c-1}.\\
g_{33}(\theta) &= \frac{1}{a+b+c-1} - \frac{c-\frac{3}{2}}{(c-1)^2}.
\end{aligned}
$$

□

For all $(a,b,c) \in \mathbb{R}^3 - \{Y\}$
with $Y = \{(a,b,c) \in \mathbb{R}^3 \mid 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1\}$.
We have a dual coordinate $\eta_i = \partial_i \Phi(\theta)$. So,

$$
\begin{cases}
\eta_1 = \log(a+b+c-1) - \log(a-1) + \frac{-\frac{1}{2}}{a-1}\\
\eta_2 = \log(a+b+c-1) - \log(b-1) + \frac{-\frac{1}{2}}{b-1}\\
\eta_3 = \log(a+b+c-1) - \log(c-1) + \frac{-\frac{1}{2}}{c-1}
\end{cases}
$$

we have the following system

$$
\begin{cases}
\frac{\partial}{\partial a}\frac{\partial \varphi(\theta)}{\partial a} = \frac{1}{a+b+c-1} - \frac{a-\frac{3}{2}}{(a-1)^2}\\
\frac{\partial}{\partial a}\frac{\partial \varphi(\theta)}{\partial b} = \frac{1}{a+b+c-1}\\
\frac{\partial}{\partial a}\frac{\partial \varphi(\theta)}{\partial c} = \frac{1}{a+b+c-1}\\
\frac{\partial}{\partial b}\frac{\partial \varphi(\theta)}{\partial b} = \frac{1}{a+b+c-1} - \frac{b-\frac{3}{2}}{(b-1)^2}\\
\frac{\partial}{\partial b}\frac{\partial \varphi(\theta)}{\partial c} = \frac{1}{a+b+c-1}\\
\frac{\partial}{\partial c}\frac{\partial \varphi(\theta)}{\partial c} = \frac{1}{a+b+c-1} - \frac{c-\frac{3}{2}}{(c-1)^2}
\end{cases}. \tag{18}
$$

we have

$$
\begin{aligned}
\Phi(\theta) &= (a+b+c-\frac{1}{2})\log(a+b+c-1) + (\frac{1}{2}-a)\log(a-1) + (\frac{1}{2}-b)\log(b-1)\\
&\quad + (\frac{1}{2}-c)\log(c-1) + k.\ k \in \mathbb{R}
\end{aligned}
$$

We have

$$
\Psi(\eta) = a\eta_1 + b\eta_2 + c\eta_3 - \Phi(\theta) \tag{19}
$$

so, we have the following Lemma

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Proposition 2. *The dual potential function associated with the first species beta variety with three parameters is defined for all* $(a,b,c) \in \mathbb{R}^3 - \{\Upsilon\}$ *is given by*

$$\begin{aligned}\Psi(\eta) &= -\frac{a}{2(a-1)} - \frac{b}{2(b-1)} - \frac{c}{2(c-1)} + \frac{1}{2}\log(a+b+c-1) - \frac{1}{2}\log(a-1) \\ &\quad -\frac{1}{2}\log(b-1) - \frac{1}{2}\log(c-1) - k\end{aligned}$$

Proof. By applying (19) we have the result. □

We start by computing the determinant, and we have:

$$det(G) = -\frac{1}{8}\frac{4cab - 8ca + 15a - 8ba + 15c - 8bc - 27 + 15b}{(a-1)^2(b-1)^2(c-1)^2(a+b+c-1)}$$

So, we have the following theorem:

Theorem 3. *Let* Υ *the cube such that* $\Upsilon = \{(a,b,c) \in \mathbb{R}^3 \mid 0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1\}$, *let* $(a,b,c) \in \mathbb{R}^3 - \{\Upsilon\}$, *let* $D =< A(0, \frac{3}{2}, \frac{3}{2});\ \vec{e}\begin{pmatrix}1\\0\\0\end{pmatrix} >$ *the vector line,*
let $\mathbb{V} =< \left(\frac{8bc-15b-15c+27}{4cb-8b-8c+15}, b, c\right) >$

