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Article

Compact Orbit and Topological Sensitivity on Locally Compact Spaces

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Abstract

A cover-based notion of topological sensitivity and sensitivity at a point is introduced, which is the same as the popular concept of sensitivity in metric spaces. The basin of attraction of infinity, the point of compactification of locally compact, non-compact topological spaces, is studied in this paper. It is proved that, under certain conditions on the underlying map f , the set of points whose orbits have compact support, the basin of attraction of infinity, and the set of sensitive points are identical, thus generalizing the standard Julia sets on the Riemann sphere.

Keywords: topological dynamics; escaping set; one-point compactification; proper maps; topological sensitivity; compact orbits

MSC: 37B02; 37B25; 37F10

1. Introduction

The study of dynamical systems on non-compact spaces requires tools that distinguish between bounded and unbounded orbit behaviour. While much of classical topological dynamics has been developed for compact phase spaces [1,2], several important classes of systems arise naturally on locally compact but non-compact spaces, including iterations of polynomials, translation-type maps, and systems admitting escape to infinity.

A common approach in this setting is to pass to the one-point compactification $\hat{X} = X \cup \{\infty\}$ and to regard infinity as a fixed point of the extended map. This allows divergence of orbits to be interpreted as convergence to the point ∞ and enables the use of compactness arguments [3]. In this framework, two sets play a fundamental role. The first is the set $K(f)$, consisting of points whose forward orbits are contained in some compact subset of X . The second is the basin of attraction $A(\infty)$ of the point at infinity, consisting of all points whose orbits escape to infinity under iteration.

In complex dynamics, analogous constructions appear in the theory of polynomial iteration, where the filled Julia set consists of points with bounded orbits and the basin of infinity describes escaping behaviour [4]. Motivated by this analogy, we investigate the topological properties of $K(f)$ and $A(\infty)$ for general continuous self-maps of locally compact spaces, without assuming any algebraic or metric structure beyond continuity.

The main objectives of this paper are threefold. First, we study basic topological properties of the set $K(f)$, including forward invariance, closedness, and local compactness under suitable hypotheses on the phase space. Second, we analyze the basin $A(\infty)$ as an open, absorbing, forward invariant set and establish its relationship with $K(f)$ via complementarity and boundary results. Third, we connect these global invariant sets with notions of sensitivity, which play a central role in topological dynamics and chaos theory [5,6].

A key observation is that, in general, the inclusion $K(f) \subset \hat{X} \setminus A(\infty)$ may be strict. We provide explicit examples showing that non-escaping behaviour does not necessarily imply the existence of a compact trapping region for the orbit.

The results presented here provide a topological framework for studying boundedness, escape to infinity, and sensitivity in non-compact dynamical systems, extending several classical ideas from compact dynamics and complex iteration to a broader setting.

2. Preliminaries

2.1. Historical Development of Julia Sets

The study of Julia sets originated in the early investigations of iteration theory in complex analysis during the first decades of the twentieth century. The subject was developed independently by **Gaston Julia** and **Pierre Fatou**, whose pioneering works between 1917 and 1920 initiated the systematic study of the dynamics of rational functions on the Riemann sphere $\hat{\mathbb{C}}$.

Given a rational map $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of degree at least two, Julia and Fatou observed that the qualitative behavior of the iterates $\{f^n\}_{n \geq 1}$ naturally divides the phase space into two invariant regions. The first region, later known as **the Fatou set** $F(f)$, consists of points where the iterates behave in a stable and regular manner. Its complement, **the Julia set** $J(f)$, captures points where arbitrarily small perturbations lead to drastically different dynamical outcomes. This dichotomy marked one of the earliest rigorous distinctions between regular and chaotic behavior in deterministic systems.

Gaston Julia focused on the geometric and topological complexity of the unstable set $J(f)$, establishing its complete invariance under f and demonstrating that it often forms highly irregular, infinitely detailed structures [7]. In parallel, Pierre Fatou developed powerful analytical tools based on normal family theory, proving fundamental results concerning the structure and classification of components of $F(f)$ and their boundaries [8]. Their combined contributions showed that the Julia set is either the entire Riemann sphere or a perfect, nowhere dense subset of it.

Following this early period, research activity in complex dynamics slowed considerably for several decades. A renewed interest arose in the late twentieth century, largely motivated by computational experiments and visualizations. In particular, the work of Benoît Mandelbrot revealed the striking self-similar geometry of Julia sets associated with quadratic polynomials and uncovered their intimate relationship with the Mandelbrot set [9]. These discoveries played a crucial role in popularizing the subject and highlighting its relevance to fractal geometry and nonlinear science.

The modern mathematical framework of Julia sets was further strengthened by the seminal work of Douady and Hubbard, who introduced polynomial-like mappings and developed renormalization techniques to analyze local and global properties of complex dynamical systems [10]. Their results clarified the topological organization of Julia sets and established deep connections between local dynamics and global parameter spaces.

Today, Julia sets are regarded as fundamental examples of chaotic invariant sets in holomorphic dynamics. They exhibit characteristic features of chaos, including topological transitivity, dense periodic points, and sensitivity to initial conditions. As such, they continue to play a central role in contemporary research, linking complex analysis with ergodic theory, fractal geometry, and the general theory of dynamical systems.

2.2. Basic Properties of Julia Sets

Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. The Julia set $J(f)$ plays a central role in the qualitative description of the dynamics generated by the iterates of f . We summarize below some of its fundamental properties that are well established in the literature.

2.3. The Julia Set

The Julia set is completely invariant under the action of f , that is,

$$f(J(f)) = J(f) = f^{-1}(J(f)).$$

Moreover, $J(f)$ is a closed subset of the Riemann sphere and hence compact. These properties follow directly from the definition of $J(f)$ as the complement of the Fatou set and the invariance of normality under holomorphic mappings [4,11].

2.4. Perfectness and Density of Repelling Periodic Points

For any rational map of degree at least two, the Julia set is a perfect set; that is, it is closed and has no isolated points. Furthermore, the set of repelling periodic points of f is dense in $J(f)$. This density result highlights the intrinsic instability of the dynamics on the Julia set and provides a strong link between local expansion and global chaotic behavior [12,13].

2.5. Sensitivity to Initial Conditions [2]

The map f exhibits sensitive dependence on initial conditions; that is, there exists $\delta > 0$ such that for any open set U and for any point $x \in U$, there exists a point $y \in U$ and an integer $k \geq 1$ such that $d(f^k(x), f^k(y)) > \delta$. The positive number δ is called a sensitivity constant; it only depends on the space X and the function f .

The dynamics restricted to the Julia set exhibits sensitive dependence on initial conditions. More precisely, for any $z \in J(f)$ and any neighborhood U of z , there exist points $w \in U$ and an integer $n \geq 1$ such that the spherical distance between $f^n(z)$ and $f^n(w)$ exceeds a fixed positive constant. This sensitivity property is a hallmark of chaotic dynamics and contrasts sharply with the equicontinuous behavior observed on Fatou components [2,4].

2.6. Topological Transitivity

The map f is topologically transitive on its Julia set. That is, for any pair of nonempty open subsets $U, V \subset J(f)$, there exists an integer $n \geq 1$ such that

$$f^n(U) \cap V \neq \emptyset.$$

Topological transitivity reflects the indecomposable nature of the dynamics on $J(f)$ and ensures that the orbit of a typical point in the Julia set is distributed throughout the entire set. [14].

2.7. Relation to the Fatou Set

The Julia set coincides with the boundary of every Fatou component. In particular, if U is a connected component of $F(f)$, then

$$\partial U = J(f).$$

This boundary characterization emphasizes the role of the Julia set as the interface between stable and unstable dynamics and underscores its significance as the locus of chaotic behavior [11].

3. Examples of Julia Sets

We present several classical examples of Julia sets arising from polynomial and rational maps. These examples illustrate the wide variety of geometric and dynamical behaviors exhibited by Julia sets.

3.1. Quadratic Polynomial $f(z) = z^2$

Consider the map $f(z) = z^2$. In this case, the Fatou set consists of two invariant components: the basin of attraction of 0 and the basin of attraction of ∞ . The Julia set is given explicitly by

$$J(f) = \{z \in \mathbb{C} : |z| = 1\},$$

the unit circle. On $J(f)$, the dynamics is conjugate to the angle-doubling map on the circle, which is expanding and chaotic. This example demonstrates that Julia sets can be smooth manifolds, although this is exceptional among polynomial maps [4,11].

3.2. Quadratic Polynomial $f(z) = z^2 - 2$

For the polynomial $f(z) = z^2 - 2$, the Julia set is the real interval

$$J(f) = [-2, 2].$$

Despite its simple geometric appearance, the dynamics on this set is highly chaotic. The restriction of f to $[-2, 2]$ is topologically conjugate to the tent map, and hence exhibits sensitive dependence on initial conditions and dense periodic points [2,4].

