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Article

# The $\omega^\sharp$ -Operator in Ideal Topological Spaces and Its Associated Topology

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## Abstract

In this paper, we introduce a new set-theoretic operator  $(\cdot)^\sharp_\omega$  in the framework of ideal topological spaces and investigate its fundamental properties, including its connections with the classical  $\sharp$ -operator and the  $\omega$ -local function. Using this operator, we define a closure-type operator  $Cl^\sharp_\omega$  and show that it satisfies the Kuratowski closure axioms. Consequently, a topology  $\mathcal{T}^\sharp_\omega$  is obtained, which is strictly finer than the topology induced by the  $\sharp$ -operator. Furthermore, the structural relationships among these topologies are examined, and some applications of the  $\omega^\sharp$ -operator are presented. Finally, we introduce the notions of  $\omega^*$ -continuity and  $\omega^\sharp$ -continuity, investigate their relationship, and establish a new decomposition of continuity. We also compare these notions with related concepts such as  $*$ -continuity and  $\sharp$ -continuity.

**Keywords:** ideal topological space;  $\omega$ -open set;  $(\cdot)^\sharp_\omega$ -operator;  $Cl^\sharp_\omega$ -operator;  $\mathcal{T}^\sharp_\omega$ -topology;  $\mathcal{T}^*_\omega$ -topology;  $\omega^*$ -continuity;  $\omega^\sharp$ -continuity

**MSC:** 54A05; 54A10; 54C10

## 1. Introduction and Preliminaries

The interaction between algebraic structures and topology such as ideals [1], filters [2], primals [3], and grills [4] has played an important role in general topology. Among these structures, ideals provide a convenient framework for formalizing notions of smallness.

Let  $(\mathbb{X}, \mathcal{T})$  be a topological space. A nonempty family  $\mathfrak{I} \subseteq 2^\mathbb{X}$  is called an *ideal* if it is hereditary and closed under finite unions. The triple  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  is called an *ideal topological space* (briefly, an ideal Top space), with no separation axioms assumed. For an ideal topological space, the associated local function (or  $*$ -operator) is defined by

$$H^* = \{x \in \mathbb{X} : \forall U \in \mathcal{T}(x), U \cap H \notin \mathfrak{I}\},$$

where  $\mathcal{T}(x) = \{U \in \mathcal{T} : x \in U\}$  and  $H \subseteq \mathbb{X}$ . This operator, originating in Kuratowski's work [5], was further developed by Vaidyanathaswamy [6]. Moreover, the operator

$Cl^*(H) = H \cup H^*$ , defines a Kuratowski closure operator on  $2^\mathbb{X}$ . Later contributions include the complementary operator  $\Psi$  introduced by Natkaniec [7] and the systematic study of ideal topological spaces by Janković and Hamlett [8]. Islam and Modak (2018) provided an in-depth study of the  $*$  and  $\Psi$  operators and their properties [9].

Recent developments in the theory of ideal topological spaces have emphasized the interaction between ideals and various generalized topological notions. In particular, investigations have addressed weakened forms of separation axioms as well as extensions of closedness concepts within this framework [10,11]. Moreover, approaches based on nano-topology, especially those employing covering-generated neighborhood systems in the presence of multiple ideals, have further enriched the structure of such spaces [12].

Earlier contributions have also examined decompositions of continuity in the setting of ideal topologies and  $\mathcal{I}$ -Alexandroff spaces, providing a deeper understanding of how classical continuity notions can be refined through the incorporation of ideal-related constraints [13,14].

More recently, attention has shifted toward the formulation of new operators and localized functions in ideal topological spaces, such as sharp-type operators and aura-based local functions. These constructions serve as effective tools for producing finer or alternative topological structures, thereby extending the scope of operator-driven methodologies [15–18]. Collectively, these advancements reflect a growing interest in operator-oriented frameworks and their role in shaping modern generalizations of ideal topological spaces.

Throughout this paper, a pair  $(\mathbb{X}, \mathcal{T})$ , or simply  $\mathbb{X}$  when the topology is clear, denotes a topological space (*Top-space*) without assuming separation axioms. For any  $H \subseteq \mathbb{X}$ ,  $\text{Cl}(H)$  and  $\text{Int}(H)$  represent the closure and interior of  $H$  relative to  $\mathcal{T}$ , respectively. Let  $(\mathbb{X}, \mathcal{T})$  be a Top-space and  $H \subseteq \mathbb{X}$ . A point  $x \in \mathbb{X}$  is called a *condensation point* of  $H$  if

$$\forall U \in \mathcal{T} (x \in U \Rightarrow U \cap H \text{ is uncountable}).$$

The set  $H$  is said to be  $\omega$ -closed [19] if it contains all of its condensation points. A subset  $W \subseteq \mathbb{X}$  is  $\omega$ -open if  $\mathbb{X} \setminus W$  is  $\omega$ -closed. Equivalently,  $W$  is  $\omega$ -open if and only if

$$\forall x \in W \exists U \in \mathcal{T} \text{ such that } x \in U \text{ and } U \setminus W \text{ is countable.}$$

The family of all  $\omega$ -open subsets of  $\mathbb{X}$  is denoted by  $\mathcal{T}_\omega$  forms a topology on  $X$  finer than  $\tau$ . The  $\omega$ -closure and  $\omega$ -interior, which can be defined in the same way as  $\text{Cl}(A)$  and  $\text{Int}(A)$ , respectively, will be denoted by  $\text{Cl}_\omega(A)$  and  $\text{Int}_\omega(A)$ , respectively.

Motivated by the concept of  $\omega$ -open sets, Ahmad Al-Omari and Hanan Al-Saadi [20] introduced the notion of  $\omega$ -local functions with respect to a topology  $\mathcal{T}$  and an ideal  $\mathcal{I}$  on a set  $\mathbb{X}$ . For any subset  $H \subseteq \mathbb{X}$ , define  $H_\omega^*(\mathcal{T}, \mathcal{I}) = \{x \in \mathbb{X} : U \cap H \notin \mathcal{I} \text{ for every } U \in \mathcal{T}_\omega(x)\}$ , where  $\mathcal{T}_\omega(x) = \{U \in \mathcal{T}_\omega : x \in U\}$ . For brevity, we write  $H_\omega^*$  instead of  $H_\omega^*(\mathcal{T}, \mathcal{I})$ . Moreover, for an ideal  $\mathcal{I}$  on  $\mathbb{X}$ , the operator  $\text{Cl}_\omega^*(H) = H \cup H_\omega^*$  defines a Kuratowski closure operator on  $2^\mathbb{X}$ . The topology generated by the operator  $\text{Cl}_\omega^*$  is  $\mathcal{T}_\omega^*(\mathbb{X}, \tau) = \{H \subseteq \mathbb{X} : \text{Cl}_\omega^*(\mathbb{X} \setminus H) = \mathbb{X} \setminus H\}$ . This topology is called the  $\omega^*$ -topology, and it is finer than  $\mathcal{T}_\omega$ .

Recently, Issaka, F. Y.; Özköç, M. [21] developed a framework based on extremal ideals and investigated their structural behavior within ideal-induced topologies. Let  $\mathbb{X} \neq \emptyset$  and let  $\mathcal{I} \subsetneq 2^\mathbb{X}$  be an ideal. The ideal  $\mathcal{I}$  is said to be *maximal* if

$$\forall \mathcal{K} (\mathcal{I} \subseteq \mathcal{K} \subseteq 2^\mathbb{X} \Rightarrow (\mathcal{K} = \mathcal{I} \vee \mathcal{K} = 2^\mathbb{X})).$$

Dually, assuming  $\mathcal{I} \neq \{\emptyset\}$ , the ideal  $\mathcal{I}$  is called *minimal* whenever

$$\forall \mathcal{K} (\{\emptyset\} \subseteq \mathcal{K} \subseteq \mathcal{I} \Rightarrow (\mathcal{K} = \mathcal{I} \vee \mathcal{K} = \{\emptyset\})).$$

Let  $\mathcal{T}$  be a topology on  $\mathbb{X}$ . For any  $H \subseteq \mathbb{X}$ , define the operator

$$H^\sharp(\mathcal{I}, \mathcal{T}) = \{x \in \mathbb{X} : \forall U \in \mathcal{T}, x \in U \Rightarrow \exists J \in \mathcal{I} \setminus \{\emptyset\} \text{ such that } I \subseteq U \cap H\}.$$

This operator is referred to as the *sharp transform* of  $H$  relative to  $(\mathcal{I}, \mathcal{T})$ , and is briefly denoted by  $H^\sharp$ . Moreover, define the dual operator by  $\Psi^\sharp(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)^\sharp$ . Using  $H^\sharp$ , one obtains a Kuratowski-type closure operator  $\text{Cl}^\sharp(H) = H \cup H^\sharp$ ,  $H \subseteq \mathbb{X}$ . The family of open sets induced by  $\text{Cl}^\sharp$  is given by  $\mathcal{T}^\sharp = \{U \subseteq \mathbb{X} : \text{Cl}^\sharp(\mathbb{X} \setminus U) = \mathbb{X} \setminus U\}$ , which defines a topology on  $\mathbb{X}$  satisfying  $\mathcal{T} \subseteq \mathcal{T}^\sharp$ .

In Section 2, we introduce a new operator  $(\cdot)_\omega^\sharp$  and investigate its principal properties, emphasizing its connections with the classical sharp operator and the  $\omega$ -local function.

In Section 3, using this operator, we define a Kuratowski-type closure  $\text{Cl}_\omega^\sharp$  and analyze the topology it induces, proving that this topology is strictly finer than  $\mathcal{T}^\sharp$ . Furthermore, we examine the relationships between  $\mathcal{T}_\omega^\sharp$  and other related topologies, highlighting both implications and independence results among them.

In Section 4, we present some applications of the operator  $\omega^\sharp$ .

Finally, in Section 5, we define the notions of  $\omega^*$ -continuity and  $\omega^\sharp$ -continuity for functions, and examine their relationships with other established forms of continuity, such as  $\omega$ -continuity and  $\sharp$ -continuity.

In this study,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$  denote the sets of integers, rational numbers, and real numbers, respectively.

We first present a number of preliminary results needed in the sequel.

**Definition 1.** [21] Let  $\mathbb{X}$  be a nonempty set and  $H \subseteq \mathbb{X}$  with  $H \neq \emptyset$ . The ideal generated by  $H$  is defined as  $\mathfrak{I}(H) = \{J \subseteq \mathbb{X} \mid J \subseteq H\}$ .

**Theorem 1.** [22] Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top – space and let  $H, G \subseteq \mathbb{X}$ . Then the following hold.