- *If* $(a,b,c) \in \mathbb{V} \cup \{D\}$, *then* S *then is an algebraic manifold.*
- *If* $(a,b,c) \in \mathbb{R}^3 - \{\Upsilon; \mathbb{V} \cup \{D\}\}$ *then* G *is a pseudo riemannian metric.*

Proof. For all $(a,b,c) \in \mathbb{R}^3 - \{\Upsilon\}$

- we solve $det(G) = 0$ and we have $a = \frac{27-15b-15c+8cb}{4cb-8b-8c+15},\ b = b,\ c = c$ where $a = a,\ b = \frac{3}{2}, c = \frac{3}{2}$. Or $a = a,\ b = \frac{3}{2}, c = \frac{3}{2}$ define the vector line D and $\frac{27-15b-15c+8cb}{4cb-8b-8c+15},\ b = b,\ c = c$ define $\mathbb{V}$. So if $(a,b,c) \in \mathbb{V} \cup \{D\}$, then S then is algebraic manifold.
- If $(a,b,c) \in \mathbb{R}^3 - \{\Upsilon; \mathbb{V} \cup \{D\}\}$ then $det(G) \neq 0$ and G is a symmetric matrix then G is a pseudo riemannian metric.

□

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Theorem 4. *Let G is a Pseudo-Riemannian matrix. Let S be the statistical manifold, where p_θ is the density function of the Bivariate Beta family of the first kind with three parameters family. The gradient system on S is given by*

$$\begin{cases} \dot{a} = \frac{2(a-1)^2(-6ba-6ca+9a+4cab-c-b+3)}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}\right) \\ +\frac{4(2c-3)(b-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}\right) \\ +\frac{4(2b-3)(c-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}\right) \\ \dot{b} = \frac{4(2c-3)(b-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}\right) \\ +\frac{2(b-1)^2(-6ba-a+4cab+9b-c-6bc+3)}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}\right) \\ +\frac{4(2a-3)(b-1)^2(c-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}\right) \\ \dot{c} = \frac{4(2b-3)(c-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}\right) \\ +\frac{4(2a-3)(b-1)^2(c-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}\right) \\ +\frac{2(c-1)^2(-6ca-a+4cab+9c+3-b-6bc)}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}\right) \end{cases} . \tag{20}$$

Proof. In the case, G is a Pseudo-Riemannian matrix, using Lemma 4 we have the inverse of the matrix will be given by

$$G^{-1} = \begin{pmatrix} a1 & a2 & a3 \\ b1 & b2 & b3 \\ c1 & c2 & c3 \end{pmatrix} \tag{21}$$

(22)

with
$$\begin{cases} a1 = -\frac{2(a-1)^2(-6ba-6ca+9a+4cab-c-b+3)}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & ; \\ a2 = -\frac{4(2c-3)(b-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & ; \\ a3 = -\frac{4(2b-3)(c-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & ; \\ b1 = -\frac{4(2c-3)(b-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & ; \\ b2 = -\frac{2(b-1)^2(-6ba-a+4cab+9b-c-6bc+3)}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & ; \\ b3 = -\frac{4(2a-3)(b-1)^2(c-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & ; \\ c1 = -\frac{4(2b-3)(c-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & ; \\ c2 = -\frac{4(2a-3)(b-1)^2(c-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & ; \\ c3 = -\frac{2(c-1)^2(-6ca-a+4cab+9c+3-b-6bc)}{4cab-8ca+15a-8ba+15c-8bc-27+15b} & . \end{cases}$$

So, we have

$$\partial_\theta \Phi(\theta) = \begin{pmatrix} \log(a+b+c-1)-\log(a-1)+\frac{-\frac{1}{2}}{a-1} \\ \log(a+b+c-1)-\log(b-1)+\frac{-\frac{1}{2}}{b-1} \\ \log(a+b+c-1)-\log(c-1)+\frac{-\frac{1}{2}}{c-1} \end{pmatrix}. \tag{23}$$

