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Article

Hierarchical Set Constructions via Multi-Iterated Powersets and the Signed Iterated Power Multiset

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Abstract

This paper develops a unified, hierarchy-aware framework for set constructions that scale from items to templates, libraries, catalogs, and beyond. We introduce the Signed Power Multiset and the Signed Iterated Power Multiset, defined by coordinatewise factorization. We prove that the signed constructions reduce to classical powersets and power multisets when signs are nonnegative or repetitions disappear, and that multiplicities factor across disjoint supports. For finite bases we obtain size recurrences forming exponential towers and identify cancellation laws triggered by negative multiplicities. We also formalize multi–iterated powersets indexed by block vectors and establish a flattening law showing that only the total height matters. Worked examples from inventory reconciliation and planning illustrate how the framework captures layered selections, recalls, and multi-stage decisions. We also extend our investigation to the concept of *Named Sets*, considering their generalizations such as the *Named Power Set* and the *Named Iterated PowerSet*.

Keywords: n-th powerset; power multiset; multi-iterated powersets; signed iterated power multiset

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1. Preliminaries

This section outlines the key concepts and definitions required for understanding the content of this paper. The n-th iterated powerset constructs sets repeatedly; each step applies the powerset operation, producing higher-order collections of subsets.

Definition 1 (Universe). *Let U be a nonempty finite set, called the* universe *or* base set. *All subsequent powerset constructions are formed relative to U.*

Definition 2 (Powerset [1]). *The* powerset *of a set S, denoted $\mathcal{P}(S)$, is the family of all subsets of S, including both the empty set and S itself:*

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

Definition 3 (n-th iterated Powerset [2–6]). *For a nonempty set H and integer $n \geq 1$, the n*-th powerset *is defined recursively by*

$$\mathcal{P}_1(H) := \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) := \mathcal{P}(\mathcal{P}_n(H)).$$

Analogously, the n-th nonempty powerset, *denoted $\mathcal{P}^*_n(H)$, is constructed by*

$$\mathcal{P}^*_1(H) := \mathcal{P}^*(H), \qquad \mathcal{P}^*_{n+1}(H) := \mathcal{P}^*(\mathcal{P}^*_n(H)),$$

where $\mathcal{P}^*(H) := \mathcal{P}(H) \setminus \{\emptyset\}$.

Example 1 (Trip-planning templates as an n-th iterated powerset (take $n = 3$)). *Consider building reusable packing templates in a travel department.*

Base level (items). *Let the base set be*

$$H = \{Passport, \ Adapter, \ Jacket\} \quad (|H| = 3).$$

Level 1: day packing lists. $\mathcal{P}_1(H) = \mathcal{P}(H)$ *is the set of all* packing lists *(subsets of items). There are* $2^3 = 8$ *possible lists, e.g.*

$$L_{\text{city}} = \{Passport, Adapter\}, \qquad L_{\text{winter}} = \{Passport, Jacket\}, \qquad \varnothing \text{ (no extra items)}.$$

Level 2: trip-type libraries. $\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}(H))$ *is the set of all* libraries of packing lists *for distinct trip types. There are* $2^8 = 256$ *such libraries. A concrete library might be*

$$\Lambda = \{\, L_{\text{city}},\ L_{\text{winter}},\ \varnothing \,\},$$

meaning: for corporate use, keep templates for city trips, winter trips, and an empty baseline.

Level 3: corporate catalogs. $\mathcal{P}_3(H) = \mathcal{P}(\mathcal{P}_2(H))$ *is the set of all* corporate catalogs, *each a set of libraries selected for regions or business units. Its size is* 2^{256}. *A concrete catalog could be*

$$\mathcal{C} = \{\, \Lambda,\ \{\, L_{\text{city}},\ \varnothing \,\}\,\},$$

i.e., the catalog offers the three-template library Λ *and, in addition, a lean library for short city trips.*

Interpretation across levels.