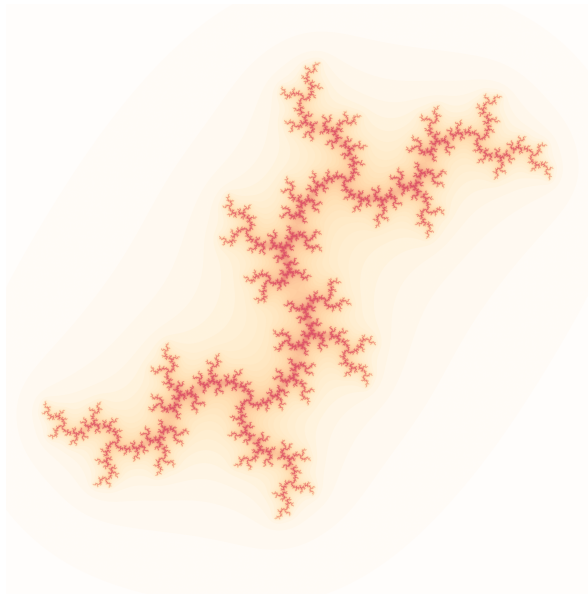


Figure 1. Julia set for $z^2 + i$.

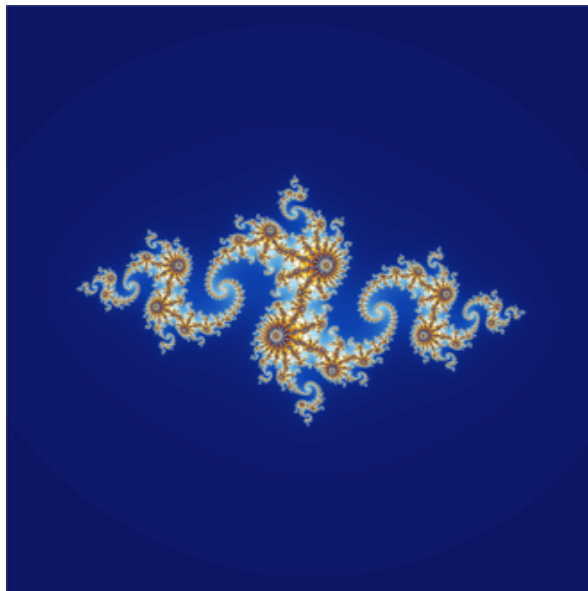


Figure 2. Julia set for $z^2 + 0.8 + 0.156i$.

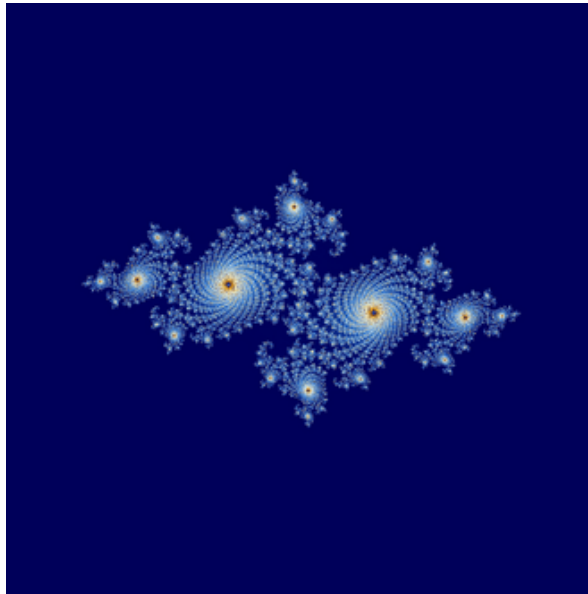


Figure 3. Julia set for $z^2 + 0.7269 + 0.1889i$.

3.3. Quadratic Polynomial $f(z) = z^2 + i$

For $f(z) = z^2 + i$, the Julia set is connected but has a highly intricate fractal structure. It is neither smooth nor locally connected at many points. This example highlights how small changes in the parameter c can dramatically alter the topology and geometry of the Julia set [4,10].

3.4. Rational Map $f(z) = z + \frac{1}{z}$

Consider the rational function $f(z) = z + \frac{1}{z}$. The Julia set of this map is the real line together with the point at infinity. The dynamics exhibits strong expansion away from critical points, and the Julia set forms the boundary between regions of regular and chaotic behavior [11].

3.5. Lattès Maps

Lattès maps arise from affine transformations on complex tori projected onto the Riemann sphere. The Julia set of a Lattès map is the entire Riemann sphere $\hat{\mathbb{C}}$. These examples are distinguished by the fact that chaotic behavior occurs everywhere, with no nontrivial Fatou components [4,15].

One example is the map $f(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$.

4. Orbits with Compact Support

Definition 1. Let X be a non-compact locally compact Hausdorff space and let $f : X \rightarrow X$ be continuous. Define

$$K(f) = \{x \in X : O_f(x) \subset C \text{ for some compact subset } C \subset X\}.$$

$K(f)$ is the set of points whose orbit has compact support.

4.1. Forward Invariance

Let X be a topological space and let $f : X \rightarrow X$ be continuous. A subset $A \subseteq X$ is said to be forward invariant under f if

$$f(A) \subseteq A.$$

Definition 2 (Completely Invariant Set). Let X be a topological space and let $f : X \rightarrow X$ be a map. A subset $A \subset X$ is said to be completely invariant under f if

$$f(A) = A = f^{-1}(A).$$

Equivalently,

$$x \in A \iff f(x) \in A.$$

Remark 1. Complete invariance is stronger than forward invariance. A set A is:

- Forward invariant if $f(A) \subset A$.
- Backward invariant if $f^{-1}(A) \subset A$.

A set is completely invariant if and only if it is both forward and backward invariant.

4.2. We Prove Some Properties of $K(f)$

Proposition 1. $K(f)$ is forward invariant. ie $f(K(f)) \subset K(f)$

Proof. Let $x \in K(f)$, then $O_f(x) \subset K$, for some compact subset K .

$$\implies \{x, f(x), f^2(x), f^3(x), \dots\} \subset K$$

$$\implies \{f(x), f^2(x), f^3(x), \dots\} \subset K$$

$$\implies O_f(f(x)) \subset K$$

Hence $K(f)$ is forward invariant. $\square$

Proposition 2. Let X be a locally compact Hausdorff space and let $f : X \rightarrow X$ be a homeomorphism. Then the boundary $\partial K(f)$ is completely invariant, i.e.,

$$f(\partial K(f)) = \partial K(f) = f^{-1}(\partial K(f)).$$

Proof. Step 1: $K(f)$ is completely invariant.

Forward invariance. Let $x \in K(f)$. Then there exists a compact set $C \subset X$ such that

$$f^n(x) \in C \quad \text{for all } n \geq 0.$$

For $f(x)$ we have

$$f^n(f(x)) = f^{n+1}(x) \in C,$$

so the forward orbit of $f(x)$ is also contained in C . Hence $f(x) \in K(f)$, and therefore

$$f(K(f)) \subset K(f).$$

Backward invariance. Let $x \in K(f)$ and let C be a compact set containing the forward orbit of x . Since f is a homeomorphism, f^{-1} is continuous and $f^{-1}(C)$ is compact.

For $n \geq 1$,

$$f^n(f^{-1}(x)) = f^{n-1}(x) \in C,$$

and for $n = 0$, $f^0(f^{-1}(x)) = f^{-1}(x) \in f^{-1}(C)$. Hence

$$O_f(f^{-1}(x)) \subset C \cup f^{-1}(C),$$

which is compact. Thus $f^{-1}(x) \in K(f)$, so

$$f^{-1}(K(f)) \subset K(f).$$

Combining both inclusions,

$$f(K(f)) = K(f) = f^{-1}(K(f)).$$

Step 2: Closure and interior are invariant.

Since f is a homeomorphism, for any subset $A \subset X$,

$$f(\overline{A}) = \overline{f(A)}, \quad f(A^\circ) = (f(A))^\circ.$$

Applying this to $A = K(f)$ and using the invariance of $K(f)$,

$$f(\overline{K(f)}) = \overline{K(f)}, \quad f(K(f)^\circ) = K(f)^\circ.$$

Step 3: Invariance of the boundary.

Recall that

$$\partial K(f) = \overline{K(f)} \setminus K(f)^\circ.$$

Since f is bijective,

$$f(A \setminus B) = f(A) \setminus f(B)$$

for all subsets $A, B \subset X$. Therefore,

$$f(\partial K(f)) = f(\overline{K(f)} \setminus K(f)^\circ) = f(\overline{K(f)}) \setminus f(K(f)^\circ).$$

Using Step 2,

$$f(\partial K(f)) = \overline{K(f)} \setminus K(f)^\circ = \partial K(f).$$

Step 4: Backward invariance of the boundary.

Since f is bijective and $f(\partial K(f)) = \partial K(f)$, applying f^{-1} gives

$$f^{-1}(\partial K(f)) = \partial K(f).$$

Hence,

$$f(\partial K(f)) = \partial K(f) = f^{-1}(\partial K(f)).$$

; see [1]. Applying the same argument to f^{-1} gives $f^{-1}(\partial K(f)) = \partial K(f)$ $\square$

Proposition 3. $K(f) = \bigcup_{C \in K(X)} \bigcap_{n \geq 0} f^{-n}(C)$, where $K(X)$ is the family of compact subsets of X .