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The gradient system will therefore be written as follows

$$\begin{cases} \dot{a} = \frac{2(a-1)^2(-6ba-6ca+9a+4cab-c-b+3)}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}\right) \\ +\frac{4(2c-3)(b-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}\right) \\ +\frac{4(2b-3)(c-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}\right) \\ \dot{b} = \frac{4(2c-3)(b-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}\right) \\ +\frac{2(b-1)^2(-6ba-a+4cab+9b-c-6bc+3)}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}\right) \\ +\frac{4(2a-3)(b-1)^2(c-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}\right) \\ \dot{c} = \frac{4(2b-3)(c-1)^2(a-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}\right) \\ +\frac{4(2a-3)(b-1)^2(c-1)^2}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}\right) \\ +\frac{2(c-1)^2(-6ca-a+4cab+9c+3-b-6bc)}{4cab-8ca+15a-8ba+15c-8bc-27+15b}\left(\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}\right) \end{cases}.$$

□

4.2. Linearization and Hamiltonian function.

Let's calculate now

$$\dot{\vec{\eta}} = G\vec{\theta}$$

so, we have

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{pmatrix} = \begin{bmatrix} \frac{1}{a+b+c-1}-\frac{a-\frac{3}{2}}{(a-1)^2} & \frac{1}{a+b+c-1} & \frac{1}{a+b+c-1} \\ \frac{1}{a+b+c-1} & \frac{1}{a+b+c-1}-\frac{b-\frac{3}{2}}{(b-1)^2} & \frac{1}{a+b+c-1} \\ \frac{1}{a+b+c-1} & \frac{1}{a+b+c-1} & \frac{1}{a+b+c-1}-\frac{c-\frac{3}{2}}{(c-1)^2} \end{bmatrix} \begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix}$$

we have

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{pmatrix} = \begin{pmatrix} \log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1} \\ \log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1} \\ \log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1} \end{pmatrix}$$

which clearly shows that system (20) is linearized in the form

$$\frac{d\eta}{dt} = -\eta \tag{24}$$

we have the following Lemma.

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Proposition 3. *Let S be the statistical manifold. The Hamiltonian of the gradient system (20) is given on the beta family of the first species manifold is given by*

$$\mathcal{H}(a,b,c) = \frac{\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}}{\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}} + \frac{\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}}{\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}} \tag{25}$$

Proof. We obtain these values in the appendix. So, we have $\{\mathcal{H},\mathcal{H}\}=0$. $\mathcal{H}$ is involution. We have: $\frac{d\mathcal{H}}{dt} = \frac{\partial H}{\partial a}\frac{da}{dt} + \frac{\partial \mathcal{H}}{\partial b}\frac{db}{dt} + \frac{\partial \mathcal{H}}{\partial c}\frac{dc}{dt}$ we have $\frac{d\mathcal{H}}{dt} = 0$. □ □

5. Integrability of Gradient Systems.

After calculation we have $\dot{P}_1 = P_1$, $\dot{Q}_1 = -Q_1$, $\dot{P'}_1 = P'_1$, $\dot{Q'}_1 = -Q'_1$
So, given that $\frac{\partial \mathcal{H}}{\partial Q_1} = P_1$, $\frac{\partial \mathcal{H}}{\partial P_1} = Q_1$, $\frac{\partial \mathcal{H}}{\partial Q_2} = P_2$, $\frac{\partial \mathcal{H}}{\partial P'_1} = Q_1$
We have the following system:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial Q_1} &= \dot{P}_1 \\ -\frac{\partial \mathcal{H}}{\partial P_1} &= \dot{Q}_1 \\ \frac{\partial \mathcal{H}}{\partial Q'_1} &= \dot{P'}_1 \\ -\frac{\partial \mathcal{H}}{\partial P'_1} &= \dot{Q'}_1 \end{aligned} \tag{26}$$

the system (26) takes the form

$$\begin{pmatrix} \dot{P}_1 \\ \dot{Q}_1 \\ \dot{P'}_1 \\ \dot{Q'}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial P_1} \\ \frac{\partial \mathcal{H}}{\partial Q_1} \\ \frac{\partial \mathcal{H}}{\partial P'_1} \\ \frac{\partial \mathcal{H}}{\partial Q'_2} \end{pmatrix}$$

we obtain the following Theorem:

Theorem 5. *On a statistical manifold S of dimension 3 defined by the Bivariate Beta family of the first kind with three parameters, the gradient system obtained is a sub-dynamical system of 4-dimensional system and is a completely integrable Hamiltonian system, such that*

$$\begin{pmatrix} \dot{P}_1 \\ \dot{Q}_1 \\ \dot{P'}_1 \\ \dot{Q'}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial P_1} \\ \frac{\partial \mathcal{H}}{\partial Q_1} \\ \frac{\partial \mathcal{H}}{\partial P'_1} \\ \frac{\partial \mathcal{H}}{\partial Q'_1} \end{pmatrix} \tag{27}$$

where $\overline{\wedge} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ *is a Poisson tensor.*

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Proof. We show that (27) is equivalent to the Hamiltonian system

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial Q_1} &= \dot{P}_1 \\
-\frac{\partial \mathcal{H}}{\partial P_1} &= \dot{Q}_1 \\
\frac{\partial \mathcal{H}}{\partial Q'_1} &= \dot{P'}_1 \\
-\frac{\partial \mathcal{H}}{\partial P'_1} &= \dot{Q'}_1
\end{aligned} \qquad (28)
$$

on S. Thus (Q_1, P_1, Q'_1, P'_1) is a set of canonical variables. Since $\mathcal{H} \in C^1(S)$ does not depend on t explicitly. According to Liouville-Arnol'd [8]system (20) the complete integrability is proven. □

6. Lax pair representation.

In [5], it is given the definition of the Lax-pair. In this section, we will construct the pair of lax associated with the gradient system. We recall that the pair of Lax must respect two important conditions: $tr\,(L) = \mathcal{H}$ and $\frac{dtr(L)}{dt} = 0$. With $\frac{dtr(L)}{dt} = tr\,[L, N]$.
So, we have the Theorem

Theorem 6. *The gradient system on the manifold S of Beta family of the first kind Distribution is represented by the following Lax pair:*

$$
\dot{L} = [L, N] \qquad (29)
$$

$$
L = \begin{pmatrix}
\frac{\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}}{\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}} & 0 & \sqrt{\frac{\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}}{\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}}} \\
0 & 0 & 0 \\
\sqrt{\frac{\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}}{\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}}} & 0 & \frac{\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}}{\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}}
\end{pmatrix}
$$

and

$$
N = \begin{pmatrix}
\ell & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \ell
\end{pmatrix}
$$

with, $\ell \in \mathbb{R}$.

Proof. we are given a symmetric matrix L such that its trace is equal to the Hamiltonian and whose unknowns are on the second diagonal as a sequence.

$$
L = \begin{pmatrix}
\frac{\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}}{\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}} & 0 & m_1 \\
0 & 0 & 0 \\
m_1 & 0 & \frac{\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}}{\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}}
\end{pmatrix} \qquad (30)
$$

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The matrix (30) is a symmetric matrix and real-valued, so it is diagonalizable. And according to Zakharov shabat the determinant of this matrix is a prime integral therefore its eigenvalues are prime integrals. Thus to determine the values of m, we pose the condition that these values are prime integrals. We have the matrix (**??**) becomes

$$L = \begin{pmatrix} \frac{\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}}{\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}} & 0 & \sqrt{\frac{\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}}{\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}}} \\ 0 & 0 & 0 \\ \sqrt{\frac{\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}}{\log(a+b+c-1)-\log(a-1)-\frac{\frac{1}{2}}{a-1}}} & 0 & \frac{\log(a+b+c-1)-\log(c-1)-\frac{\frac{1}{2}}{c-1}}{\log(a+b+c-1)-\log(b-1)-\frac{\frac{1}{2}}{b-1}} \end{pmatrix}$$

The matrix that allows us to have $\frac{dtr(L)}{dt} = 0$ is given by

$$N = \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \ell_2 \end{pmatrix}$$

To determine the values of ℓ_1 and ℓ_2, we solve the equation: $\dot{L} = [L, N]$. So we obtain

$$N = \begin{pmatrix} \ell & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \ell \end{pmatrix}$$

for all $\ell \in \mathbb{R}$. So, we have

$$\frac{dL}{dt} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and we have

$$[L, N] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

□

7. Discussion

Authors should discuss the results and how they can be interpreted from the perspective of previous studies and of the working hypotheses. The findings and their implications should be discussed in the broadest context possible. Future research directions may also be highlighted.