- (1)  $(H_\omega^*)_\omega^* \subseteq H_\omega^*$ ,
- (2)  $(H \cup G)_\omega^* = H_\omega^* \cup G_\omega^*$ ,
- (3)  $H_\omega^* \subseteq \text{Cl}_\omega(H)$ ,
- (4)  $H_\omega^* \setminus G_\omega^* \subseteq (H \setminus G)_\omega^*$ .

**Lemma 1.** [21] Let  $\mathfrak{I}$  be an ideal on  $\mathbb{X}$ , and let  $\mathcal{K} \subseteq 2^\mathbb{X}$ . Define

$$(\mathfrak{I} : \mathcal{K}) = \{H \subseteq \mathbb{X} \mid (H \cap K \in \mathfrak{I})(\forall K \in \mathcal{K})\}.$$

Then  $(\mathfrak{I} : \mathcal{K})$  is itself an ideal on  $\mathbb{X}$ .

**Definition 2.** [21] [Ideal Quotient and Annihilator] Let  $\mathbb{X} \neq \emptyset$  and let  $\mathfrak{I}$  be an ideal on  $\mathbb{X}$ . Suppose  $\mathcal{K} \subseteq 2^\mathbb{X}$ .

- (1) The quotient of the ideal  $\mathfrak{I}$  by  $\mathcal{K}$  is the collection

$$(\mathfrak{I} : \mathcal{K}) = \{H \subseteq \mathbb{X} \mid H \cap K \in \mathfrak{I} \text{ for every } K \in \mathcal{K}\}.$$

- (2) In the special case where  $\mathfrak{I} = \{\emptyset\}$ , the quotient  $(\{\emptyset\} : \mathcal{K})$  is called the annihilator of  $\mathcal{K}$  relative to  $\mathbb{X}$ , denoted by  $\mathcal{ANN}(\mathcal{K}) = (\{\emptyset\} : \mathcal{K})$ .
- (2) For a singleton subset  $H \subseteq \mathbb{X}$ , we write  $\mathcal{ANN}_H = \mathcal{ANN}(\{H\})$  for brevity.

**Definition 3.** [21] Let  $\mathbb{X} \neq \emptyset$  and  $\mathfrak{I} \subseteq 2^\mathbb{X}$ . The ideal  $\mathfrak{I}$  is called faithful if  $\mathcal{ANN}(\mathfrak{I}) = \{\emptyset\}$ .

**Lemma 2.** [21] Let  $\mathfrak{I}$  be an ideal on  $\mathbb{X}$ . Then  $\mathfrak{I} \cap \mathcal{ANN}(\mathfrak{I}) = \{\emptyset\}$ .

**Theorem 2.** [21] A proper ideal  $\mathfrak{I}$  on a non-empty set  $\mathbb{X}$  is maximal if and only if for every  $H \subseteq \mathbb{X}$ , either  $H \in \mathfrak{I}$  or  $\mathbb{X} \setminus H \in \mathfrak{I}$ .

**Corollary 1.** [21] Let  $\mathbb{X} \neq \emptyset$  with an ideal  $\mathfrak{I}$ . Then the following hold.

- (1) For every  $H \subseteq \mathbb{X}$ , one has  $\mathcal{ANN}_H = \mathcal{ANN}(\mathfrak{I}(H))$ .
- (2) If  $\mathfrak{I}$  is a minimal ideal, then  $\mathcal{ANN}(\mathfrak{I})$  is a maximal ideal.

## 2. Basic Properties of the $\omega^\sharp$ -Operator

**Definition 4.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top-space. For any subset  $H \subseteq \mathbb{X}$ , define the operator  $(\cdot)_\omega^\sharp : 2^\mathbb{X} \rightarrow 2^\mathbb{X}$  by  $H_\omega^\sharp(\mathfrak{I}, \mathcal{T}) = \left\{x \in \mathbb{X} \mid \forall W \in \mathcal{T}_\omega(x), \exists J \in \mathfrak{I} \setminus \{\emptyset\} (J \subseteq W \cap H)\right\}$ , where  $\mathcal{T}_\omega(x) = \{W \in \mathcal{T}_\omega \mid x \in W\}$ . The set  $H_\omega^\sharp(\mathfrak{I}, \mathcal{T})$ , also denoted by  $H_\omega^\sharp$ , is called the  $\omega^\sharp$ -operator of  $H$  relative to the ideal  $\mathfrak{I}$  and the topology  $\mathcal{T}$ . When the context is clear, we simply write  $H_\omega^\sharp$ .

**Corollary 2.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top-space and let  $H \subseteq \mathbb{X}$ . Then  $H_{\omega}^{\sharp} \subseteq H^{\sharp}$ .

**Proof.** Since every  $\mathcal{T}$ -open is also  $\mathcal{T}_{\omega}$ -open, the inclusion  $H_{\omega}^{\sharp} \subseteq H^{\sharp}$  follows immediately by definition.  $\square$

The next example illustrates that, in general, the containment  $H^{\sharp} \subseteq H_{\omega}^{\sharp}$  fails to be valid.

**Example 1.** Consider  $\mathbb{R}$  endowed with the trivial (indiscrete) topology  $\mathcal{T}_{ind} = \{\emptyset, \mathbb{R}\}$ . Define an ideal on  $\mathbb{R}$  by  $\mathfrak{I} = \mathfrak{I}(\{\emptyset, \sqrt{2}\}) = \{\emptyset, \{0\}, \{\sqrt{2}\}, \{0, \sqrt{2}\}\}$ . Let  $H = \mathbb{P}$  be the set of irrational numbers. Then the sharp operator associated with  $\mathfrak{I}$  satisfies  $H^{\sharp} = \mathbb{R}$ . Indeed, observe that the set  $\mathbb{R} \setminus \{\sqrt{2}\}$  is  $\omega$ -open. Moreover, for every  $J \in \mathfrak{I} \setminus \{\emptyset\}$ , we have  $(\mathbb{R} \setminus \{\sqrt{2}\}) \cap J = \emptyset$ . Consequently, for each  $x \in \mathbb{R} \setminus \{\sqrt{2}\}$ , we obtain  $x \notin H_{\omega}^{\sharp}$ . Hence,  $H^{\sharp} = \mathbb{R} \not\subseteq H_{\omega}^{\sharp} = \{\sqrt{2}\}$ .

**Remark 1.** From Example 1, we have  $H_{\omega}^{\sharp} = \{\sqrt{2}\}$ , and hence  $H = \mathbb{P} \not\subseteq H_{\omega}^{\sharp}$ . Next, let  $\mathfrak{I} = \mathfrak{I}(\mathbb{Z})$  be the ideal on  $\mathbb{R}$  generated by the set of integer numbers  $\mathbb{Z}$ , and consider the set  $H = \mathbb{R} \setminus \mathbb{Z}$ .

For every  $W \in \mathcal{T}_{\omega}$ , we obtain  $(\mathbb{R} \setminus \mathbb{Z}) \cap W \in \mathcal{ANN}(\mathfrak{I})$ , which implies that  $H_{\omega}^{\sharp} = \emptyset$ . Consequently,  $H = \mathbb{R} \setminus \mathbb{Z} \not\subseteq H_{\omega}^{\sharp} = (\mathbb{R} \setminus \mathbb{Z})_{\omega}^{\sharp} = \emptyset$ .

Finally, consider the ideal  $\mathcal{K} = \{H \subseteq \mathbb{R} : 1 \notin H\}$ . Then let  $H = \mathbb{R} \setminus \{1\}$ . It follows that  $H_{\omega}^{\sharp} = (\mathbb{R} \setminus \{1\})_{\omega}^{\sharp} = \mathbb{R}$ , and hence  $H_{\omega}^{\sharp} = \mathbb{R} \not\subseteq \mathbb{R} \setminus \{1\} = H$ . Therefore, in general, there is no inclusion relationship between a set  $H$  and its associated set  $H_{\omega}^{\sharp}$ .

Based on Theorem 2, we obtain the following corollary.

**Corollary 3.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top-space. If the ideal  $\mathfrak{I}$  is maximal, then for every subset  $H \subseteq \mathbb{X}$ , we have either  $H_{\omega}^* = \emptyset$  or  $(\mathbb{X} \setminus H)_{\omega}^* = \emptyset$ .

**Theorem 3.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top-space and let  $H \subseteq \mathbb{X}$ . If  $\mathfrak{I}$  is a maximal ideal, then  $H$  is either  $\mathcal{T}_{\omega}^*$ -closed or  $\mathcal{T}_{\omega}^*$ -open.

**Proof.** Let  $\mathfrak{I}$  be a maximal ideal on  $\mathbb{X}$ , and consider an arbitrary subset  $H \subseteq \mathbb{X}$ . Because  $\mathfrak{I}$  is maximal, it follows from the Theorem 2, that

$$H \in \mathfrak{I} \vee \mathbb{X} \setminus H \in \mathfrak{I}.$$

**Case 1.** If  $H \in \mathfrak{I}$ , then  $H_{\omega}^* = \emptyset$ ,  $\text{Cl}_{\omega}^*(H) = H$ . Thus  $H$  is  $\mathcal{T}_{\omega}^*$ -closed, and hence  $\mathbb{X} \setminus H$  is  $\mathcal{T}_{\omega}^*$ -open.

**Case 2.** If  $\mathbb{X} \setminus H \in \mathfrak{I}$ , then  $(\mathbb{X} \setminus H)_{\omega}^* = \emptyset$ ,  $\text{Cl}_{\omega}^*(\mathbb{X} \setminus H) = \mathbb{X} \setminus H$ . Thus  $\mathbb{X} \setminus H$  is  $\mathcal{T}_{\omega}^*$ -closed, and hence  $H$  is  $\mathcal{T}_{\omega}^*$ -open. Therefore,  $H$  is either  $\mathcal{T}_{\omega}^*$ -closed or  $H$  is  $\mathcal{T}_{\omega}^*$ -open.  $\square$

**Theorem 4.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top-space. If the ideal  $\mathfrak{I}$  is maximal, then the space  $(\mathbb{X}, \mathcal{T}_{\omega}^*)$  satisfies the  $T_0$  separation axiom.

**Proof.** Let  $\alpha, \beta \in \mathbb{X}$  with  $\alpha \neq \beta$ . Define  $H = \{\alpha\}$ . Since the ideal  $\mathfrak{I}$  is maximal, we have  $H \in \mathfrak{I} \vee (\mathbb{X} \setminus H) \in \mathfrak{I}$ .