- *Level 0 (H): individual items.*
- *Level 1 ($\mathcal{P}(H)$): packing lists (templates) as subsets of items.*
- *Level 2 ($\mathcal{P}(\mathcal{P}(H))$): libraries (collections of templates) for different trip types.*
- *Level 3 ($\mathcal{P}(\mathcal{P}_2(H))$): catalogs (collections of libraries) deployed across the organization.*

General n. *Each application of* $\mathcal{P}$ *adds one management layer: "collections of previous-level objects." For a finite base* H *with* $|H| = m$, *the sizes follow*

$$|\mathcal{P}_1(H)| = 2^m, \quad |\mathcal{P}_2(H)| = 2^{2^m}, \quad |\mathcal{P}_3(H)| = 2^{2^{2^m}}, \ \ldots, \ |\mathcal{P}_n(H)| = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n \text{ times}}{}^{m}.$$

Thus the n*-th iterated powerset cleanly models real hierarchies: items* $\to$ *templates* $\to$ *libraries* $\to$ *catalogs* $\to \cdots$.

1.1. Power Multiset

A power multiset extends the classical powerset, assigning multiplicities to submultisets, thus capturing all possible multiplicity-aware subsets [7–12].

Definition 4 (Submultiset). *Let* A *be a (finite) multiset on a universe* U *with multiplicity function* $m_A : U \to \mathbb{N}_0$. *A multiset* B *on* U *is called a* submultiset *of* A, *written* $B \subseteq A$, *if*

$$\forall x \in U \quad m_B(x) \le m_A(x).$$

The set of submultisets *of* A *is*

$$\mathcal{P}(A) \ :=\ \{\, B \text{ multiset on } U \mid B \subseteq A \,\}.$$

Note that $\mathcal{P}(A)$ *is an (ordinary) set.*

Definition 5 (Power multiset). *Let* A *be a (finite) multiset on* U. *The* power multiset *of* A, *denoted* $\mathbb{P}(A)$, *is the multiset whose underlying universe is* $\mathcal{P}(A)$ *and whose multiplicity function* $M_A : \mathcal{P}(A) \to \mathbb{N}_0$ *is given by*

$$M_A(B) \ :=\ \prod_{x \in \operatorname{supp}(A)} \binom{m_A(x)}{m_B(x)} \qquad (B \in \mathcal{P}(A)).$$

Equivalently, $\mathbb{P}(A)$ *"lists" every submultiset* $B \subseteq A$ *as many times as there are ways to choose, for each* $x \in \operatorname{supp}(A)$*, exactly* $m_B(x)$ *copies out of the* $m_A(x)$ *copies of* x*.*

Example 2. *Let* $U = \{x, y\}$ *and let* A *be the multiset with* $m_A(x) = 2$*,* $m_A(y) = 1$*. Then the submultisets are determined by pairs* (i, j) *with* $i \in \{0, 1, 2\}$*,* $j \in \{0, 1\}$*, i.e.*

$$\mathcal{P}(A) = \{\ \varnothing = (0,0),\ \{x\} = (1,0),\ \{x,x\} = (2,0),\ \{y\} = (0,1),\ \{x,y\} = (1,1),\ \{x,x,y\} = (2,1)\ \}.$$

Their multiplicities in $\mathbb{P}(A)$ *are*

$$\begin{aligned} M_A(0,0) &= \binom{2}{0}\binom{1}{0} = 1, & M_A(1,0) &= \binom{2}{1}\binom{1}{0} = 2, & M_A(2,0) &= \binom{2}{2}\binom{1}{0} = 1, \\ M_A(0,1) &= \binom{2}{0}\binom{1}{1} = 1, & M_A(1,1) &= \binom{2}{1}\binom{1}{1} = 2, & M_A(2,1) &= \binom{2}{2}\binom{1}{1} = 1. \end{aligned}$$

Thus (writing duplicates explicitly) the power multiset is

$$\mathbb{P}(A) = [\ \varnothing,\ \{x\},\ \{x\},\ \{x,x\},\ \{y\},\ \{x,y\},\ \{x,y\},\ \{x,x,y\}\ ].$$

1.2. Signed Multiset

A signed multiset assigns integer multiplicities, allowing both positive and negative counts, thus generalizing sets and ordinary multisets [13].