Proof. Let $x \in K(f)$. Then there exists a compact set $C \subset X$ such that $f^n(x) \in C, \forall n \geq 0$,

$$\implies x \in f^{-n}(C), \forall n \geq 0,$$

$$\implies x \in \bigcap_{n \geq 0} f^{-n}(C), \implies x \in \bigcup_{C \in K(X)} \bigcap_{n \geq 0} f^{-n}(C).$$

Conversely, let $x \in \bigcup_{C \in K(X)} \bigcap_{n \geq 0} f^{-n}(C)$, for some compact C .

$$\implies f^n(x) \in C, \forall n \geq 0, \implies O_f(x) \subset C$$

$$\implies x \in K(f).$$

$$\text{Hence } K(f) = \bigcup_{C \in K(X)} \bigcap_{n \geq 0} f^{-n}(C). \quad \square$$

Proposition 4. If X is locally compact Hausdorff and σ -compact and $f : X \rightarrow X$ is continuous, then $K(f)$ is an F_σ subset of X .

Proof. Choose a compact exhaustion $(C_m)_{m \geq 1}$ of X , i.e. compact sets $C_1 \subset C_2 \subset \dots$ with $X = \bigcup_{m \geq 1} C_m$. We claim that

$$K(f) = \bigcup_{m \geq 1} \bigcap_{n \geq 0} f^{-n}(C_m).$$

If $x \in K(f)$, then $O_f^+(x) \subset C$ for some compact $C \subset X$. Since $C \subset \bigcup_{m \geq 1} C_m$ and C is compact, there exists m with $C \subset C_m$. Hence $f^n(x) \in C_m$ for all $n \geq 0$, i.e. $x \in \bigcap_{n \geq 0} f^{-n}(C_m)$.

Conversely, if $x \in \bigcap_{n \geq 0} f^{-n}(C_m)$ for some m , then $f^n(x) \in C_m$ for all n , so $x \in K(f)$.

Finally, each $f^{-n}(C_m)$ is closed and hence $\bigcap_{n \geq 0} f^{-n}(C_m)$ is closed. Therefore $K(f)$ is a countable union of closed sets, i.e. an F_σ set. $\square$

4.3. Proper Map

Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a continuous map. The map f is said to be *proper* if for every compact set $K \subseteq Y$, the preimage $f^{-1}(K)$ is compact in X .

Proposition 5. *Let X be a non-compact locally compact Hausdorff space and let $f : X \rightarrow X$ be a continuous proper map. Define*

$$K(f) = \{x \in X : \overline{\{f^n(x) : n \geq 0\}} \text{ is compact in } X\}.$$

Then $K(f)$ is closed in X .

Let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification. Since f is proper, it extends to a continuous map $\hat{f} : \hat{X} \rightarrow \hat{X}$ with $\hat{f}(\infty) = \infty$.

Define the escaping set

$$I(f) := \{x \in X : \hat{f}^n(x) \rightarrow \infty \text{ in } \hat{X}\}.$$

Claim 1: $I(f)$ is open in X . Let $x \in I(f)$. Choose a neighbourhood U of ∞ such that $\hat{f}(U) \subset U$ (which exists by continuity of $\hat{f}$ at ∞ and $\hat{f}(\infty) = \infty$). Since $\hat{f}^n(x) \rightarrow \infty$, there exists N such that $\hat{f}^N(x) \in U$. By continuity of $\hat{f}^N$, there is a neighbourhood W of x with $\hat{f}^N(W) \subset U$. Then for every $y \in W$ and every $k \geq 0$,

$$\hat{f}^{N+k}(y) \in \hat{f}^k(U) \subset U,$$

so $\hat{f}^n(y) \rightarrow \infty$, i.e. $W \subset I(f)$.

Claim 2: $K(f) = X \setminus I(f)$. If $x \in I(f)$, then the orbit eventually leaves every compact subset of X , so $O_f^+(x)$ cannot be compact; hence $x \notin K(f)$.

Conversely, if $x \notin I(f)$, then ∞ is not a limit point of $\{\hat{f}^n(x)\}$ in the compact space $\hat{X}$. Therefore the orbit closure in $\hat{X}$ is a compact subset of $\hat{X}$ contained in X , so the orbit closure in X is compact and $x \in K(f)$.

Thus $K(f) = X \setminus I(f)$, and since $I(f)$ is open, $K(f)$ is closed.

4.3.1. Note: Closedness of $K(f)$ May Fail without Local Compactness

Let $X = \mathbb{Q} \subset \mathbb{R}$ with the subspace topology, which is not locally compact and define $f(x) = x$, identity map.

Then for any $x \in \mathbb{Q}$, $O_f(x) = x$, which is compact. Hence $K(f) = \mathbb{Q}$, but $\mathbb{Q}$ is not closed in $\mathbb{R}$.

Proposition 6. *Let X be a locally compact Hausdorff space and $f : X \rightarrow X$ continuous. If $K(f) \subseteq X$ is closed, then $K(f)$ is locally compact (and Hausdorff). In particular, if X is non-compact and f is proper, then $K(f)$ is locally compact.*

Proof. Since $K(f)$ is closed in the Hausdorff space X , it is Hausdorff. Let $x \in K(f)$. Choose an open neighbourhood $U \subset X$ of x with $\bar{U}$ compact (local compactness of X). Then $U \cap K(f)$ is an open neighbourhood of x in $K(f)$ and

$$\overline{U \cap K(f)}^{K(f)} \subset \bar{U} \cap K(f),$$

which is compact as a closed subset of the compact set $\bar{U}$. Hence $K(f)$ is locally compact.

If f is proper and X is non-compact locally compact Hausdorff, then $K(f)$ is closed by Proposition 5, hence locally compact. $\square$

Proposition 7. $K(f)$ is the largest forward invariant subset of X on which f acts as a proper map.

Proof. First, $K(f)$ is forward invariant. Moreover, $f|_{K(f)}$ is proper: if $C \subset K(f)$ is compact, then C is compact in X , and $(f|_{K(f)})^{-1}(C) = K(f) \cap f^{-1}(C)$ is compact as a closed subset of the compact set $f^{-1}(C)$ (when f is proper on X ; if you do not assume global properness, state and prove the needed restriction-property carefully).

Now let $Y \subset X$ be forward invariant and assume $f|_Y$ is proper. Fix $y \in Y$. Choose a compact neighbourhood C of y in Y . Then each $(f|_Y)^{-n}(C)$ is compact, hence their intersection is compact and contains y . Thus $f^n(y) \in C$ for all $n \geq 0$, so $O_f^+(y) \subset C$ and $y \in K(f)$. Hence $Y \subset K(f)$. $\square$

Proposition 8. If f is a homeomorphism, $f(K(f)) = K(f) = f^{-1}(K(f))$

Proof. The Proposition follows from **Proposition 4.7** $\square$

Proposition 9. If $h : X \rightarrow Y$ is a homeomorphism, $f : X \rightarrow X, g : Y \rightarrow Y$ are continuous, and $h \circ f = g \circ h$ (ie h is a topological conjugacy), then $h(K(f)) = K(g)$

Proof. Let $x \in K(f)$, then $\exists$ a compact set $C \subset X$ such that $f^n(x) \in C, \forall n \geq 0$
 $\implies g^n(h(x)) = h(f^n(x)) \in h(C), \forall n \geq 0$.
 $\implies O_g(h(x)) \subset h(C)$, since $h(C)$ is compact.
 $\implies h(x) \in K(g)$
 $\implies h(K(f)) \subset K(g)$.

Applying the above argument for h^{-1} , gives the reverse inclusion.
Hence $h(K(f)) = K(g)$. $\square$

Proposition 10. $(\partial K(f))^0 = \phi$

Proof. Assume that $(\partial K(f))^0 \neq \phi$,
Then $\exists$ a non empty open set $U \subset X$ such that $U \subset \partial K(f)$
But $\partial K(f) \subset \overline{K(f)}$
ie $U \subset \overline{K(f)}$, $\implies U \cap K(f) \neq \phi$
So U is an open set such that $U \cap K(f) \neq \phi$ and $U \subset (X - (K(f)))^0$
Let $x \in U \cap K(f)$, $\implies x \in (K(f))^0$, $\implies U \cap (K(f))^0 \neq \phi$
Hence Our assumption is false, so $(\partial K(f))^0 = \phi$. $\square$

4.4. $K(f)$ Depends on the Metric

Consider $\mathbb{R}^2$. Let $f(x, y) = (2x, 2y)$.
Let $x = (x_1, x_2), y = (y_1, y_2)$.

1. Let $d_1(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, then clearly $K(f) = \{(0, 0)\}$.
2. Let $d_2(x, y) = |x_1 - y_1| + |x_2 - y_2|$ then $K(f) = \{(0, 0)\}$.
3. Let $d_3(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, then $K(f) = \{(0, 0)\}$.
4. Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R} \times (-1, 1)$ by $\phi(x, y) = (x, \tanh y)$, then ϕ is a homeomorphism. Let $d_4(p, q) = \|\phi(p) - \phi(q)\|_2$, $f^n(x, y) = (2^n x, 2^n y)$, then $\phi(f^n(x, y)) = (2^n x, \tanh(2^n y))$. Then $K(f) = \{(0, y) | y \in \mathbb{R}\}$, which is the y -axis.
5. Define

$$d_5(x, y) = |\arctan x - \arctan y|.$$

Since $\arctan : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ is a homeomorphism onto a bounded interval, the metric space $(\mathbb{R}, d_2)$ is bounded.

Moreover, every closed and bounded subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$ is compact, and compactness in $(\mathbb{R}, d_2)$ corresponds to compactness in that interval.

Now observe that

$$\arctan(2^n x) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \text{for all } n.$$

Hence the set

$$\{\arctan(2^n x) : n \geq 0\}$$

is bounded in a compact interval, and therefore relatively compact in $(\mathbb{R}, d_5)$.