**Case 1:**  $\mathbb{X} \setminus H \in \mathfrak{I}$ . By Theorem 3,  $\mathbb{X} \setminus H \in \mathfrak{I} \Rightarrow H \in \mathcal{T}_{\omega}^*$ . Hence,  $H \in \mathcal{T}_{\omega}^* \wedge (\alpha \in H \wedge \beta \notin H)$ .

**Case 2:**  $H \in \mathfrak{I}$ . Then  $H \in \mathfrak{I} \Rightarrow \mathbb{X} \setminus H \in \mathcal{T}_{\omega}^*$ . Hence,  $(\mathbb{X} \setminus H) \in \mathcal{T}_{\omega}^* \wedge (\beta \in \mathbb{X} \setminus H \wedge \alpha \notin \mathbb{X} \setminus H)$ .

Thus in all cases,  $\exists U \in \mathcal{T}_{\omega}^* [(\alpha \in U \wedge \beta \notin U) \vee (\beta \in U \wedge \alpha \notin U)]$ . Hence  $(\mathbb{X}, \mathcal{T}_{\omega}^*)$  satisfies the  $T_0$  separation axiom.  $\square$

**Remark 2.** Consider a topological space  $(\mathbb{X}, \mathcal{T})$  and two ideals  $\mathcal{K}$  and  $\mathfrak{I}$  on  $\mathbb{X}$ . Assume that  $\mathcal{K} \subseteq \mathfrak{I}$ . Then, for any  $H \subseteq \mathbb{X}$ , the following implication holds:

$$\mathcal{K} \subseteq \mathfrak{I} \implies H_{\omega}^{\sharp}(\mathcal{K}, \mathcal{T}) \subseteq H_{\omega}^{\sharp}(\mathfrak{I}, \mathcal{T}).$$

**Remark 3.** In general, there is no intrinsic correspondence between the  $\omega$ -local function and the  $\omega^\sharp$ -operator. This lack of dependence can be demonstrated through the following construction.

Let  $(\mathbb{X}, \mathcal{U})$  be the usual topological space where  $\mathbb{X} = \mathbb{R}$ . Let  $\mathfrak{I} = \mathfrak{I}(\mathbb{Q})$  be the ideal on  $\mathbb{R}$ , and consider the set  $F = \mathbb{R} \setminus \mathbb{Q}$ . Then  $F \cap W \in \mathcal{ANN}(\mathfrak{I})$  for every  $W \in \mathcal{T}_\omega$ .

In this setting, one observes that

$$F_\omega^\sharp(\mathfrak{I}, \mathcal{T}) = \emptyset \quad \text{while} \quad F_\omega^*(\mathfrak{I}, \mathcal{T}) = \mathbb{R}.$$

Conversely, since  $F \in \mathcal{ANN}(\mathfrak{I})$ , the behavior of the corresponding operators is reversed. Indeed,

$$(\mathbb{X} \setminus F)_\omega^\sharp(\mathfrak{I}, \mathcal{T}) = \mathbb{Q} \quad \text{and} \quad (\mathbb{X} \setminus F)_\omega^*(\mathfrak{I}, \mathcal{T}) = \emptyset.$$

This example illustrates that the two operators behave independently..

The following theorem establishes a connection between the  $\omega$ -localized function and the  $\omega^\sharp$ -operator.

**Theorem 5.** For any ideal Top-space  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  and any  $H \subseteq X$ , we have

$$H_\omega^\sharp(\mathfrak{I}, \mathcal{T}) = H_\omega^*(\mathcal{ANN}(\mathfrak{I}), \mathcal{T}).$$

**Proof.** Let  $H \subseteq X$ . Then, for any  $x \in \mathbb{X}$ :

$$\begin{aligned} x \in H_\omega^\sharp(\mathfrak{I}, \mathcal{T}) &\iff \forall W \in \mathcal{T}_\omega(x), \exists J \in \mathfrak{I} \setminus \{\emptyset\}, J \subseteq W \cap H \\ &\iff \forall W \in \mathcal{T}_\omega(x), \exists J \in \mathfrak{I} \setminus \{\emptyset\}, J \cap (W \cap H) \neq \emptyset \\ &\iff \forall W \in \mathcal{T}_\omega(x), W \cap H \notin \mathcal{ANN}(\mathfrak{I}) \\ &\iff x \in H_\omega^*(\mathcal{ANN}(\mathfrak{I}), \mathcal{T}). \end{aligned}$$

Hence, we conclude,  $H_\omega^\sharp(\mathfrak{I}, \mathcal{T}) = H_\omega^*(\mathcal{ANN}(\mathfrak{I}), \mathcal{T})$ .  $\square$

**Theorem 6.** Consider an ideal Top-space  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$ . For any subsets  $H$  and  $G$  of  $\mathbb{X}$ , the operator  $\omega^\sharp$  satisfies the following properties:

- (1)  $H \subseteq G \Rightarrow H_\omega^\sharp \subseteq G_\omega^\sharp$ ;
- (2)  $H_\omega^\sharp = \text{Cl}_\omega(H_\omega^\sharp) \subseteq \text{Cl}_\omega(H)$  and  $H_\omega^\sharp$  is  $\omega$ -closed in  $(\mathbb{X}, \mathcal{T})$ ;
- (3)  $(H \cap G)_\omega^\sharp \subseteq H_\omega^\sharp \cap G_\omega^\sharp$ ;
- (4)  $(H \cup G)_\omega^\sharp = H_\omega^\sharp \cup G_\omega^\sharp$ ;
- (5)  $H_\omega^\sharp \setminus G_\omega^\sharp \subseteq (H \setminus G)_\omega^\sharp$ ;
- (6) If  $H \in \mathcal{ANN}(\mathfrak{I})$ , then  $H_\omega^\sharp = \emptyset$ ;
- (7) If  $H \in \mathcal{ANN}(\mathfrak{I})$ , then  $(H \cup G)_\omega^\sharp = (G \setminus H)_\omega^\sharp$ ;
- (8) If  $\mathfrak{I}$  is faithful, then  $H_\omega^\sharp = \text{Cl}_\omega(H)$ ;
- (9) If  $H \in \mathcal{T}_\omega$ , then  $H \cap G_\omega^\sharp \subseteq (H \cap G)_\omega^\sharp$ ;
- (10)  $(H_\omega^\sharp)_\omega^\sharp \subseteq H_\omega^\sharp$ .

**Proof.** We verify the properties of the operator  $\omega^\sharp$  as follows:

$$\begin{aligned} (1) \quad x \in H_\omega^\sharp &\implies \forall W \in \mathcal{T}_\omega(x), \exists J \in \mathfrak{I}, J \neq \emptyset, \text{ satisfying } J \subseteq W \cap H \\ &\implies H \subseteq G \text{ implies } J \subseteq W \cap G \\ &\implies x \in G_\omega^\sharp \end{aligned}$$

$$\implies H_\omega^\# \subseteq G_\omega^\#.$$

(2) Clearly,  $H_\omega^\# \subseteq \text{Cl}_\omega(H_\omega^\#)$ . To verify the converse inclusion  $\text{Cl}_\omega(H_\omega^\#) \subseteq H_\omega^\#$ , we proceed as follows:

$$\begin{aligned} x \in \text{Cl}_\omega(H_\omega^\#) &\implies \forall W \in \mathcal{T}_\omega(x), W \cap H_\omega^\# \neq \emptyset, \\ \exists z \in W \cap H_\omega^\# &\implies z \in W \text{ and } z \in H_\omega^\#, \\ \text{since } z \in H_\omega^\#, \exists J \in \mathcal{J} \setminus \{\emptyset\} \text{ with } J \subseteq W \cap H &\implies x \in H_\omega^\#, \\ \therefore \text{Cl}_\omega(H_\omega^\#) &= H_\omega^\#, \\ x \notin \text{Cl}_\omega(H) &\implies \exists W \in \mathcal{T}_\omega(x) \text{ such that } W \cap H = \emptyset, \\ &\implies W \cap H \in \mathcal{ANN}(\mathcal{J}) \implies x \notin H_\omega^\# = \text{Cl}_\omega(H_\omega^\#), \\ \therefore H_\omega^\# &= \text{Cl}_\omega(H_\omega^\#) \subseteq \text{Cl}_\omega(H). \end{aligned}$$

(3) This follows directly from (1).

(4) From Theorem 1(2) and Theorem 5, we infer that

$$\begin{aligned} (H \cup G)_\omega^\#(\mathcal{J}, \mathcal{T}) &= (H \cup G)_\omega^*(\mathcal{ANN}(\mathcal{J}), \mathcal{T}) \\ &= H_\omega^*(\mathcal{ANN}(\mathcal{J}), \mathcal{T}) \cup G_\omega^*(\mathcal{ANN}(\mathcal{J}), \mathcal{T}) \\ &= H_\omega^\#(\mathcal{J}, \mathcal{T}) \cup G_\omega^\#(\mathcal{J}, \mathcal{T}). \end{aligned}$$

(5) Since  $H = (H \setminus G) \cup (H \cap G)$ , we have

$$\begin{aligned} H_\omega^\#(\mathcal{J}, \mathcal{T}) &= (H \setminus G)_\omega^\#(\mathcal{J}, \mathcal{T}) \cup (H \cap G)_\omega^\#(\mathcal{J}, \mathcal{T}) && \text{by (4),} \\ &\subseteq (H \setminus G)_\omega^\#(\mathcal{J}, \mathcal{T}) \cup G_\omega^\#(\mathcal{J}, \mathcal{T}) && \text{by (1),} \\ \implies H_\omega^\#(\mathcal{J}, \mathcal{T}) \setminus G_\omega^\#(\mathcal{J}, \mathcal{T}) &\subseteq (H \setminus G)_\omega^\#(\mathcal{J}, \mathcal{T}) \setminus G_\omega^\#(\mathcal{J}, \mathcal{T}) \end{aligned} \quad (1)$$

$$\begin{aligned} &\text{By axiom (1) } (H \setminus G)_\omega^\#(\mathcal{J}, \mathcal{T}) \subseteq H_\omega^\#(\mathcal{J}, \mathcal{T}), \\ \implies (H \setminus G)_\omega^\#(\mathcal{J}, \mathcal{T}) \setminus G_\omega^\#(\mathcal{J}, \mathcal{T}) &\subseteq H_\omega^\#(\mathcal{J}, \mathcal{T}) \setminus G_\omega^\#(\mathcal{J}, \mathcal{T}) \end{aligned} \quad (2)$$

Combining (1) and (2), we get:

$$\begin{aligned} H_\omega^\#(\mathcal{J}, \mathcal{T}) \setminus G_\omega^\#(\mathcal{J}, \mathcal{T}) &= (H \setminus G)_\omega^\#(\mathcal{J}, \mathcal{T}) \setminus G_\omega^\#(\mathcal{J}, \mathcal{T}) \\ &\subseteq (H \setminus G)_\omega^\#(\mathcal{J}, \mathcal{T}). \end{aligned}$$

**Equivalent Expression via Theorems 1(4) and 5.** Using the canonical forms, we also have

$$\begin{aligned} H_\omega^\#(\mathcal{J}, \mathcal{T}) \setminus G_\omega^\#(\mathcal{J}, \mathcal{T}) &= H_\omega^*(\mathcal{ANN}(\mathcal{J}), \mathcal{T}) \setminus G_\omega^*(\mathcal{ANN}(\mathcal{J}), \mathcal{T}) \quad (\text{Thm. 5}) \\ &\subseteq (H \setminus G)_\omega^*(\mathcal{ANN}(\mathcal{J}), \mathcal{T}) \quad (\text{Thm. 1}) \\ &= (H \setminus G)_\omega^\#(\mathcal{J}, \mathcal{T}). \end{aligned}$$

This completes the proof.