Definition 6 (Signed multiset)**.** *[13] Let U be a universe. A* signed multiset *(on U) is a function*

$$m_A : U \longrightarrow \mathbb{Z}$$

of integer-valued *multiplicities. Its* support *is*

$$\operatorname{supp}(A) \ := \ \{\, x \in U : \ m_A(x) \neq 0 \,\}.$$

We write $x \in A$ *iff* $m_A(x) \neq 0$*. When* $\operatorname{supp}(A)$ *is finite, the* (signed) size *is*

$$\#A \ := \ \sum_{x \in U} m_A(x)\,.$$

An ordinary set corresponds to the $0/1$*-valued case; an ordinary (unsigned) multiset corresponds to the* $\mathbb{N}_0$*-valued case.*

Remark 1 (Generalized characteristic function)**.** *The map* m_A *is the (generalized) characteristic function of* A *with range* $\mathbb{Z}$*; ordinary sets have range* $\{0, 1\}$ *and ordinary multisets have range* $\mathbb{N}_0$*.*

Example 3 (Library curation with additions (positive) and withdrawals (negative))**.** *A public library performs a monthly collection update on three titles:*

$$U = \{\mathit{Atlas},\ \mathit{Calculus},\ \mathit{Novel}\}.$$

Model the net action *as a signed multiset* $A : U \to \mathbb{Z}$*:*

$$m_A(\mathit{Atlas}) = +3 \quad (\mathit{acquire\ three}),$$

$$m_A(\mathit{Calculus}) = -2 \quad (\mathit{withdraw\ two}),$$

$$m_A(\mathit{Novel}) = +1 \quad (\mathit{acquire\ one}).$$

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Hence

$$\operatorname{supp}(A) = \{\mathit{Atlas}, \mathit{Calculus}, \mathit{Novel}\}, \qquad \#A = \sum_{x \in U} m_A(x) = 3 - 2 + 1 = 2,$$

so the shelves gain a net of 2 *copies.*

Brace notation. *One may write*

$$A = \{\underbrace{\mathit{Atlas}, \mathit{Atlas}, \mathit{Atlas}}_{+3},\ \underbrace{\mathit{Novel}}_{+1} \mid \underbrace{\mathit{Calculus}, \mathit{Calculus}}_{-2}\},$$

where the bar separates positive and negative multiplicities.

Operational meaning via a weighted sum. *Let* $t : U \to \mathbb{R}_{\geq 0}$ *be the spine thickness (cm) of each title:*

$$t(\mathit{Atlas}) = 5, \quad t(\mathit{Calculus}) = 4, \quad t(\mathit{Novel}) = 2.$$

Then the net shelf-space change is the signed sum

$$\sum_{x \in A} t(x) = \sum_{x \in U} m_A(x)\, t(x) = 3 \cdot 5 \ +\ (-2) \cdot 4 \ +\ 1 \cdot 2 = 15 - 8 + 2 = 9 \text{ cm (increase)}.$$

Thus the signed multiset compactly encodes "add these copies, remove those copies," and any linear resource (space, weight, cost) is computed by a single signed sum against the corresponding per-title function.

1.3. Named Set

Informally, a named set assigns to each element of a *support* a label drawn from a *set of names* by means of a designated map; see, e.g., [14–19]. We record a precise formulation below.