Thus every orbit is relatively compact in d_5 , and so

$$K_{d_5}(f) = \mathbb{R}.$$

The filled set $K(f)$ depends crucially on the choice of metric, and hence on the topology

Proposition 11. *Let d_1 and d_2 be two equivalent metrics on a set X , that is, they induce the same topology on X . Then a subset $C \subset X$ is compact with respect to d_1 if and only if it is compact with respect to d_2 . Consequently, for any continuous map $f : X \rightarrow X$,*

$$K_{d_1}(f) = K_{d_2}(f),$$

where $K_{d_i}(f)$ denotes the set of points whose forward orbits are relatively compact with respect to the metric d_i .

Proof. Since d_1 and d_2 induce the same topology on X , the identity map

$$\text{id} : (X, d_1) \longrightarrow (X, d_2)$$

is a homeomorphism, and its inverse is again the identity map. It is a standard topological fact that homeomorphisms preserve compactness [16, Theorem 26.6].

Let $C \subset X$ be compact in (X, d_1) . Because $\text{id} : (X, d_1) \rightarrow (X, d_2)$ is continuous, the image $\text{id}(C) = C$ is compact in (X, d_2) . Applying the same argument to the inverse identity map shows that compactness in (X, d_2) implies compactness in (X, d_1) . Hence the two metrics determine exactly the same compact subsets of X .

By definition,

$$K_{d_i}(f) = \{x \in X : \{f^n(x)\}_{n \geq 0} \subset C \text{ for some compact set } C \subset X\}.$$

Since the families of compact sets induced by d_1 and d_2 coincide, a forward orbit is relatively compact with respect to d_1 if and only if it is relatively compact with respect to d_2 . This proves that

$$K_{d_1}(f) = K_{d_2}(f).$$

□

4.5. Example 4.1 : $K(f)$ on a Non-Metrizable Locally Compact Space

Let $X = I \times \mathbb{R}$, where I is an index discrete space and $\mathbb{R}$ is the usual metric.

Then clearly X is locally compact, not compact, and non-metrizable.

We define $f_1 : X \rightarrow X$ as $f_1(i, t) = (i, t + 1)$

Then clearly $f_1^n(i, t) = (i, t + n)$,

Then $K(f_1) = \emptyset$

Next we set $f_2 : X \rightarrow X$ by $f_2(i, t) = (i, t/2)$

Clearly $f_2^n(i, t) = (i, t/2^n)$.

Hence $K(f_2) = X$.

We define $f_3 : X \rightarrow X$, by $f_3(i, t) = (i, t/2)$, if $i \in J$ and $f_3(i, t) = (i, t+1)$, $i \notin J$, where $J \subset I$

Then $K(f_3) = J \times \mathbb{R}$. $\square$

4.6. Example 4.2: $K(f)$ Can Be Disconnected

Let $X = C[0, 1]$, we define $d(f, g) = \sup_{x \in X} \|f(x) - g(x)\|$

Let $\psi : X \rightarrow X$ be defined by $\psi(f) = f^2, \forall f \in X$

If $\|f\| < 1$, then $\|\psi^n(f)\| \rightarrow 0$, if $\|f\| = 1$ then $\|\psi^n(f)\| = 1$ and if $\|f\| > 1$, then $\|\psi^n(f)\| \rightarrow \infty$

Hence $K(f) = \{f \mid \|f\| < 1\} \cup \{1, -1\}$

$\therefore K(f)$ is disconnected. $\square$

4.7. Example 4.3

Let $X = C^\infty(S^1)$ of smooth, 2π - periodic functions.

Define $d(f, g) = \sup_{x \in [0, 2\pi]} |f(x) - g(x)|$.

Let $\psi : X \rightarrow X$ be defined as $\psi(f) = f'$.

Using Fourier Series, $f(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx}$

Hence $\psi^n(f) = \sum_{k \in \mathbb{Z}} (ik)^n a_k e^{ikx}$. $\therefore$ orbit of f lies in a finite dimensional space and hence is relatively compact.

$\therefore K(\psi) = C^\infty(S^1)$. $\square$

4.7.1. Example 4.4

Let ω_1 be the first uncountable ordinal. Let $\omega < \omega_1$.

Let $X = [0, \omega_1] \times [0, \omega] - \{(\omega_1, \omega_1)\}$, with the product order topology.

$f : X \rightarrow X$ is defined as $f(\alpha, n) = \begin{cases} (\alpha, n-1) & n > 0 \\ (\alpha, 0) & n = 0 \end{cases}$

Then $K(f) = X$.

Let $g : X \rightarrow X$ be defined as $g(\alpha, n) = \begin{cases} (\alpha, n) & n = 0 \\ (\alpha+1, n) & n > 0 \end{cases}$

Then $K(g) = [0, \omega_1] \times \{0\}$.

These examples shows that even in a non-metrizable locally compact space, the non-escaping set $K(f)$ can range from empty to the entire space and can realize arbitrary “vertical slices” of the form $J \times \mathbb{R}$. The structure of $K(f)$ is therefore highly sensitive to the dynamics along different components of the space.

5. Basin of Attraction of Infinity

Definition 3. Let $\hat{X} = X \cup \{\infty\}$ be the one point compactification of X , so we can extend the continuous function $f : X \rightarrow X$ to $\hat{X}$ by $\hat{f} : \hat{X} \rightarrow \hat{X}, \hat{f}(x) = f(x), \forall x \in X$ and $\hat{f}(\infty) = \infty$. We define the basin of attraction of ∞ as

$$A(\infty) = \{x \in \hat{X} \mid \hat{f}^n(x) \rightarrow \infty\}.$$

Proposition 12. $A(\infty) \neq \emptyset$

Proof. Since $\hat{f}(\infty) = \infty$, we have $\hat{f}^n(\infty) = \infty$ for all $n \geq 0$, and hence $\hat{f}^n(\infty) \rightarrow \infty$. Hence $\infty \in A(\infty)$. $\square$

Proposition 13. Let X be a locally compact Hausdorff space and let $f : X \rightarrow X$ be continuous and proper. Let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification and let $\hat{f} : \hat{X} \rightarrow \hat{X}$ be the extension with $\hat{f}(\infty) = \infty$. Define

$$A(\infty) := \{x \in \hat{X} : \hat{f}^n(x) \rightarrow \infty\}.$$

Then for every neighbourhood U of ∞ in $\hat{X}$,

$$A(\infty) = \bigcup_{N \geq 0} \bigcap_{n \geq N} \hat{f}^{-n}(U).$$

In particular, $A(\infty)$ is a G_δ subset of $\hat{X}$, and $A(\infty) \cap X$ is a G_δ subset of X .

Proof. Fix a neighbourhood U of ∞ . A point $x \in \hat{X}$ satisfies $\hat{f}^n(x) \rightarrow \infty$ if and only if there exists N such that $\hat{f}^n(x) \in U$ for all $n \geq N$. Equivalently, $x \in \bigcap_{n \geq N} \hat{f}^{-n}(U)$ for some N . Taking the union over all $N \geq 0$ yields the stated formula.

Since each $\hat{f}^{-n}(U)$ is open (as U is open and $\hat{f}$ is continuous), $\bigcap_{n \geq N} \hat{f}^{-n}(U)$ is a G_δ , and the union over N is again a G_δ . $\square$

Proposition 14. Let X be a locally compact Hausdorff space and let $f : X \rightarrow X$ be a continuous map. Let $A(\infty)$ denote the basin of attraction of the point ∞ in the one-point compactification $\hat{X} = X \cup \{\infty\}$. Then

$$K(f) \subseteq X - A(\infty).$$

Proof. Let $x \in K(f)$. By definition, $O_f(x) \subset C$ for some compact subset $C \subset X$ of X .

Hence there exists a compact set $C \subset X$ such that

$$f^n(x) \in C \quad \text{for all } n \geq 0.$$

Suppose, on the contrary, that $x \in A(\infty)$. Then, by the definition of the basin of attraction of ∞ , for every compact set $K \subset X$ there exists $N \in \mathbb{N}$ such that

$$f^n(x) \notin K \quad \text{for all } n \geq N.$$

Taking $K = C$, this contradicts the fact that $f^n(x) \in C$ for all $n \geq 0$.

Therefore $x \notin A(\infty)$, and hence $x \in X \setminus A(\infty)$. Since x was arbitrary, we conclude that

$$K(f) \subseteq X - A(\infty).$$

$\square$

Proposition 15. $K(f) = \hat{X} - A(\infty)$

Proof. For $x \in X$, $\hat{f}^n(x) \rightarrow \infty$ iff for every compact $C \subset X$, $\exists N \geq 0$ such that $f^n(x) \notin C, \forall n \geq N$.

$\iff O_f(x) \not\subset C$, for any compact set C of X .