(6) It follows directly from the definition of  $\mathcal{ANN}(\mathfrak{J})$ .

(7)

$$\begin{aligned} (H \cup G)_\omega^\# &= [(H \setminus G) \cup (H \cap G) \cup (G \setminus H)]_\omega^\# \\ &\stackrel{\text{by(4)}}{=} (H \setminus G)_\omega^\# \cup (H \cap G)_\omega^\# \cup (G \setminus H)_\omega^\#. \end{aligned}$$

Since  $H \in \mathcal{ANN}(\mathfrak{J})$ , by (6) we get

$$(H \cup G)_\omega^\# = \emptyset \cup G_\omega^\# = G_\omega^\# = (G \setminus H)_\omega^\#.$$

(8) Since  $\mathfrak{J}$  is assumed to be faithful, it follows that  $\mathcal{ANN}(\mathfrak{J}) = \{\emptyset\}$ . From Theorem 5, we deduce that

$$H_\omega^\#(\mathfrak{J}, \mathcal{T}) = H_\omega^*(\{\emptyset\}, \mathcal{T}) = \{x \in \mathbb{X} \mid (\forall W \in \mathcal{T}_\omega(x)) (W \cap H \neq \emptyset)\} = \text{Cl}_\omega(H).$$

(9) Let  $x \in H \cap G_\omega^\#$ . Then

$$x \in H \cap G_\omega^\# \Rightarrow \begin{cases} x \in H, \\ x \in G_\omega^\# \Rightarrow (\forall W \in \mathcal{T}_\omega(x)) (\exists J \in \mathfrak{J} \setminus \{\emptyset\}) (J \subseteq W \cap G). \end{cases}$$

$$\begin{aligned} H \in \mathcal{T}_\omega(x) &\Rightarrow W \cap H \in \mathcal{T}_\omega(x), \\ &\Rightarrow J \subseteq (W \cap H) \cap G = W \cap (H \cap G), \\ &\Rightarrow x \in (H \cap G)_\omega^\#, \\ &\Rightarrow H \cap G_\omega^\# \subseteq (H \cap G)_\omega^\#. \end{aligned}$$

(10)

$$\begin{aligned} x &\in (H_\omega^\#)_\omega^\# \\ &\Rightarrow \forall U \in \mathcal{T}_\omega(x), \exists I \in \mathfrak{J} \setminus \{\emptyset\} : I \subseteq H_\omega^\# \cap U \\ &\Rightarrow E^{\delta\#} \cap U \neq \emptyset \Rightarrow \exists z \in E^{\delta\#} \cap U \\ &\Rightarrow z \in H_\omega^\# \text{ and } z \in U \\ &\Rightarrow \exists J \in \mathfrak{J} \setminus \{\emptyset\} : J \subseteq H \cap U \Rightarrow z \in E^{\delta\#}. \end{aligned}$$

Hence,  $(H_\omega^\#)_\omega^\# \subseteq H_\omega^\#$ .  $\square$

**Example 2.** Let  $\mathbb{R}$  be equipped with the usual topology  $\mathcal{U}$ , and let  $\mathfrak{J} = \{\emptyset, \{0\}\}$  be an ideal on  $\mathbb{R}$ . Consider the operator  $\omega^\#$ .

For any  $x \in \mathbb{R}$  with  $x \neq 0$ , there exists  $\varepsilon > 0$  such that

$$(x - \varepsilon, x + \varepsilon) \in \mathcal{T}_\omega(x) \quad \text{and} \quad 0 \notin (x - \varepsilon, x + \varepsilon).$$

Hence,

$$[\mathbb{R} \cap (x - \varepsilon, x + \varepsilon)] \cap \{0\} = \emptyset,$$

which implies  $x \notin \mathbb{R}_\omega^\#$ . Therefore,  $\mathbb{R}_\omega^\# = \{0\}$ . This shows that  $\omega^\#$  is not extensive, since  $\mathbb{R}_\omega^\# \not\supseteq \mathbb{R}$ .

**Lemma 3.** Let  $(\mathbb{X}, \mathcal{T})$  be a topological space and let  $\mathfrak{J}, \mathcal{K} \subseteq 2^{\mathbb{X}}$  be ideals on  $\mathbb{X}$ . Then, for any subset  $H \subseteq \mathbb{X}$ ,  $H_\omega^*(\mathfrak{J} \cap \mathcal{K}, \mathcal{T}) = H_\omega^*(\mathfrak{J}, \mathcal{T}) \cup H_\omega^*(\mathcal{K}, \mathcal{T})$ .

**Proof.** ( $\subseteq$ ) Let  $x \in H_{\omega}^*(\mathfrak{J} \cap \mathcal{K}, \mathcal{T})$ . If  $x \notin H_{\omega}^*(\mathfrak{J}, \mathcal{T})$  and  $x \notin H_{\omega}^*(\mathcal{K}, \mathcal{T})$ , then there exist  $E, F \in \mathcal{T}_{\omega}(x)$  such that  $E \cap H \in \mathfrak{J}$  and  $F \cap H \in \mathcal{K}$ . Hence  $(E \cap F) \cap H \in \mathfrak{J} \cap \mathcal{K}$ , a contradiction. Thus  $x \in H_{\omega}^*(\mathfrak{J}, \mathcal{T}) \cup H_{\omega}^*(\mathcal{K}, \mathcal{T})$ .

( $\supseteq$ ) If  $x \in H_{\omega}^*(\mathfrak{J}, \mathcal{T})$  or  $x \in H_{\omega}^*(\mathcal{K}, \mathcal{T})$ , then  $W \cap H \notin \mathfrak{J} \cap \mathcal{K}$  for every  $W \in \mathcal{T}_{\omega}(x)$ , and hence  $x \in H_{\omega}^*(\mathfrak{J} \cap \mathcal{K}, \mathcal{T})$ .  $\square$

**Theorem 7.** In an ideal Top-space  $(\mathbb{X}, \mathcal{T}, \mathfrak{J})$ , the  $\omega$ -closure of any subset  $H \subseteq \mathbb{X}$  decomposes as the union of the sets  $H_{\omega}^{\sharp}$  and  $H_{\omega}^*$ . Explicitly,

$$\text{Cl}_{\omega}(H) = H_{\omega}^{\sharp}(\mathfrak{J}, \mathcal{T}) \cup H_{\omega}^*(\mathfrak{J}, \mathcal{T}).$$

**Proof.**

By Theorem 5

$$H_{\omega}^{\sharp}(\mathfrak{J}, \mathcal{T}) \cup H_{\omega}^*(\mathfrak{J}, \mathcal{T}) = H_{\omega}^*(\mathcal{ANN}(\mathfrak{J}), \mathcal{T}) \cup H_{\omega}^*(\mathfrak{J}, \mathcal{T})$$

By Lemma 3

$$\begin{aligned} &= H_{\omega}^*(\mathcal{ANN}(\mathfrak{J}) \cap \mathfrak{J}, \mathcal{T}), \text{ since } \mathcal{ANN}(\mathfrak{J}) \cap \mathfrak{J} = \emptyset \\ &= H_{\omega}^*(\{\emptyset\}, \mathcal{T}) = \text{Cl}_{\omega}(H). \quad \square \end{aligned}$$

$\square$

**Corollary 4.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{J})$  be an ideal Top-space and let  $H \subseteq \mathbb{X}$ . Then:

- (1)  $H \in \mathfrak{J} \implies H_{\omega}^{\sharp} = \text{Cl}_{\omega}(H)$ .
- (2)  $H \in \mathcal{ANN}(\mathfrak{J}) \implies H_{\omega}^* = \text{Cl}_{\omega}(H)$ .

### 3. On the $\omega^{\sharp}$ -Closure Operator and the Associated Sharp Topology

**Definition 5.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{J})$  be an ideal Top-space. The  $\omega^{\sharp}$ -closure operator

$$\text{Cl}_{\omega}^{\sharp} : 2^{\mathbb{X}} \rightarrow 2^{\mathbb{X}} \text{ is defined by } \forall H \subseteq \mathbb{X}, \quad \text{Cl}_{\omega}^{\sharp}(H) = H \cup H_{\omega}^{\sharp}.$$

**Theorem 8.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{J})$  be an ideal Top-space, and  $H, G \subseteq \mathbb{X}$ . Then  $\text{Cl}_{\omega}^{\sharp}$  satisfies:

- (1)  $\text{Cl}_{\omega}^{\sharp}(\emptyset) = \emptyset$
- (2)  $\text{Cl}_{\omega}^{\sharp}(\mathbb{X}) = \mathbb{X}$
- (3)  $H \subseteq \text{Cl}_{\omega}^{\sharp}(H)$
- (4)  $H \subseteq G \implies \text{Cl}_{\omega}^{\sharp}(H) \subseteq \text{Cl}_{\omega}^{\sharp}(G)$
- (5)  $\text{Cl}_{\omega}^{\sharp}(H \cup G) = \text{Cl}_{\omega}^{\sharp}(H) \cup \text{Cl}_{\omega}^{\sharp}(G)$
- (6)  $\text{Cl}_{\omega}^{\sharp}(\text{Cl}_{\omega}^{\sharp}(H)) = \text{Cl}_{\omega}^{\sharp}(H)$
- (7) If  $H \in \mathcal{T}_{\omega}$ , then  $H \cap \text{Cl}_{\omega}^{\sharp}(G) \subseteq \text{Cl}_{\omega}^{\sharp}(H \cap G)$ .