Definition 7 (Named set)**.** *[14–17] Fix an ambient category (typically* Set*) together with a specified class* $\mathcal{M}$ *of admissible morphisms. A* named set *is a triple*

$$\Gamma \ = \ (X,\ a,\ I),$$

consisting of

- *a support object* X *(the* carrier of elements*),*
- *a name object* I *(the* pool of labels*), and*
- *a morphism* $a\colon\ X \to I$ *from the fixed class* $\mathcal{M}$*, called the* naming map.

For $x \in X$*, the value* $a(x) \in I$ *is the* name *(or label) attached to* x*. When the ambient category is* Set *and* $\mathcal{M}$ *is the class of all functions, a named set is simply a function* $a\colon X \to I$*.*

Definition 8 (Support, names, and naming map)**.** *For a named set* $\Gamma = (X, a, I)$*, we use the following notation:*

$$S(\Gamma) := X \quad (\text{support}), \qquad N(\Gamma) := I \quad (\text{set of names}), \qquad n(\Gamma) := a \quad (\text{naming map}).$$

Example 4 (University roster with student ID numbers)**.** *Let the* support *(students enrolled in a course) be*

$$X = \{\mathit{Ayame},\ \mathit{Bob},\ \mathit{Chen}\}.$$

Let the set of names *be the assigned ID codes*

$$I = \{\text{S-0428},\ \text{S-1359},\ \text{S-2718}\}.$$

Define the naming map $a : X \to I$ *by*

$$a(\mathit{Ayame}) = \text{S-0428}, \qquad a(\mathit{Bob}) = \text{S-1359}, \qquad a(\mathit{Chen}) = \text{S-2718}.$$

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Then $\Gamma = (X, a, I)$ *is a* named set *in which each student carries exactly one unique label (the student ID). Here* $S(\Gamma) = X$*,* $N(\Gamma) = I$*, and* $n(\Gamma) = a$*.*

Example 5 (Desktop files labeled by MIME type)**.** *Let the* support *be a finite set of files on a laptop:*

$$X = \{\texttt{paper.pdf},\ \texttt{logo.png},\ \texttt{photo.png},\ \texttt{notes.txt}\}.$$

Let the set of names *be the MIME types*

$$I = \{\texttt{application/pdf},\ \texttt{image/png},\ \texttt{text/plain}\}.$$

Define the naming map $a : X \to I$ *by*

$$a(\texttt{paper.pdf}) = \texttt{application/pdf}, \quad a(\texttt{logo.png}) = \texttt{image/png},$$

$$a(\texttt{photo.png}) = \texttt{image/png}, \quad a(\texttt{notes.txt}) = \texttt{text/plain}.$$

Then $\Gamma = (X, a, I)$ *is a* named set *where each file has exactly one label (its type). Unlike the first example,* a *is* many-to-one *because two different files share the same name* `image/png`*. Again* $S(\Gamma) = X$*,* $N(\Gamma) = I$*, and* $n(\Gamma) = a$*.*

2. Main Results

This section presents the main results of the paper.

2.1. Iterated Power Multiset

An iterated power multiset repeatedly applies the power multiset construction, producing hierarchical layers of submultisets with multiplicities tracking combinatorial realizations.

Definition 9 (Iterated power multiset $\mathbb{P}^n(A)$)**.** *Let* A *be a finite multiset (base level). Define recursively*

$$\mathbb{P}^0(A) \ := \ A, \qquad \mathbb{P}^{n+1}(A) \ := \ \mathbb{P}\big(\mathbb{P}^n(A)\big) \quad (n \geq 0).$$

Thus, the underlying set of $\mathbb{P}^{n+1}(A)$ *is* $\mathcal{P}\big(\mathbb{P}^n(A)\big)$*, the set of all submultisets of* $\mathbb{P}^n(A)$*. Writing* $M_A^{(n)}$ *for the multiplicity function of* $\mathbb{P}^n(A)$ *and* $S_A^{(n)} := \operatorname{supp}(\mathbb{P}^n(A))$ *for its support, we have explicitly*

$$M_A^{(n+1)}(Y) \ = \ \prod_{Z \in S_A^{(n)}} \binom{M_A^{(n)}(Z)}{m_Y(Z)} \qquad (Y \in \mathcal{P}(\mathbb{P}^n(A))),$$

where $m_Y : S_A^{(n)} \to \mathbb{N}_0$ *is the multiplicity function of the submultiset* $Y \subseteq \mathbb{P}^n(A)$*.*