Hence $x \notin K(f) \iff x \in A(\infty)$. $\square$

5.0.1. An Example to Show That $K(f) = \hat{X} - A(\infty)$

$$f(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 \leq x \leq 1, \\ 2x - 1, & 1 \leq x \leq 2, \\ x + 1, & x \geq 2. \end{cases}$$

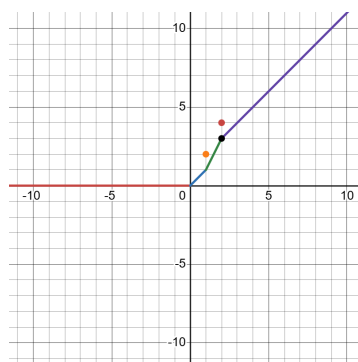


Figure 4. graph of $f(x)$.

clearly $K(f) = (-\infty, 2]$, for $x \geq 2$, $f^n(x) \rightarrow \infty$, hence $A(\infty) = (2, \infty)$.
Hence in this case $K(f) = X - A(\infty)$. $\square$

Proposition 16. $A(\infty)$ is forward invariant.

Proof. Let $x \in A(\infty)$, then $f^n(x) \rightarrow \infty$.

$f^{n-1}(f(x)) \rightarrow \infty$ as $n \rightarrow \infty$.

Hence $f(x) \in A(\infty)$.

ie $f(A(\infty)) \subset A(\infty)$. $\square$

Proposition 17. If f is proper, then $A(\infty)$ is open in X .

Proof. Let $x \in A(\infty)$, then $f^n(x) \rightarrow \infty$ in $\hat{X} = X \cup \{\infty\}$.

Since the neighborhood's of ∞ in $\hat{X}$ are of the form $\hat{X} - C$, for some compact set $K \subset X$, therefore

$\exists n \in \mathbb{N}, f^n(x) \in \hat{X} - C, \forall n \geq N$

ie $f^n(x) \notin C, \forall n \geq N$

Let $x \in A(\infty)$ and let $C \subset X$ be compact. By the characterization of escape, there exists $n \geq 0$ such that $f^n(x) \notin C$. Since f^n is proper, the preimage $f^{-n}(C)$ is compact. Thus $X \setminus f^{-n}(C)$ is an open neighbourhood of x consisting entirely of points whose n -th iterate lies outside C . Hence this neighbourhood is contained in $A(\infty)$.

5.1. Properness Is Essential in the above Result

If f is not proper, the preimage of a compact set may be non-compact. In this case, sequences of points arbitrarily close to an escaping point may have iterates that return to compact regions. Consequently, the set $A(\infty)$ need not be open. $\square$

5.1.1. Example (Failure of Openness Without Properness)

Let $X = \mathbb{R}$ and define

$$f(x) = x + \frac{\sin(x^2)}{x+1}.$$

Then f is continuous but not proper. Indeed, there exist sequences $x_n \rightarrow \infty$ such that $f(x_n)$ remains bounded, so the preimage of a compact set need not be compact.

For large x , the map satisfies $f(x) \approx x$, so many points escape to $+\infty$. However, due to the oscillatory term, there exist arbitrarily large points whose orbits repeatedly return to bounded regions.

Thus escaping points accumulate at non-escaping points, and the basin of attraction $A(\infty)$ is not open.

This shows that properness is essential for openness of $A(\infty)$.

Proposition 18. $\partial K(f)$ is closed and nowhere dense.

Proof. We have $\partial K(f) = \overline{K(f)} \cap (X \setminus K(f)^\circ)$.

Clearly as the intersection of two closed sets is closed, $\partial K(f)$ is closed.

Also $\partial K(f) \subset \overline{K(f)} - (K(f))^\circ$ and $\partial K(f) \subset \overline{A(\infty)} - (A(\infty))^\circ$

Since, $(K(f))^\circ \cap (A(\infty))^\circ = \emptyset$, any open subset $U \subset \partial K(f)$ have to be in both $(K(f))^\circ$ and $(A(\infty))^\circ$,

Hence $U = \emptyset$. $\square$

5.1.2. Examples 5.2

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 1$, then $A(\infty) = \mathbb{R}$
2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x/2$, then $A(\infty) = \emptyset$.
3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 1$, then $A(\infty) = \mathbb{R}$

Proposition 19. (Characterization of escape).

A point $x \in A(\infty)$ if and only if for every compact set $C \subset X$, $\exists n \geq 0$ such that $f^n(x) \notin C$
see [3, Chapter 18]

Proof. Suppose $x \in A(\infty)$. By definition, $\hat{f}^n(x) \rightarrow \infty$ in the one-point compactification $\hat{X} = X \cup \{\infty\}$. Let $C \subset X$ be compact. Then $\hat{X} \setminus C$ is a neighbourhood of ∞ . Hence there exists $n \geq 0$ such that $f^n(x) \in X \setminus C$.

Conversely, suppose that for every compact set $C \subset X$, there exists $n \geq 0$ such that $f^n(x) \notin C$. Let U be an arbitrary neighbourhood of ∞ in $\hat{X}$. Then $U = \hat{X} \setminus C$ for some compact $C \subset X$. By assumption, there exists n such that $f^n(x) \in U$. Hence $\hat{f}^n(x) \rightarrow \infty$, and therefore $x \in A(\infty)$. $\square$

6. Sensitivity

6.1. Metric Sensitivity

Definition 4. *nDefinition[2] Let (X, d) be a metric space and $f : X \rightarrow X$ be continuous. We say that f is sensitive on X if there exists $\delta > 0$ such that for every $x \in X$ and every $\varepsilon > 0$, there exist $y \in B_\varepsilon(x)$ and $n \in \mathbb{N}$ such that*

$$d(f^n(x), f^n(y)) \geq \delta.$$

The number δ is called a sensitivity constant.

6.1.1. L

Let (X, d) be a metric space, and let $f : X \rightarrow X$ be continuous. f is sensitive at a point $x \in X$ if there exist $\delta_x > 0$ such that for every $\varepsilon > 0$, there exist $y \in B_\varepsilon(x)$ and $n \in \mathbb{Z}_+$ such that $d(f^n(x), f^n(y)) \geq \delta_x$. Set of all sensitive points at which f is sensitive is denoted by $S(f)$.

6.1.2. Remark

Sensitivity depends on the defined metric

Let $X = \mathbb{R}$, and let $f(x) = x + 1$.

Let $d_1(x, y) = |x - y|$, $\rho(x, y) = |e^x - e^y|$.

Then f is sensitive on $(\mathbb{R}, \rho)$ but not on $(\mathbb{R}, d)$.

If (X, d) is a compact metric space, the sensitivity is preserved within the class of equivalent metrics.

6.1.3. Examples

1. It is well known that the tent map

$$f : [0, 1] \rightarrow [0, 1], \quad f(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2(1-x), & 1/2 \leq x \leq 1 \end{cases}$$

is sensitive on $[0,1]$ [2].

2. Let $X = \{-1/n\}_{n \geq 1} \cup \{0\}$ and $f : X \rightarrow X$ be $f(1/n) = \frac{1}{n+1}, f(0) = 0, f(-1/n) = \frac{-1}{n-1}; n \geq 2, f(-1) = -1$.
 f is continuous on X .
 If $x \in X - 0$, then f is not sensitive at x , since x is an isolated point. But for $x = 0$, given $\epsilon > 0$ there exists $y = -1/n \in (x - \epsilon, x + \epsilon)$ such that $d(f^n(x), f^n(y)) = d(0, 1) = 1$.
 Hence $S(f) = \{0\}$.
3. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be defined as $f(z) = z^2$
 If $|z| < 1, |f^n(z)| \rightarrow 0$
 If $|z| > 1, |f^n(z)| \rightarrow \infty$
 So if $|z| < 1$ and $|w| < 1$, then $|f^n(z) - f^n(w)| \rightarrow 0$.
 $\therefore z \notin S(f)$.
 if $|z| > 1, |w| > 1$, then $|f^n(z) - f^n(w)| = |z|^{2^n} |1 - (\frac{w}{z})^{2^n}|$
 $\therefore z \notin S(f)$.
 If $|z| = 1, z \in S(f)$.
 $\therefore S(f) = S^1$, which is the Julia set of f .

Proposition 20. This forward invariance property of the sensitive set is standard in topological dynamics [2,5].

Proposition 21. Let (X, d) be a metric space and $f : X \rightarrow X$ be continuous. Assume that the sensitive set $S(f)$ is closed. If $x \in S(f)$, then

$$\overline{O_f(x)} \subset S(f).$$

Proof. By Proposition 20, $S(f)$ is forward invariant. Hence $O_f(x) \subset S(f)$ and $S(f)$ is closed, we obtain

$$\overline{O_f(x)} \subset S(f).$$

Hence every point in the orbit closure of x is sensitive. $\square$

Proposition 22. $S(f)$ is a F_σ -set.

Proof. For a fixed $\delta > 0$, let $S_\delta = \{x \in X | f \text{ is sensitive at } x \text{ with sensitivity constant } \geq \delta\}$.

We prove that $S(f) = \bigcup_{n \in \mathbb{Z}_+} \overline{S_{1/n}(f)}$.

We have $\overline{S_\delta(f)} \subset S_{\delta/2}(f) \subset S(f)$.

$\therefore \bigcup_{n \in \mathbb{Z}_+} \overline{S_{1/n}(f)} \subset S(f)$.

Let $x \in S(f)$. Then there exists $n \in \mathbb{Z}_+$ such that $x \in S_\delta(f)$ where $\delta > 1/n$ (if $\delta < 1/n, \forall n \in \mathbb{Z}_+$ then $\inf\{\delta | x \in S_\delta(f)\} = 0$, contradicts $x \in S(f)$).

So $x \in S_{1/n}(f) \subset \overline{S_{1/n}(f)}$.