**Proof.** We verify the properties of  $\text{Cl}_{\omega}^{\sharp}$  as follows:

- (1) By definition,  $\emptyset_{\omega}^{\sharp} = \emptyset$ , so

$$\text{Cl}_{\omega}^{\sharp}(\emptyset) = \emptyset \cup \emptyset_{\omega}^{\sharp} = \emptyset.$$

- (2) Since  $\mathbb{X}_{\omega}^{\sharp} \subseteq \mathbb{X}$ ,

$$\text{Cl}_{\omega}^{\sharp}(\mathbb{X}) = \mathbb{X} \cup \mathbb{X}_{\omega}^{\sharp} = \mathbb{X}.$$

- (3) Clearly,  $H \subseteq H \cup H_{\omega}^{\sharp} = \text{Cl}_{\omega}^{\sharp}(H)$ .

- (4) If  $H \subseteq G$ , then  $H_{\omega}^{\sharp} \subseteq G_{\omega}^{\sharp}$  (Theorem 6 (2)), so

$$\text{Cl}_{\omega}^{\sharp}(H) = H \cup H_{\omega}^{\sharp} \subseteq G \cup G_{\omega}^{\sharp} = \text{Cl}_{\omega}^{\sharp}(G).$$

(5) For unions, using Theorem 6 (4),

$$\begin{aligned}\text{Cl}_\omega^\sharp(H \cup G) &= (H \cup G) \cup (H \cup G)_\omega^\sharp \\ &= (H \cup G) \cup (H_\omega^\sharp \cup G_\omega^\sharp) \\ &= (H \cup H_\omega^\sharp) \cup (G \cup G_\omega^\sharp) \\ &= \text{Cl}_\omega^\sharp(H) \cup \text{Cl}_\omega^\sharp(G).\end{aligned}$$

(6) Using Theorem 6 (10) and items (3) and (4), we obtain that

$$\begin{aligned}\text{Cl}_\omega^\sharp(\text{Cl}_\omega^\sharp(H)) &= \text{Cl}_\omega^\sharp(H) \cup (\text{Cl}_\omega^\sharp(H))_\omega^\sharp \\ &= \text{Cl}_\omega^\sharp(H) \cup (H \cup H_\omega^\sharp)_\omega^\sharp \\ &= \text{Cl}_\omega^\sharp(H) \cup H_\omega^\sharp \cup (H_\omega^\sharp)_\omega^\sharp \\ &\subseteq \text{Cl}_\omega^\sharp(H).\end{aligned}$$

(7) If  $H \in \mathcal{T}_\omega$ , then by Theorem 6 (9),

$$H \cap \text{Cl}_\omega^\sharp(G) = (H \cap G) \cup (H \cap G)_\omega^\sharp \subseteq (H \cap G) \cup (H \cap G)_\omega^\sharp = \text{Cl}_\omega^\sharp(H \cap G).$$

□

**Proposition 1.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{J})$  be an ideal Top  $\omega$ -space and  $H \subseteq X$ . If  $H \subseteq H_\omega^\sharp$ , then the following properties hold.

- (1)  $\text{Cl}_\omega(H) = \text{Cl}_\omega^\sharp(H)$ ;
- (2)  $\text{Int}_\omega(\mathbb{X} \setminus H) = \text{Int}_\omega^\sharp(\mathbb{X} \setminus H)$ .

**Proof.** (1) By Theorem 6 (2),  $H_\omega^\sharp = \text{Cl}_\omega(H_\omega^\sharp) \subseteq \text{Cl}_\omega(H)$ . Since  $H \subseteq H_\omega^\sharp$ , it follows that  $\text{Cl}_\omega(H) \subseteq \text{Cl}_\omega(H_\omega^\sharp)$ . Therefore,  $\text{Cl}_\omega(H) = \text{Cl}_\omega(H_\omega^\sharp) = \text{Cl}_\omega^\sharp(H)$ .

(2) From (1), we obtain  $\mathbb{X} \setminus \text{Cl}_\omega(H) = \mathbb{X} \setminus \text{Cl}_\omega^\sharp(H) \Leftrightarrow \text{Int}_\omega(\mathbb{X} \setminus H) = \text{Int}_\omega^\sharp(\mathbb{X} \setminus H)$ . Hence, the result follows. □

**Corollary 5.** If  $(\mathbb{X}, \mathcal{T}, \mathfrak{J})$  is an ideal Top  $\omega$ -space, then for every  $H \subseteq \mathbb{X}$ ,  $\text{Cl}_\omega^\sharp(H) = H \cup H_\omega^*(\mathcal{ANN}(\mathfrak{J}), \mathcal{T})$ .

**Remark 4.** By Theorem 8, the operator  $\text{Cl}_\omega^\sharp(H) = H \cup H_\omega^\sharp$  is a Kuratowski closure. Accordingly,

$$\mathcal{T}_\omega^\sharp = \{W \subseteq \mathbb{X} \mid \text{Cl}_\omega^\sharp(\mathbb{X} \setminus W) = \mathbb{X} \setminus W\}$$

defines a topology on  $\mathbb{X}$ , called the  $\omega^\sharp$ -topology. When needed, we write  $\mathcal{T}_\omega^\sharp(\mathfrak{J})$ .

**Theorem 9.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{J})$  be an ideal Top  $\omega$ -space. Then the inclusions  $\mathcal{T}_\omega \subseteq \mathcal{T}_\omega^\sharp$  and  $\mathcal{T}^\sharp \subseteq \mathcal{T}_\omega^\sharp$  hold.

**Proof.** To show that  $\mathcal{T}_\omega \subseteq \mathcal{T}_\omega^\sharp$ . Let  $W \in \mathcal{T}_\omega$ . Then,  $(\forall x \in W)(\forall J \in \mathfrak{J} \setminus \{\emptyset\}) [J \cap (W \cap (\mathbb{X} \setminus W)) = \emptyset]$ . Hence  $W \cap (\mathbb{X} \setminus W) \in \mathcal{ANN}(\mathfrak{J})$ , it follows that

$$[(\forall x \in W) (x \notin (\mathbb{X} \setminus W)_\omega^\sharp)] \implies (\mathbb{X} \setminus W)_\omega^\sharp \subseteq \mathbb{X} \setminus W.$$

By the definition of the closure operator  $\text{Cl}_\omega^\sharp$ ,  $\text{Cl}_\omega^\sharp(\mathbb{X} \setminus W) = (\mathbb{X} \setminus W) \cup (\mathbb{X} \setminus W)_\omega^\sharp = \mathbb{X} \setminus W$ . Therefore,  $\mathbb{X} \setminus W$  is  $\mathcal{T}_\omega^\sharp$ -closed, and consequently  $W \in \mathcal{T}_\omega^\sharp$ . Hence,  $\mathcal{T}_\omega \subseteq \mathcal{T}_\omega^\sharp$ .

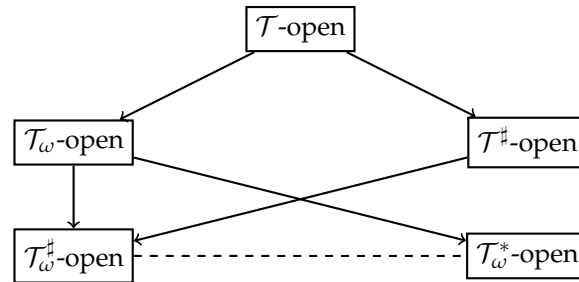
To show that  $\mathcal{T}^\sharp \subseteq \mathcal{T}_\omega^\sharp$ . Let  $W \in \mathcal{T}^\sharp$ . Then  $\mathbb{X} \setminus W$  is  $\mathcal{T}^\sharp$ -closed, so that  $(\mathbb{X} \setminus W)^\sharp \subseteq \mathbb{X} \setminus W$ .

By Corollary 2,  $(\mathbb{X} \setminus W)_\omega^\sharp \subseteq (\mathbb{X} \setminus W)^\sharp \subseteq \mathbb{X} \setminus W$ . Thus,  $\text{Cl}_\omega^\sharp(\mathbb{X} \setminus W) = (\mathbb{X} \setminus W) \cup (\mathbb{X} \setminus W)_\omega^\sharp = \mathbb{X} \setminus W$ . Therefore,  $\mathbb{X} \setminus W$  is  $\mathcal{T}_\omega^\sharp$ -closed, and hence  $W \in \mathcal{T}_\omega^\sharp$ . It follows that  $\mathcal{T}^\sharp \subseteq \mathcal{T}_\omega^\sharp$ . □

**Corollary 6.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal topological space. Then

$$(\forall H \subseteq \mathbb{X}) \left( \text{Cl}_\omega^\sharp(H) \subseteq \text{Cl}^\sharp(H) \wedge \text{Cl}_\omega^\sharp(H) \subseteq \text{Cl}_\omega(H) \right).$$

**Remark 5.** The relationships induced by the definitions of the  $\sharp$ -topology and the  $\omega^\sharp$ -topology are summarized in the diagram below. The example that follows shows that these implications do not admit converses. Moreover, the notions of  $\mathcal{T}_\omega^\sharp$ -open sets and  $\mathcal{T}_\omega^*$ -open sets are independent.



**Example 3.** Consider  $\mathbb{R}$  equipped with the topology  $\mathcal{T} = \{\emptyset, \mathbb{R}\}$ , and let  $\mathfrak{I} = \mathfrak{I}(\mathbb{Q})$  denote the ideal generated by  $\mathbb{Q}$ . Since  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{ANN}(\mathfrak{I}) \implies (\mathbb{R} \setminus \mathbb{Q})^\sharp = \emptyset$ , it follows that  $\mathbb{Q} \in \mathcal{T}^\sharp \wedge \mathbb{Q} \notin \mathcal{T}_\omega$ . Moreover,  $\mathcal{T}^\sharp \subseteq \mathcal{T}_\omega^\sharp \implies \mathbb{Q} \in \mathcal{T}_\omega^\sharp \wedge \mathbb{Q} \notin \mathcal{T}_\omega$ . On the other hand,  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_\omega \wedge (\mathbb{R} \setminus \mathbb{Q})^\sharp = \emptyset \implies \mathbb{R} \setminus \mathbb{Q} \notin \mathcal{T}^\sharp$ . Hence,  $\mathcal{T}_\omega$ -open set and  $\mathcal{T}^\sharp$ -open set are independent concepts. Furthermore, since  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_\omega \wedge \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset \in \mathcal{ANN}(\mathfrak{I})$ , we obtain  $\mathbb{Q}_\omega^\sharp \subseteq \mathbb{Q}$ . Consequently,  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_\omega^\sharp \wedge \mathbb{R} \setminus \mathbb{Q} \notin \mathcal{T}^\sharp$ .