8. Conclusions

The gradient system defined by the Bivariate Beta family of the first kind with three parameters odd-dimensional manifold is a completely integrable Hamiltonian system.

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References

1. Stanisław,P and Kasperczuk. Quantum deformations of algebras associated with integrable hamiltonian systems. Japan journal of industrial and applied mathematics,transformation, *Quantum deformations of algebras associated with integrable hamiltonian systems* **2009**, *100:4*.
2. A. NW Hone. Painlevé tests, singularity structure and Integrability. Japan journal of industrial and applied mathematics, *Painlevé tests, singularity structure and Integrability* **2009**, *10(2)*, 245–277.https://doi.org/10.48550/arXiv.nlin/0502017.
3. Nakamura,Y. Completely integrable systems on the manifolds of gaussian and multinomial distribution. Japan journal of industrial and applied mathematics, *Completely integrable systems on the manifolds of gaussian and multinomial distribution* **1993**, *10(2)*, 179–189.https://doi.org/10.1007/BF03167571.
4. Mortici, C. Ramanujan formula for the generalized Stirling approximation. Applied mathematics and computation.Elsevier, *Ramanujan formula for the generalized Stirling approximation* **2010**, *257*, 2579–2585.
5. Mama Assandje,P.R; Dongho,J and Bouetou Bouetou,T. On the complete integrability of gradient systems on manifold of the lognormal family. Elsevier, Chaos, Solitons and Fractals. *On the complete integrability of gradient systems on manifold of the lognormal family* **2023**, *173*, 1–6.https://doi.org/10.1016/j.chaos.2023.113695
6. Akio,F. Dynamical systems on statistical models (state of art and perspectives of studies on nonliear integrable systems).RIMS Kkyuroku. *Dynamical systems on statistical* **1993**, *822*, 32–42.http://hdl.handle.net/2433/83219
7. Amari, S and Nagaoka,H. Methods of information geometry. In *Methods of information geometry*; Springer-Verlag, Shoshichi,K and Masamichi, T.; American Mathematic Society,: Oxford, volume 191; United States of America, 2000; pp. 1–216.https://doi.org/10.1090/mmono/191
8. Arnol'd,V.I; Givental,A. B and Novikov, S.P. *Symplectic geometry. Dynamical systems IV*, Gamkrelidze.R.V.; Publisher: Springer-Verlag , Berlin Heidelberg New York London Paris Tokyo Hong Kong, volume 191 , 2001; pp. 1–138.https://doi.org/10.1007/978-3-662-06791-8
9. Ovidiu, C and Udriste,C.*Geometric modeling in probability and statistic*, Gamkrelidze.R.V.; Publisher: Springer , Springer Cham Heidelberg New York Dordrecht London, volume 121 , 2014; pp. 1–389.https://doi.org/10.1007/978-3-319-07779-6
10. Hirsch, Morris W and Smale, Stephen and Devaney, Robert L.*Differential equations, dynamical systems, and an introduction to chaos*, The United States Of America.; Publisher: Springer , Elsevier Academic Press, 2004
11. Mama Assandje,P.R, Dongho,J. and Bouetou Bouetou,T. Complete Integrability of Gradient Systems on a Manifold Admitting a Potential in Odd Dimension. International Conference on Geometric Science of Information, Springer, France, du 30/08/2023 au 01/09/2023; Abstract Number (optional), pp. 423-432, 2023.$https://doi.org/10.1007/978-3-031-38299-4_4$4