Hence $S(f) \subset \bigcup_{n \in \mathbb{Z}_+} \overline{S_{1/n}(f)}$.

Hence the result.

$\square$

6.1.4. Note

Sensitivity can be generalized to non-metrizable topological spaces.

Definition 5. topological sensitivity

Let X be a topological space and $f : X \rightarrow X$ be continuous. The function f is topologically sensitive at a point

$x \in X$ with respect to an open cover $\mathcal{U}$ if for every neighbourhood G of x , there exists $y \in G$ and $n \in \mathbb{Z}_+$, $n \geq 1$, such that

$$(f^n(x), f^n(y)) \notin U \times U, \quad \forall U \in \mathcal{U}.$$

.

ie $(f^n(x), f^n(y)) \notin \bigcup_{U \in \mathcal{U}} U \times U$.

$\mathcal{U}$ is called sensitivity cover for f [17].

If f is sensitive at every point of X w.r.t $\mathcal{U}$, we say that f is topologically sensitive on X , w.r.t $\mathcal{U}$.

Let $S_{\mathcal{U}}(f) = \{x \in X \mid f \text{ is topologically sensitive at } x \text{ w.r.t open cover } \mathcal{U}\}$.

f is topologically sensitive on X w.r.t open cover $\mathcal{U} \Leftrightarrow S_{\mathcal{U}}(f) = X$.

Proposition 23. *The following statements are equivalent.*

- (1) $S_{\mathcal{U}}(f) = X$.
- (2) For every non-empty open set G of X , there exist $x, y \in G$ and $n \in \mathbb{Z}_+$ such that $(f^n(x), f^n(y)) \notin U \times U$ for all $U \in \mathcal{U}$.

Proof. (1) $\Rightarrow$ (2):

Assume $S_{\mathcal{U}}(f) = X$. Let $G \subset X$ be a non-empty open set. Choose $x \in G$. Since $x \in S_{\mathcal{U}}(f)$, by definition, for every neighbourhood H of x there exist $y \in H$ and $n \in \mathbb{N}$ such that

$$(f^n(x), f^n(y)) \notin \bigcup_{U \in \mathcal{U}} U \times U.$$

Taking $H = G$, we obtain $y \in G$ and $n \in \mathbb{N}$ with the required property. Thus (2) holds.

(2) $\Rightarrow$ (1):

Assume (2). Let $x \in X$ and let G be any open neighbourhood of x . Since G is non-empty and open, by (2) there exist $x_1, y_1 \in G$ and $n \in \mathbb{N}$ such that

$$(f^n(x_1), f^n(y_1)) \notin \bigcup_{U \in \mathcal{U}} U \times U.$$

We claim that x is sensitive.

Since

$$\bigcup_{U \in \mathcal{U}} U \times U$$

is open in $X \times X$, its complement is closed. Let

$$\phi_n : X \times X \rightarrow X \times X, \quad \phi_n(a, b) = (f^n(a), f^n(b)).$$

Then ϕ_n is continuous. Hence

$$\phi_n^{-1}\left((X \times X) \setminus \bigcup_{U \in \mathcal{U}} U \times U\right)$$

is closed in $X \times X$.

Since (x_1, y_1) belongs to this closed set, there exists an open neighbourhood $W \subset X \times X$ of (x_1, y_1) such that

$$\phi_n(W) \subset (X \times X) \setminus \bigcup_{U \in \mathcal{U}} U \times U.$$

Because W is open in $X \times X$, there exist open sets $U_0, V_0 \subset X$ such that

$$(x_1, y_1) \in U_0 \times V_0 \subset W.$$

Since $x \in G$ and $x_1, y_1 \in G$, we may choose an open neighbourhood $G' \subset G$ with $x \in G'$ and $G' \subset U_0 \cap V_0$.

Then for every $y \in G'$,

$$(f^n(x), f^n(y)) \notin \bigcup_{U \in \mathcal{U}} U \times U.$$

Thus x is topologically sensitive. Since x was arbitrary, $S_{\mathcal{U}}(f) = X$.

□

Proposition 24. Let X, d be a metric space and $f : X \rightarrow X$ be continuous. Let T_d denote the topology on X generated by metric d . Then if f is metrically sensitive on X , then f is topologically sensitive.

Proof. Let δ be a sensitive constant for f .

Let $\epsilon = \frac{\delta}{2}$, then $\mathcal{U} = \{B_\epsilon(p) | p \in X\}$ is a sensitivity cover for f . So if G is an open set in X and $x \in G$, then there exists $y \in G, n \in \mathbb{Z}_+$ such that $d(f^n(x), f^n(y)) \geq \delta$.

$\therefore (f^n(x), f^n(y)) \notin B_\epsilon(p) \times B_\epsilon(p)$, for every $p \in X$.

$\therefore \mathcal{U}$ is a sensitivity cover for f . Hence f is topologically sensitive. □

Proposition 25. If (X, d) is compact then f is topologically sensitive $\implies f$ is metrically sensitive.

Proof. Let $\mathcal{U}$ be a sensitive cover for f and let ϵ be a Lebesgue number of $\mathcal{U}$ (Since X is compact every open cover has a Lebesgue number).

There exists $\epsilon > 0$ such that if $d(a, b) < \epsilon$, then a, b lie in some common $U \in \mathcal{U}$.

Let G be any non-empty subset of X . By assumption, $\exists x, y \in G, n \in \mathbb{Z}_+$ such that $(f^n(x), f^n(y)) \notin \bigcup_{U \in \mathcal{U}} U \times U$.

$\implies d(f^n(x), f^n(y)) \geq \epsilon$.

$\therefore f$ is metrically sensitive. □

6.1.5. Example 7.1

Compactness in the above theorem is necessary

Let $X = \mathbb{R}, d(x, y) = |\tan^{-1}x - \tan^{-1}y|$.

Note that $d(x, y) \leq \pi$.

Let $f(x) = 2x$, Let $\mathcal{U} = \{(-\infty, -1), (-2, 2), (1, \infty)\}$ be an open cover of $\mathbb{R}$.

Let $G \subset \mathbb{R}$ be an open interval.

then $f^n(x) = 2^n x, f^n(y) = 2^n y$.

As $n \rightarrow \infty$, two points land in different elements of $\mathcal{U}$.

$\therefore \mathcal{U}$ is a sensitivity cover for f .

ie f is topologically sensitive.

Suppose f is metrically sensitive. $\exists \delta > 0$ such that $d(f^n(x), f^n(y)) > \epsilon$

Now $d(f^n(x), f^n(y)) = |\tan^{-1}(2^n x) - \tan^{-1}(2^n y)|$.

As $t \rightarrow \infty, \tan^{-1}t \rightarrow \pi/2$.

For large n , $\tan^{-1}(2^n x)$ and $\tan^{-1}(2^n y)$ tends to $\pi/2$.

$\therefore d(f^n(x), f^n(y)) \rightarrow 0$

$\therefore f$ is not metrically sensitive. □

Proposition 26. f is topologically sensitive iff f^n is topologically sensitive.

Proof. If $\mathcal{U}$ is a sensitive cover for f^n , then $\mathcal{U}$ is a sensitive cover for f .

Assume that $\mathcal{U}$ is a sensitivity cover for f .

For every $k \in \{0, 1, \dots, n-1\}$ let $\mathcal{U}_k = \{f^{-k}(U) | U \in \mathcal{U}\}$.

Let $\mathcal{V}$ be the join of $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_k$.

ie $\mathcal{V} = \{U_0 \cap f^{-1}(U_1) \cap \dots \cap f^{-(n-1)}(U_{n-1}) | U_k \in \mathcal{U}\}$.

We now prove that $\mathcal{V}$ is a sensitivity cover for f^n .

Let G be a non-empty open subset of X . Then there exist $x, y \in G$ and $- \in \mathbb{Z}_+$ such that $(f^-(x), f^-(y)) \notin \bigcup_{U \in \mathcal{U}} U \times U$.

Let $q \in \mathbb{Z}_+$ and $0 \leq p < n$ be such that $m = qn + p$.

$$((f^n)^q(x), (f^n)^q(y)) = (f^{m-p}(x), f^{m-p}(y)) \notin \bigcup_{v \in \mathcal{V}} V \times V$$

Because if $(f^{m-p}(x), f^{m-p}(y)) \in V \times V$ for some $V = U_0 \cap f^{-1}(U_1) \cap \dots \cap f^{-(n-1)}(U_{n-1}) \in \mathcal{V}$, then $(f^m(x), f^m(y)) \in U_p \times U_p, U_p \in \mathcal{U}$, which is a contradiction.

So $\mathcal{V}$ is a sensitivity cover for f^n .

Hence f^n is topologically sensitive.

□

Proposition 27. $S_{\mathcal{U}}(f)$ is forward invariant.

Proof. Let $x \in S_{\mathcal{U}}(f)$.

Let $f(x) \in G$ be open.

Then $f^{-1}(G)$ is a neighbourhood of x .

$\therefore \exists y \in f^{-1}(G), n \geq 1$ with $(f^n(x), f^n(y)) \notin \bigcup_{U \in \mathcal{U}} U \times U$.

$\therefore (f^n(f(x), f^n(f(y))) = (f^{n+1}(x), f^{n+1}(y)) \in \bigcup_{U \in \mathcal{U}} U \times U$.