**Example 4.** Let  $(\mathbb{R}, \mathcal{T}_{ind})$  be an indiscrete topological space, and let  $\mathfrak{I} = \mathfrak{I}((0, \infty))$  denote the ideal generated by the interval  $(0, \infty)$ . First, consider the set  $H = (0, \infty)$ . Then

$$\mathbb{R} \setminus H = (-\infty, 0] \in \mathcal{ANN}(\mathfrak{I}).$$

By Corollary 4 (2), we obtain  $(\mathbb{R} \setminus H)_\omega^\sharp = \emptyset$  and  $(\mathbb{R} \setminus H)_\omega^* = \text{Cl}_\omega(\mathbb{R} \setminus H) = \mathbb{R}$ . Therefore,  $H \in \mathcal{T}_\omega^\sharp$ , while  $H \notin \mathcal{T}_\omega^*$ . Next, let  $G = (-\infty, 0]$ . Then

$$\mathbb{R} \setminus G = (0, \infty) \in \mathfrak{I}.$$

Hence, by Corollary 4 (1),  $(\mathbb{R} \setminus G)_\omega^* = \emptyset$  and  $(\mathbb{R} \setminus G)_\omega^\sharp = \text{Cl}_\omega(\mathbb{R} \setminus G) = \mathbb{R}$ . Thus,  $G \in \mathcal{T}_\omega^*$ , whereas  $G \notin \mathcal{T}_\omega^\sharp$ .

**Definition 6.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal topological space and let  $H \subseteq \mathbb{X}$ . The following operators are defined:

- (1) The  $\Psi_\omega^*$ -operator is defined by  $\Psi_\omega^*(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)_\omega^*$ .
- (2) The  $\Psi_\omega^\sharp$ -operator is defined by  $\Psi_\omega^\sharp(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)_\omega^\sharp$ .

**Remark 6.** Let  $(X, \mathcal{T}, \mathfrak{I})$  be an ideal Top -space and let  $H \subseteq \mathbb{X}$ . According to the definition of the  $\Psi_\omega^\sharp$ -operator together with the Theorem 5, we have  $\Psi_\omega^*(H(\mathcal{ANN}(\mathfrak{I}), \mathcal{T})) = \Psi_\omega^\sharp(H(\mathfrak{I}, \mathcal{T}))$ .

**Theorem 10.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top -space and let  $H \subseteq \mathbb{X}$ . Then  $H \in \mathcal{T}_\omega^\sharp \iff H \subseteq \Psi_\omega^\sharp(H)$ .

**Proof.** Let  $H \subseteq \mathbb{X}$ . Then

$$\begin{aligned} H \in \mathcal{T}_\omega^\sharp &\iff \mathbb{X} \setminus H \text{ is } \mathcal{T}_\omega^\sharp\text{-closed} \\ &\iff \text{Cl}_\omega^\sharp(\mathbb{X} \setminus H) = \mathbb{X} \setminus H \\ &\iff (\mathbb{X} \setminus H)_\omega^\sharp \subseteq \mathbb{X} \setminus H \\ &\iff \mathbb{X} \setminus (\mathbb{X} \setminus H)_\omega^\sharp \supseteq H \\ &\iff \Psi_\omega^\sharp(H) \supseteq H. \end{aligned}$$

Hence,

$$H \in \mathcal{T}_\omega^\# \iff H \subseteq \Psi_\omega^\#(H).$$

□

**Theorem 11.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top  $\mathfrak{I}$ -space and  $H \subseteq \mathbb{X}$ . Then

$$\Psi_\omega^\#(H) \cap \Psi_\omega^*(H) = \text{Int}_\omega(H).$$

**Proof.** Consider any subset  $H \subseteq \mathbb{X}$ . By Theorem 7, We have

$$\begin{aligned} \text{Cl}_\omega(\mathbb{X} \setminus H) &= (\mathbb{X} \setminus H)_\omega^\#(\mathfrak{I}, \mathcal{T}) \cup (\mathbb{X} \setminus H)_\omega^*(\mathfrak{I}, \mathcal{T}) \\ \iff \mathbb{X} \setminus \text{Cl}_\omega(\mathbb{X} \setminus H) &= \mathbb{X} \setminus \left[ (\mathbb{X} \setminus H)_\omega^\#(\mathfrak{I}, \mathcal{T}) \cup (\mathbb{X} \setminus H)_\omega^*(\mathfrak{I}, \mathcal{T}) \right] \\ \iff \text{Int}_\omega(H) &= \mathbb{X} \setminus (\mathbb{X} \setminus H)_\omega^\#(\mathfrak{I}, \mathcal{T}) \cap \mathbb{X} \setminus (\mathbb{X} \setminus H)_\omega^*(\mathfrak{I}, \mathcal{T}) \\ \iff \text{Int}_\omega(H) &= \Psi_\omega^\#(H) \cap \Psi_\omega^*(H). \end{aligned}$$

□

From the Theorem 11, we obtain the following:

**Corollary 7.** For an ideal Top  $\mathfrak{I}$ -space  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  and a subset  $H \subseteq \mathbb{X}$ . Then:

- (1) If  $\mathbb{X} \setminus H \in \mathfrak{I}$ , then it follows that  $\Psi_\omega^\#(H) = \text{Int}_\omega(H)$ .
- (2) If  $\mathbb{X} \setminus H \in \mathcal{ANN}(\mathfrak{I})$ , then  $\Psi_\omega^*(H) = \text{Int}_\omega(H)$ .

**Theorem 12.** For any ideal Top  $\mathfrak{I}$ -space  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$ ,  $\mathcal{T}_\omega = \mathcal{T}_\omega^*(\mathfrak{I}, \mathcal{T}) \cap \mathcal{T}_\omega^\#(\mathfrak{I}, \mathcal{T})$ .

**Proof.** Obviously  $\mathcal{T}_\omega \subseteq \mathcal{T}_\omega^*(\mathfrak{I}, \mathcal{T}) \cap \mathcal{T}_\omega^\#(\mathfrak{I}, \mathcal{T})$ . Let  $W \in \mathcal{T}_\omega^*(\mathfrak{I}, \mathcal{T}) \cap \mathcal{T}_\omega^\#(\mathfrak{I}, \mathcal{T})$ . Then

$$W \in \mathcal{T}_\omega^*(\mathfrak{I}, \mathcal{T}) \wedge W \in \mathcal{T}_\omega^*(\mathcal{ANN}(\mathfrak{I}), \mathcal{T}) \Rightarrow W \in \mathcal{T}_\omega^*(\mathfrak{I} \cap \mathcal{ANN}(\mathfrak{I}), \mathcal{T}).$$

Since  $\mathfrak{I} \cap \mathcal{ANN}(\mathfrak{I}) = \{\emptyset\}$ , we obtain  $W \in \mathcal{T}_\omega$ . Hence,

$$\mathcal{T}_\omega^*(\mathfrak{I}, \mathcal{T}) \cap \mathcal{T}_\omega^\#(\mathfrak{I}, \mathcal{T}) \subseteq \mathcal{T}_\omega.$$

Therefore,

$$\mathcal{T}_\omega = \mathcal{T}_\omega^*(\mathfrak{I}, \mathcal{T}) \cap \mathcal{T}_\omega^\#(\mathfrak{I}, \mathcal{T}).$$

□

**Theorem 13.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top  $\mathfrak{I}$ -space with  $\mathfrak{I}$  a proper minimal ideal. Then, for every  $H \subseteq \mathbb{X}$ ,

$$H_\omega^\# = \emptyset \quad \text{or} \quad (\mathbb{X} \setminus H)_\omega^\# = \emptyset.$$

**Proof.** By the duality between ideals and their annihilators (Corollary 1 (2))  $\mathfrak{I}$  minimal  $\iff \mathcal{ANN}(\mathfrak{I})$  is maximal. We conclude that

$$\forall H \subseteq \mathbb{X}, (H \in \mathcal{ANN}(\mathfrak{I}) \vee \mathbb{X} \setminus H \in \mathcal{ANN}(\mathfrak{I})).$$

From Theorem 6 (6), we obtain

$$(H \in \mathcal{ANN}(\mathfrak{I}) \Rightarrow H_\omega^\# = \emptyset) \vee (\mathbb{X} \setminus H \in \mathcal{ANN}(\mathfrak{I}) \Rightarrow (\mathbb{X} \setminus H)_\omega^\# = \emptyset).$$

Therefore

$$\forall H \subseteq \mathbb{X}, (H_{\omega}^{\sharp} = \emptyset \vee (\mathbb{X} \setminus H)_{\omega}^{\sharp} = \emptyset).$$

□

**Corollary 8.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top –space with  $\mathfrak{I}$  minimal. Then

$$\forall H \subseteq \mathbb{X}, H \in \mathcal{T}_{\omega}^{\sharp}\text{-closed} \vee H \in \mathcal{T}_{\omega}^{\sharp}\text{-open}.$$

**Proof.** The proof proceeds analogously to that of Theorem 3. □

**Corollary 9.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top –space. If  $\mathfrak{I}$  is minimal, then the space  $(\mathbb{X}, \mathcal{T}_{\omega}^{\sharp})$  satisfies the  $T_0$  separation axiom.

**Theorem 14.** Assume that  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  is an ideal Top –space such that  $\mathfrak{I}$  is a maximal ideal on  $\mathbb{X}$ . Then, for any subset  $H \subseteq \mathbb{X}$ ,

$$H_{\omega}^{\sharp} = \text{Cl}_{\omega}(H) \vee \Psi_{\omega}^{\sharp}(H) = \text{Int}_{\omega}(H).$$

**Proof.** Let  $H \subseteq \mathbb{X}$ . Since  $\mathfrak{I}$  is a maximal ideal on  $\mathbb{X}$ , it follows that

$$H \in \mathfrak{I} \vee \mathbb{X} \setminus H \in \mathfrak{I}.$$

Consequently,

$$H_{\omega}^* = \emptyset \vee (\mathbb{X} \setminus H)_{\omega}^* = \emptyset.$$

From Theorem 7, we have

$$\text{Cl}_{\omega}(H) = H_{\omega}^* \cup H_{\omega}^{\sharp} \quad \forall H \subseteq \mathbb{X}.$$

Hence,

$$(H_{\omega}^* = \emptyset \Rightarrow \text{Cl}_{\omega}(H) = H_{\omega}^{\sharp}) \vee ((\mathbb{X} \setminus H)_{\omega}^* = \emptyset \Rightarrow \text{Cl}_{\omega}(\mathbb{X} \setminus H) = (\mathbb{X} \setminus H)_{\omega}^{\sharp}).$$

Using the identity

$$\text{Cl}_{\omega}(\mathbb{X} \setminus H) = \mathbb{X} \setminus \text{Int}_{\omega}(H),$$

we deduce

$$\Psi_{\omega}^{\sharp}(H) = \mathbb{X} \setminus (\mathbb{X} \setminus H)_{\omega}^{\sharp} = \text{Int}_{\omega}(H).$$

Therefore,

$$\text{Cl}_{\omega}(H) = H_{\omega}^{\sharp} \vee \Psi_{\omega}^{\sharp}(H) = \text{Int}_{\omega}(H),$$

which completes the proof. □

**Corollary 10.** Suppose that  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  is an ideal Top –space such that  $\mathfrak{I}$  is a proper minimal ideal on  $\mathbb{X}$ . Then, for every subset  $H \subseteq \mathbb{X}$ ,

$$H_{\omega}^* = \text{Cl}_{\omega}(H) \vee \Psi_{\omega}^*(H) = \text{Int}_{\omega}(H).$$

**Proof.** Applying Theorems 7 and 13, the statement is established. □

#### 4. Some Applications of $\omega^{\sharp}$ -Operator

**Theorem 15.** [21] Let  $(\mathbb{X}, \mathcal{T})$  be a topological space and let  $H$  be a subset of  $\mathbb{X}$ . Then  $H$  is dense in  $\mathbb{X}$  if and only if the ideal topological space  $(\mathbb{X}, \mathcal{T}, \mathcal{ANN}_H)$  is a Hayashi–Samuel space.

**Remark 7.** Let  $\mathcal{U}$  be the usual topology on  $\mathbb{R}$ , and let the annihilator ideal of the rationals be

$$\mathcal{ANN}(\{\mathbb{Q}\}) = 2^{\mathbb{P}}, \quad \mathbb{P} = \mathbb{R} \setminus \mathbb{Q}.$$

The associated  $\omega$ -topology is

$$\mathcal{U}_\omega = \left\{ W \subseteq \mathbb{R} \mid \forall x \in W, \exists U \in \mathcal{U} \text{ with } x \in U \text{ and } U \setminus W \text{ countable} \right\}.$$

Since  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{U}_\omega \cap \mathcal{ANN}(\{\mathbb{Q}\}) \neq \emptyset$ , we can write symbolically that the space

$(\mathbb{R}, \mathcal{U}_\omega, \mathcal{ANN}(\{\mathbb{Q}\}))$  fails to be a Hayashi–Samuel space as  $\mathcal{U}_\omega \cap \mathcal{ANN}(\{\mathbb{Q}\}) \neq \emptyset$ . Furthermore, applying the  $\text{Cl}_\omega^\#$  operator, we have  $\text{Cl}_\omega^\#(\mathbb{Q}) = \mathbb{Q} \cup \mathbb{Q}_\omega^\# = \mathbb{Q}$ , because  $\mathbb{Q}_\omega^\# \subseteq \mathbb{Q}$ .

**Remark 8.** In [21], Issaka and Özköç established that the set of rational numbers  $\mathbb{Q}$  is  $\mathcal{U}^\#$ -dense in the ideal topological space or dense in  $(\mathbb{R}, \mathcal{U}^\#)$   $(\mathbb{R}, \mathcal{U}, \mathfrak{I}(\mathbb{Q}))$ , where  $\mathcal{U}$  denotes the usual topology on  $\mathbb{R}$ .

**Question 1.** Does there exist a Hausdorff topology on  $\mathbb{R}$  such that  $\mathbb{Q}$  fails to be dense in  $\mathbb{R}$  with respect to the ideal  $\mathfrak{I}(\mathbb{Q})$ ?

**Example 5.** Let  $\mathcal{U}$  is the usual topology on  $\mathbb{R}$ . Let  $\mathfrak{I} = \mathfrak{I}(\mathbb{Q})$ . Since  $\mathbb{Q} \in \mathfrak{I}$ , then by Corollary 4 (1), we obtain  $\mathbb{Q}_\omega^* = \emptyset$  and  $\mathbb{Q}_\omega^\# = \text{Cl}_\omega(\mathbb{Q}) = \mathbb{Q}$  because  $\mathbb{R} \setminus \mathbb{Q}$  is  $\omega$ -open. Thus  $\text{Cl}_\omega^\#(\mathbb{Q}) = \mathbb{Q} \cup \mathbb{Q}_\omega^\# = \mathbb{Q} \cup \mathbb{Q} = \mathbb{Q}$ . Thus  $\mathbb{Q}$  is not  $\mathcal{U}_\omega^\#$ -dense or  $\mathbb{Q}$  is not dense in  $(\mathbb{R}, \mathcal{U}_\omega^\#)$ . Finally, it is obvious that  $(\mathbb{R}, \mathcal{U}_\omega^\#)$  is Hausdorff since  $\mathcal{U} \subseteq \mathcal{U}_\omega^\#$ .

**Theorem 16.** Let  $(\mathbb{X}, \mathcal{T})$  be a topological space. Assume that there is a set  $F \in \mathcal{T}_\omega^c \setminus \{\emptyset, \mathbb{X}\}$ . Then the ideal Top –space  $(\mathbb{X}, \mathcal{T}, \mathfrak{I}(F))$  is  $\mathcal{T}_\omega^\#$ -disconnected.

**Proof.** Let  $F \in \mathcal{T}_\omega^c \setminus \{\emptyset, \mathbb{X}\}$ . Then  $\mathbb{X} \setminus F \in \mathcal{T}_\omega \setminus \{\emptyset, \mathbb{X}\}$ . Since  $\mathcal{T}_\omega \subseteq \mathcal{T}_\omega^\#$ , it follows that  $\mathbb{X} \setminus F \in \mathcal{T}_\omega^\# \setminus \{\emptyset, \mathbb{X}\}$ . Moreover,  $\mathbb{X} \setminus F \in \mathcal{ANN}_F = \mathcal{ANN}(\mathfrak{I}(F))$ , and hence, by Theorem 6 (6),  $(\mathbb{X} \setminus F)_\omega^\# = \emptyset$ . By the definition of  $\text{Cl}_\omega^\#$ ,

$$\begin{aligned} \text{Cl}_\omega^\#(\mathbb{X} \setminus F) &= (\mathbb{X} \setminus F)_\omega^\# \cup (\mathbb{X} \setminus F) = \mathbb{X} \setminus F \\ &\implies \mathbb{X} \setminus F \in (\mathcal{T}_\omega^\#)^c \setminus \{\emptyset, \mathbb{X}\} \\ &\implies F \in \mathcal{T}_\omega^\# \cap (\mathcal{T}_\omega^\#)^c \\ &\implies (\mathbb{X}, \mathcal{T}, \mathfrak{I}(F)) \text{ is } \mathcal{T}_\omega^\# \text{-disconnected.} \end{aligned}$$

□

**Corollary 11.** Let  $(\mathbb{X}, \mathcal{T})$  be a topological space. If  $F \in \mathcal{T}^c \setminus \{\emptyset, \mathbb{X}\}$ , then  $(\mathbb{X}, \mathcal{T}, \mathfrak{I}(F))$  is  $\mathcal{T}_\omega^\#$ -disconnected.

**Example 6.** Consider the real line  $\mathbb{R}$  endowed with its usual topology  $\mathcal{U}$ . Let  $\mathfrak{I} = \mathfrak{I}(F)$  be the ideal generated by the finite set  $F = \{0, 1\}$ . The set  $F$  is  $\omega$ -closed in the topological space  $(\mathbb{R}, \mathcal{U})$ . By Theorem 16, the corresponding ideal topological space  $(\mathbb{R}, \mathcal{U}, \mathfrak{I}(F))$  is  $\mathcal{T}_\omega^\#$ -disconnected.

## 5. Decomposition Results for Continuity

**Definition 7.** A function  $f : (\mathbb{X}, \mathcal{T}, \mathfrak{I}) \rightarrow (\mathbb{Y}, \Gamma)$  is said to be  $\omega$ -continuous [23] (resp.  $\#$ -continuous [21]) if for every  $U \in \Gamma$ ,  $f^{-1}(U) \in \mathcal{T}_\omega$  (resp.  $f^{-1}(U) \in \mathcal{T}^\#$ ).

**Definition 8.** Let  $(\mathbb{X}, \mathcal{T}, \mathfrak{I})$  be an ideal Top –space and  $(\mathbb{Y}, \Gamma)$  a topological space. A function  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is called  $\omega^*$ -continuous (respectively,  $\omega^\#$ -continuous) if and only if

$$f^{-1}(U) \in \mathcal{T}_\omega^* \quad (\text{respectively, } f^{-1}(U) \in \mathcal{T}_\omega^\#) \quad \text{for every } U \in \Gamma.$$

**Corollary 12.** Let  $f : (\mathbb{X}, \mathcal{T}, \mathfrak{I}) \rightarrow (\mathbb{Y}, \Gamma)$  be a function. Then  $f$  is  $\omega^*$ -continuous iff  $f : (\mathbb{X}, \mathcal{T}_\omega^*) \rightarrow (\mathbb{Y}, \Gamma)$  is continuous, and  $f$  is  $\omega^\sharp$ -continuous iff  $f : (\mathbb{X}, \mathcal{T}_\omega^\sharp) \rightarrow (\mathbb{Y}, \Gamma)$  is continuous.

**Corollary 13.** Let  $f : (\mathbb{X}, \mathcal{T}, \mathfrak{I}) \rightarrow (\mathbb{Y}, \Gamma)$  be a function. If  $f$  is continuous, then it is also  $\omega^\sharp$ -continuous and  $\omega^*$ -continuous.

The converse of Corollary 12 is not valid in general, as illustrated by the following example.

**Example 7.** Let  $\mathcal{U}$  be the standard topology on  $\mathbb{R}$ , and let  $\mathfrak{I} = \{J \subseteq \mathbb{R} : J \text{ is countable}\}$  be the ideal of countable sets. Define the co-countable topology

$$\mathcal{T}_c = \{U \subseteq \mathbb{R} : \mathbb{R} \setminus U \text{ is countable}\} \cup \{\emptyset\}.$$

Consider the identity function  $f : (\mathbb{R}, \mathcal{U}, \mathfrak{I}) \rightarrow (\mathbb{R}, \mathcal{T}_c)$ ,  $f(x) = x$ .