Since $f(y) \in G, f(x)$ is sensitive. □

Proposition 28. $x \in S_{\mathcal{U}}(f) \implies \overline{O_f(x)} \subset S_{\mathcal{U}}(f)$.

Proof. Follows from the above proposition. □

Proposition 29. Let X be metrizable. Let $V_\epsilon = \{x \in X \mid d(x, x_0) < \epsilon\}, \mathcal{U} = \{V_\epsilon \mid \epsilon > 0\}$. Then sensitivity w.r.t $\mathcal{U}$ is equivalent to metric sensitivity at x .

Proof. Topological sensitivity w.r.t $\mathcal{U} \implies (f^n(x), f^n(y)) \neq V_\epsilon \times V_\epsilon$
 $\implies d(f^n(x), f^n(y)) \geq \epsilon \implies x$ is metric sensitive. □

Proposition 30. The sensitive set $S(f)$ is forward invariant; that is,

$$f(S(f)) \subseteq S(f).$$

Proof. Let $x \in S(f)$ and let U be an open neighbourhood of $f(x)$. By continuity of f , there exists an open neighbourhood W of x such that $f(W) \subset U$. Since x is sensitive, there exist $y \in W, n \in \mathbb{N}$, and an open cover $\mathcal{V}_x$ of X such that

$$(f^n(x), f^n(y)) \notin V \times V \quad \text{for all } V \in \mathcal{V}_x.$$

It follows that

$$(f^{n+1}(x), f^{n+1}(y)) \notin V \times V,$$

showing that $f(x)$ is topologically sensitive. Hence $f(S(f)) \subseteq S(f)$. □

6.2. Examples Illustrating Topological Sensitivity

In this section we present explicit examples that clarify the definition of topological sensitivity introduced in Definition. These examples demonstrate both the presence and absence of sensitivity in purely topological terms.

6.2.1. Example : The Full Shift

Let $X = \{0,1\}^{\mathbb{N}}$ endowed with the product topology, and let $f = \sigma : X \rightarrow X$ be the left shift defined by

$$\sigma((x_0, x_1, x_2, \dots)) = (x_1, x_2, x_3, \dots).$$

The space X is compact, totally disconnected, and metrizable, but the argument below is purely topological.

Result 1. *The shift map σ is topologically sensitive at every point of X . Hence*

$$S(\sigma) = X.$$

Proof. Let $x \in X$ and let U be any open neighbourhood of x . Then U contains a basic cylinder set determined by a finite initial block of coordinates. Choose $y \in U$ such that y differs from x at some coordinate k outside this finite block.

Let $\mathcal{V}$ be the open cover of X consisting of cylinder sets determined by the first coordinate. Then $\sigma^k(x)$ and $\sigma^k(y)$ differ in the first coordinate, and hence cannot lie in the same element of $\mathcal{V}$. Thus

$$(\sigma^k(x), \sigma^k(y)) \notin V \times V \quad \text{for all } V \in \mathcal{V}.$$

Therefore σ is topologically sensitive at x . Since x was arbitrary, $S(\sigma) = X$. $\square$

6.2.2. Example: Expanding Map on the Circle [18]

Proposition 31. *Let $X = \mathbb{S}^1$ and define $f : X \rightarrow X$ by*

$$f(z) = z^2.$$

The map $f(z) = z^2$ is topologically sensitive at every point of $\mathbb{S}^1$.

Proof. Let $x \in \mathbb{S}^1$ and let U be any open arc containing x . Since f is expanding, there exists $n \in \mathbb{N}$ such that $f^n(U)$ covers the entire circle.

Let $\mathcal{V}$ be any finite open cover of $\mathbb{S}^1$ by proper arcs. Then there exist $y \in U$ such that $f^n(x)$ and $f^n(y)$ lie in distinct elements of $\mathcal{V}$. Hence the sensitivity condition is satisfied at x . $\square$

6.2.3. Example: Translation on the Real Line

Let $X = \mathbb{R}$ with the usual topology and define

$$f(x) = x + 1.$$

Proposition 32. *The map $f(x) = x + 1$ is not topologically sensitive at any point. Thus*

$$S(f) = \emptyset.$$

Proof. Fix $x \in \mathbb{R}$. Let $\mathcal{V}$ be the open cover of $\mathbb{R}$ consisting of unit intervals $\{(k-1, k+1) : k \in \mathbb{Z}\}$. For any neighbourhood U of x and any $y \in U$, we have

$$f^n(y) - f^n(x) = y - x \quad \text{for all } n \in \mathbb{N}.$$

Hence $f^n(x)$ and $f^n(y)$ always remain in the same element of $\mathcal{V}$. Therefore the sensitivity condition fails at x . $\square$

6.2.4. Example: Constant Map

Let X be any topological space and define $f : X \rightarrow X$ by $f(x) = p$ for a fixed point $p \in X$.

Proposition 33. *A constant map is never topologically sensitive.*

Proof. For every $x \in X$ and every $n \in \mathbb{N}$,

$$f^n(x) = p.$$

Hence for any open cover $\mathcal{V}$ of X , there exists $V \in \mathcal{V}$ such that $p \in V$, and therefore

$$(f^n(x), f^n(y)) \in V \times V \quad \text{for all } y \in X \text{ and all } n.$$

Thus the sensitivity condition cannot be satisfied at any point. $\square$

Remark 2. *The above examples show that topological sensitivity captures genuine chaotic behavior. Expanding and shift-type systems exhibit sensitivity at every point, while equicontinuous or rigid systems fail to exhibit sensitivity anywhere.*

7. Relation Between $K(f)$ and $A(\infty)$

7.0.1. Lemma 1

$$\partial A(\infty) = \partial K(f)$$

Proof. Since $A(\infty) = X - K(f)$, $\partial A(\infty) = \overline{A(\infty)} \cap \overline{K(f)} = \partial K(f)$.

Suppose $x \notin \partial K(f)$, then there are two possibilities, $x \in (K(f))^0$ or $x \in (X - K(f))^0$. $\square$

Case 1

Let $x \in (K(f))^0 \implies \exists$ open neighbourhood G of x , such that $G \subset K(f)$

$\implies \exists$ a compact set $C \subset X$ such that $O_f(y) \subset C, \forall y \in G$.

Fix any $V \in \mathcal{U}$, Since f^n is continuous and C is compact, $\{f^n|_C : n \geq 0\}$ is equicontinuous on C .

Hence, $\exists$ a neighbourhood $G_0 \subset G$ of x such that $(f^n(x), f^n(y)) \in V \times V, \forall y \in G_0, n \geq 0$.

Hence f is not sensitive at x .

Case 2

Let $x \in (X - K(f))^0 \implies \exists$ open neighbourhood G of x such that $G \subset (X - K(f))$.

Thus for every $y \in G, f^n(y) \rightarrow \infty$ in $\hat{X}$

Since convergence to infinity is uniform on compact sets, $\exists N$ such that $f^n(G) \subset (X - C), \forall n \geq N$ where $C \subset X$ is compact.

Let $V \in \mathcal{U}$ be any open neighborhood of ∞ in $\hat{X}$

Then $f^n(x), f^n(y) \in V, \forall y \in G, n \geq N$.

Hence $(f^n(x), f^n(y)) \in V \times V$, for large n , So f is not sensitive at x .

ie $x \notin \partial K(f) \implies x \notin S(f)$, ie $x \in S(f) \implies x \in \partial K(f)$.

7.0.2. Counter Example

ie to show that $x \in \partial K(f) \not\Rightarrow x \in S(f)$.

Let $X = \mathbb{R}, f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases} \quad (1)$$

$K(f) = (-\infty, 0], A(\infty) = (0, \infty), \partial K(f) = \{0\}$.

Let G be a neighbourhood of 0, let $G = (-\epsilon, \epsilon)$, choose $y \in G \cap K(f) = (-\epsilon, 0)$,

Then, $f^n(0) = 0, f^n(y) = y \leq 0, \forall n$.

All iterates remain in the compact interval $[-\epsilon, 0]$, ie $f(0), f^n(y) \in V \times V, \forall V \in \mathcal{U}$, for any any cover $\mathcal{U}$. Hence f is not sensitive at 0.

Proposition 34. $x \in \partial K(f) \implies x \in S(f)$ iff the following condition (*) holds, for every neighbourhood G of x , there exists $y \in G \cap K(f), z \in G \cap A(\infty)$ and an integer $n \geq 1$, such that $\forall V \in \mathcal{U}, (\mathcal{U} \subset \tau), \{f^n(y), f^n(z)\} \not\subset V \dots (*)$.

Proof. Let f be sensitive at $x \in \partial K(f)$

Let G be any neighbourhood of x .

Since $x \in \partial K(f), G \cap K(f) \neq \emptyset$ and $G \cap A(\infty) \neq \emptyset$.

By sensitivity, $\exists y \in G, n \geq 1$ such that $\forall V \in \mathcal{U}, (f^n(x), f^n(y)) \notin V \times V$

If $y \in K(f)$, choose $z \in G \cap A(\infty)$.

If $y \in A(\infty)$ choose $z \in G \cap K(f)$.

Since $f^n(z)$ eventually leaves every compact set, $f^n(y)$ remains in a compact set, $\exists n \in \mathbf{N}$, such that $\forall V \in \mathcal{U}, \{f^n(y), f^n(z)\} \not\subset V$.

Hence (*) is proved.