For any  $U \in \mathcal{T}_c \setminus \{\emptyset\}$ ,  $f^{-1}(U) = U$ . Since  $\mathbb{R} \setminus U$  is countable. Then for any  $W \in \mathcal{U}_\omega$ ,

$$(\mathbb{R} \setminus U) \cap W \in \mathfrak{I} \Rightarrow (\mathbb{R} \setminus U)_\omega^* = \emptyset,$$

so  $U \in \mathcal{U}_\omega^*$ . Therefore,  $f$  is  $\omega^*$ -continuous. However,  $f$  is not continuous because  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_c$  but  $f^{-1}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \setminus \mathbb{Q} \notin \mathcal{U}$ . Similarly, using the  $\omega^\sharp$ -operator, we have  $\mathbb{R} \setminus (\mathbb{R} \setminus U) \in \mathcal{U}_\omega$  and

$$(\mathbb{R} \setminus U) \cap [\mathbb{R} \setminus (\mathbb{R} \setminus U)] \in \mathcal{ANN}(\mathfrak{I}) \Rightarrow (\mathbb{R} \setminus U)_\omega^\sharp \subseteq \mathbb{R} \setminus U,$$

so  $\text{Cl}_\omega^\sharp(\mathbb{R} \setminus U) = \mathbb{R} \setminus U$ , and  $U \in \mathcal{U}_\omega^\sharp$ . Hence,  $f$  is  $\omega^\sharp$ -continuous as well.

**Theorem 17.** A function  $f : (\mathbb{X}, \mathcal{T}, \mathfrak{I}) \rightarrow (\mathbb{Y}, \Gamma)$  is  $\omega$ -continuous if and only if it is both  $\omega^*$ -continuous and  $\omega^\sharp$ -continuous.

**Proof.** The proof follows directly from Theorem 12.  $\square$

**Corollary 14.** Let  $f : (\mathbb{X}, \mathcal{T}, \mathfrak{I}) \rightarrow (\mathbb{Y}, \Gamma)$  be a function. Then the following hold.

- (1) If  $f$  is  $\omega$ -continuous, it is also  $\omega^\sharp$ -continuous.
- (2) If  $f$  is  $\sharp$ -continuous, it is also  $\omega^\sharp$ -continuous.

The reverse statement of Corollary 14 is not true in general; this is shown by the following example.

**Example 8.** Let  $(\mathbb{R}, \mathcal{U}, \mathfrak{I})$  be an ideal Top  $\omega$ -space, where  $\mathcal{U}$  is the usual topology on  $\mathbb{R}$  and  $\mathfrak{I} = \mathfrak{I}(\mathbb{Q})$ . Define a function  $f : (\mathbb{R}, \mathcal{U}, \mathfrak{I}) \rightarrow (\mathbb{R}, \mathcal{U})$  by

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} \setminus \mathbb{Q}, \\ 0, & x \in \mathbb{Q}. \end{cases}$$

For any  $U \in \mathcal{U}$ ,

$$f^{-1}(U) = \begin{cases} \emptyset, & 0 \notin U, 1 \notin U, \\ \mathbb{Q}, & 0 \in U, 1 \notin U, \\ \mathbb{R} \setminus \mathbb{Q}, & 0 \notin U, 1 \in U, \\ \mathbb{R}, & 0 \in U, 1 \in U. \end{cases}$$

Since  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{ANN}(\mathfrak{I})$ , Theorem 6 (6) yields  $(\mathbb{R} \setminus \mathbb{Q})_\omega^\sharp = \emptyset$ . Hence

$$\mathbb{Q} \subseteq \Psi_\omega^\sharp(\mathbb{Q}) = \mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Q})_\omega^\sharp = \mathbb{R},$$

which implies  $\mathbb{Q} \in \mathcal{U}_\omega^\sharp$  by Theorem 10. Similarly,  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{U}_\omega^\sharp$  by Theorem 10. Indeed, since  $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$  and  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{U}_\omega$ , it follows that  $\mathbb{Q}_\omega^\sharp \subseteq \mathbb{Q}$ . Consequently,  $\Psi_\omega^\sharp(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \setminus \mathbb{Q}$ , and therefore  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{U}_\omega^\sharp$ . Therefore  $f^{-1}(U) \in \mathcal{U}_\omega^\sharp$  for every  $U \in \mathcal{U}$ , and thus  $f$  is  $\omega^\sharp$ -continuous. However,  $f$  is not  $\omega$ -continuous since  $\mathbb{Q} \notin \mathcal{U}_\omega$ .

**Remark 9.** Following Issaka and Özköç [21], for any  $H \in \mathfrak{I}$ , we have  $H^\sharp = \text{Cl}(H)$ . In Example 7, since  $\mathbb{Q} \in \mathfrak{I}$ , it holds that  $\mathbb{Q}^\sharp = \text{Cl}(\mathbb{Q}) = \mathbb{R}$ . Consequently,  $\mathbb{R} \setminus \mathbb{Q} \notin \mathcal{U}^\sharp$ , and thus the function  $f : (\mathbb{R}, \mathcal{U}, \mathfrak{I}) \rightarrow (\mathbb{R}, \mathcal{U})$  is  $\omega^\sharp$ -continuous but not  $\sharp$ -continuous.

**Remark 10.** The example below demonstrates that  $\omega$ -continuity and  $\sharp$ -continuity are independent notions.

**Example 9.** Let  $(\mathbb{R}, \mathcal{U}, \mathfrak{I})$  and  $(\mathbb{R}, \mathcal{U}, \mathcal{K})$  be ideal Top-spaces, where  $\mathcal{U}$  is the usual topology on  $\mathbb{R}$ ,  $\mathfrak{I} = \mathfrak{I}(\mathbb{Q})$ , and  $\mathcal{K} = \{\emptyset, \{0\}\}$ . Consider  $\mathbb{Y} = \{0, 1\}$  equipped with the topology  $\Gamma = \{\emptyset, \mathbb{Y}, \{0\}\}$ .

(1) Define

$$f : (\mathbb{R}, \mathcal{U}, \mathcal{K}) \longrightarrow (\mathbb{Y}, \Gamma), \quad f(x) = \begin{cases} 1, & x \in \mathbb{R} \setminus \mathbb{Q}, \\ 0, & x \in \mathbb{Q}. \end{cases}$$

Note that  $f^{-1}(\{0\}) = \mathbb{Q}$ . Since  $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{ANN}(\mathcal{K})$ , by Theorem 6 we have  $(\mathbb{R} \setminus \mathbb{Q})^\sharp = \emptyset$ , which implies  $\mathbb{Q} \in \mathcal{U}^\sharp$ . Therefore, for each  $U \in \Gamma$ ,  $f^{-1}(U) \in \mathcal{U}^\sharp$ . Hence,  $f$  is  $\sharp$ -continuous but not  $\omega$ -continuous.

(2) Define

$$f : (\mathbb{R}, \mathcal{U}, \mathfrak{I}) \longrightarrow (\mathbb{Y}, \Gamma), \quad f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1, & x \in \mathbb{Q}. \end{cases}$$

In this case,  $f$  is  $\omega$ -continuous but not  $\sharp$ -continuous.

**Remark 11.** The concepts of  $\omega^*$ -continuity and  $\omega^\sharp$ -continuity are independent as illustrated by the following examples. Let  $(\mathbb{R}, \mathcal{T}_{ind})$  be the indiscrete topological space, and consider the ideals  $\mathfrak{I} = \{\emptyset, \{1\}\}$ ,  $\mathcal{K} = \mathcal{K}((0, \infty))$  on  $\mathbb{R}$ . Define the codomain  $\mathbb{Y} = \{0, 1\}$  with topologies  $\Gamma_1 = \{\emptyset, \mathbb{Y}, \{0\}\}$ ,  $\Gamma_2 = \{\emptyset, \mathbb{Y}, \{1\}\}$ . Let

$$f : \mathbb{R} \longrightarrow \mathbb{Y}, \quad f(x) = \begin{cases} 0, & x > 0, \\ 1, & x \leq 0. \end{cases}$$

(1) When  $f$  is considered as  $f : (\mathbb{R}, \mathcal{T}_{ind}, \mathfrak{I}) \longrightarrow (\mathbb{Y}, \Gamma_1)$ , it is  $\omega^\sharp$ -continuous but not  $\omega^*$ -continuous. Observe that  $f^{-1}(\{0\}) = (0, \infty) \in \mathcal{T}_\omega^\sharp$ . Since  $(-\infty, 0] \in \mathcal{ANN}(\mathfrak{I})$ , Corollary 4 (2) implies  $(-\infty, 0]_\omega^\sharp = \emptyset$ ,  $(-\infty, 0]_\omega^* = \text{Cl}_\omega((-\infty, 0]) = \mathbb{R}$ . Therefore,  $(0, \infty) \notin \mathcal{T}_\omega^*$ , so  $f$  is not  $\omega^*$ -continuous.

(2) When  $f$  is considered as  $f : (\mathbb{R}, \mathcal{T}_{ind}, \mathcal{K}) \longrightarrow (\mathbb{Y}, \Gamma_2)$ , it is  $\omega^*$ -continuous but not  $\omega^\sharp$ -continuous. Observe that  $f^{-1}(\{1\}) = (-\infty, 0] \in \mathcal{T}_\omega^*$ . Since  $(0, \infty) \in \mathcal{K}$ , Corollary 4 (1) gives  $(0, \infty)_\omega^* = \emptyset$ ,  $(0, \infty)_\omega^\sharp = \text{Cl}_\omega((0, \infty)) = \mathbb{R}$ . Therefore,  $(-\infty, 0] \notin \mathcal{T}_\omega^\sharp$ , so  $f$  is not  $\omega^\sharp$ -continuous.

## 6. Conclusion

We introduced and analyzed the operator  $(\cdot)_\omega^\sharp$  in ideal topological spaces, showing that the associated closure  $\text{Cl}_\omega^\sharp$  satisfies the Kuratowski axioms and generates a topology  $\mathcal{T}_\omega^\sharp$  strictly finer than the classical  $\sharp$ -topology. Inclusion relations with related topologies were precisely characterized, clarifying the structural hierarchy. We also defined  $\omega^*$ - and  $\omega^\sharp$ -continuity and demonstrated their independence from existing continuity notions, providing new perspectives on operator-induced continuity. These results highlight the refinement and versatility of the  $(\cdot)_\omega^\sharp$ -framework, offering a solid foundation for further exploration in ideal and generalized topological spaces. Future work may investigate applications of these operators in generalized operators, separation axioms, and generalized continuity, as well as other extensions of topological structures.

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