Conversely assume (*), let G be an open neighbourhood of x .

Then by (*), there exist $y \in G \cap K(f), z \in G \cap A(\infty)$, and an integer $n \geq 1$ such that

$$\forall V \in \mathcal{U} (\mathcal{U} \subset \tau), \quad \{f^n(y), f^n(z)\} \not\subset V.$$

Since f^n is continuous, $(f^n(x), f^n(y)) \notin V \times V$ or $(f^n(x), f^n(z)) \notin V \times V, \forall V \in \mathcal{U}$.

ie $x \in S(f)$. $\square$

Proposition 35. Let X be a locally compact Hausdorff space. Fix a compact exhaustion (K_n) , such that $K_1 \subset K_2 \subset K_3 \dots$ with $X = \bigcup_{m \geq 1} K_m$. Let $f : X \longrightarrow X$ be continuous. We say that the property (p) as "For every $m \geq 1, \exists N(m) \geq 0$ such that $\forall x \in X, \exists t \geq 0$ with $f^t(x) \in X - K_m, \implies \forall n \geq t + N(m), f^n(x) \notin K_m$." If (p) holds w.r.t (K_m) then $K(f) = X - A(\infty)$.

Proof. We have $K(f) \subset X - A(\infty)$. Hence we have to prove that $X - A(\infty) \subset K(f)$. We will prove that if $x \notin K(f)$ then $x \in A(\infty)$.

So let $x \notin K(f)$. Since (K_m) covers X , for each $m \geq 1, O_f(x) \not\subset K_m$

ie $\forall m \geq 1, \exists t_m \geq 0$ such that $f^{t_m}(x) \in X - K_m \dots (1)$.

Fix an arbitrary $m \geq 1$, by property (p), $\exists N(m) \geq 0$ such that whenever an orbit leaves K_m at some time t , it never returns after $t + N(m)$

ie if $f^t(y) \notin K_m$ for some y and t , then $\forall n \geq t + N(m), f^n(y) \notin K_m$.

Apply this to x , from (1) pick t_m with $f^{t_m}(x) \in X - K_m$. Then by property (p), $\forall n \geq t_m + N(m), f^n(x) \in X - K_m$, ie $\exists, T_m = t_m + N(m)$ such that $\forall n \geq T_m, f^n(x) \notin K_m \dots (2)$.

But K_m is arbitrary. For any neighborhood U of ∞ in $\hat{X}$, we can choose m so large that $(X - K_m) \subset U$.

From (2), we have T_m such that $\forall n \geq T_m, f^n(x) \in (X - K_m) \subset U$.

ie the tail of the orbit eventually lies in U . Since this holds for every neighborhood U of $\infty, f^n(x) \longrightarrow \infty$, when $n \longrightarrow \infty$.

ie $x \in A(\infty)$. $\square$

Proposition 36. Let X be a locally compact T_2 space and $f : X \rightarrow X$ be continuous. Suppose there exists a continuous proper function $\phi : X \rightarrow [0, \infty)$ and constants $R \geq 0$ and $\delta > 0$ such that

$$\phi(f(x)) \geq \phi(x) + \delta \quad \text{whenever } \phi(x) \geq R \quad \dots \quad (*)$$

Then $K(f) = X - A(\infty)$.

Proof. We have, $K(f) \subseteq (X - A(\infty))$. We will prove $(X - A(\infty)) \subset K(f)$.

Let $x \notin K(f)$. Then the orbit $O_f(x)$ is not contained in any compact subset of X . Since ϕ is proper (inverse images of compact subsets are compact), $\forall M \geq 0, \phi^{-1}([0, M]) = \{y \in X / \phi(y) \leq M\}$ is compact. Since $x \notin K(f), \forall M \geq 0, \exists n \geq 0$, such that $\phi(f^n(x)) > M$ (because, if there were M with $\phi(f^n(x)) \leq M, \forall n$ then $O_f(x)$ would lie in the compact set $\phi^{-1}([0, M])$, which contradicts the fact that $x \notin K(f)$. $\phi(f^n(x)) \rightarrow \infty$.

Since the sequence $\phi(f^n(x))$ is unbounded above. So there exists N_0 such that $\phi(f^{N_0}(x)) \geq R$, (if not, ie if $\phi(f^n(x)) < R, \forall n$, hence bounded a contradiction). By the hypothesis (*), $\forall k \geq 0 \phi(f^{N_0+k}(x)) \geq \phi(f^{N_0}(x)) + k\delta$, inductively we get $\phi(f^{N_0+k}(x)) \geq \phi(f^{N_0}(x)) + k\delta$.

for $\phi(f^{N_0}(x)) \geq R$.

ie $\phi(f^n(x)) \rightarrow \infty$ linearly (at least) as $n \rightarrow \infty$. Since ϕ is proper and $\phi(f^n(x)) \rightarrow \infty$, give $f^n(x) \rightarrow \infty$.

ie $f^n(x)$ eventually leave every compact set . ie for every compact set $K \subset X, \exists N(k)$ with $f^n(x) \notin K, \forall n \geq N(k)$.

ie $f^n(x) \rightarrow \infty$ in $\hat{X}, \implies x \in A(\infty)$. $\square$

Remark

For complex polynomial $p(z)$ of degree ≥ 2 , $\phi(z) = \log(1 + |z|)$, thus for polynomials $K(f) = (X - A(\infty))$.

Proposition 37. Let X be a locally compact, non-compact, Hausdorff space and let $f : X \rightarrow X$ be a proper continuous map. Extend f to the one-point compactification $\hat{X} = X \cup \{\infty\}$ by setting $\hat{f}(\infty) = \infty$. Assume that there exists a continuous, proper function $\phi : X \rightarrow [0, \infty)$ and constants $R > 0$ and $\delta > 0$ such that $\phi(f(x)) \geq \phi(x) + \delta$ whenever $\phi(x) \geq R$ and there exists a finite open cover $\mathcal{U}$ of X , such that for every x with $\phi(x) \geq R$ and every neighborhood $x \in V$ there exists $y \in V$ and $n \geq 1$ such that $f^n(x), f^n(y) \notin U \times U, \forall U \in \mathcal{U} \dots (**)$ Then

$$S_{\mathcal{U}}(f) = A(\infty) \cap X = X \setminus K(f).$$

Proof. By previous proposition as ϕ satisfies (*), we have $A(\infty) \cap X = X \setminus K(f)$.

Next we will show that $A(\infty) \cap X \subset S_{\mathcal{U}}(f)$.

Let $x \in A(\infty)$. Then for so-e $N_0, \phi(f^{N_0}(x)) > R$.

Let V be any neighborhood of x .

Since f^{N_0} is continuous, $f^{N_0}(V)$ is a neighborhood of $f^{N_0}(x)$.

By (**) applied to $f^{N_0}(x)$, there exists $z \in f^{N_0}(V), k \geq 1$ such that $(f^k(f^{N_0}(x)), f^k(z)) \notin U \times U$.

ie $x \in S_{\mathcal{U}}(f)$.

Let $x \notin A(\infty) \cap X$. Then the orbit of x lies in a compact set $C \subset X$. Since f is continuous and C compact, all iterates remain in C .

Fix $n \geq 0$. Since each f^n is continuous at x , for each open set U containing $f^n(x)$, there exists a neighborhood V_n of x such that $f^n(V_n) \subset U$.

For each n , choose $U_{i(n)}$ such that $f^n(x) \in U_{i(n)}$.

Then by continuity, there exists a neighborhood V_n of x , such that $f^n(V_n) \subset U_{i(n)}$. Thus for every $y \in V_n, (f^n(x), f^n(y)) \in U_{i(n)} \times U_{i(n)}$.

Since C is compact define $V = \bigcap_{n=0}^N V_n$ for sufficiently large N .

We can choose a neighborhood V of x , such that $f^n(V) \subset U_{i(n)}, \forall n \geq 0$.

Thus for all $y \in V$ and all $n, (f^n(x), f^n(y)) \in U \times U$, for all $U \in \mathcal{U}$.

Hence $x \notin S_U(f)$

Hence the result.

□

8. Conclusions

In this paper we examined the relationship between three fundamental objects in topological dynamics: the escaping set $A(\infty)$ defined via the one-point compactification, the set $K(f)$ of points with relatively compact forward orbits, and the set $S(f)$ of topologically sensitive points. Working in the general setting of locally compact Hausdorff spaces, we showed that these notions are inherently topological and do not rely on any metric structure.

Our results demonstrate that escape to infinity is equivalent to the failure of relative compactness of forward orbits, yielding the identity $A(\infty) \cap X = X \setminus K(f)$. Moreover, escape to infinity always produces topological sensitivity, reflecting the intrinsic instability of unbounded dynamics. The central conclusion of the paper is that, under the additional assumption of properness, this instability is complete: sensitivity, escape to infinity, and non-compactness of forward orbits coincide. Thus, for proper maps, local and global dynamical behaviors are perfectly aligned.

Examples show that this equivalence breaks down in the absence of properness, where bounded forward orbits may coexist with sensitive dependence due to the accumulation of preimages at infinity. This highlights properness as the decisive condition ensuring local stability of non-escaping dynamics and openness of the escaping set.

The framework developed here provides a natural foundation for further study of sensitivity and escape in broader settings, including non-locally compact spaces, n -sensitivity, and applications to complex dynamical systems beyond the polynomial case.

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