Example 6 (Real-life scenario: assembling daily gift bags from limited stock)**.** *Consider a small shop that prepares daily* gift bags *from a limited inventory. Let the base multiset (level 0 stock) be*

$$A = \{Chocolate, Cookie, Juice\} \quad with \quad m_A(Chocolate) = 2, \quad m_A(Cookie) = 2, \quad m_A(Juice) = 1.$$

Here m_A *counts physically indistinguishable copies in stock (two chocolates, two cookies, one juice).*

By Definition 9, the level-1 objects $Y \in \mathbb{P}^1(A) = \mathbb{P}(A)$ *represent* possible gift-bag contents *(submultisets of the stock); their multiplicity* $M_A^{(1)}(Y)$ *counts how many distinct ways one could pick physical copies to realize* Y*:*

$$M_A^{(1)}(Y) \ = \ \prod_{x \in \{Chocolate,\, Cookie,\, Juice\}} \binom{m_A(x)}{m_Y(x)}.$$

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Two concrete daily bags:

$$Y_1: \quad m_{Y_1}(\mathit{Chocolate}) = 1,\ m_{Y_1}(\mathit{Cookie}) = 1,\ m_{Y_1}(\mathit{Juice}) = 0 \implies M_A^{(1)}(Y_1) = \binom{2}{1}\binom{2}{1}\binom{1}{0} = 4,$$

$$Y_2: \quad m_{Y_2}(\mathit{Chocolate}) = 1,\ m_{Y_2}(\mathit{Cookie}) = 0,\ m_{Y_2}(\mathit{Juice}) = 1 \implies M_A^{(1)}(Y_2) = \binom{2}{1}\binom{2}{0}\binom{1}{1} = 2.$$

Interpretation: Y_1 *can be assembled in* 4 *ways (choose which of the two chocolates and which of the two cookies);* Y_2 *can be assembled in* 2 *ways (choose which chocolate; the juice is unique).*

At level 2, an element $Z \in \mathbb{P}^2(A) = \mathbb{P}(\mathbb{P}(A))$ *is a* plan of daily bags*: it is a submultiset of level-1 bags. Its multiplicity uses the level-1 multiplicities as the new "copy counts":*

$$M_A^{(2)}(Z) \;=\; \prod_{W \in S_A^{(1)}} \binom{M_A^{(1)}(W)}{m_Z(W)}.$$

For a concrete two-day plan that prepares one Y_1*-bag and one* Y_2*-bag, set*

$$m_Z(Y_1) = 1, \qquad m_Z(Y_2) = 1, \qquad m_Z(W) = 0 \text{ for all other } W.$$

Then

$$M_A^{(2)}(Z) \;=\; \binom{M_A^{(1)}(Y_1)}{1}\binom{M_A^{(1)}(Y_2)}{1} \;=\; \binom{4}{1}\binom{2}{1} \;=\; 4 \times 2 \;=\; 8.$$

Meaning: there are 8 *distinct ways to realize the two-day plan when one distinguishes the concrete physical picks that instantiate the day-1 and day-2 bags.*

A second plan that prepares two Y_1*-bags (and no other type) has*

$$m_{Z'}(Y_1) = 2, \qquad m_{Z'}(W) = 0 \text{ for } W \neq Y_1, \quad \implies \quad M_A^{(2)}(Z') = \binom{4}{2} = 6.$$

Thus, even with tiny stocks, the iterated *power-multiset naturally models "*objects*" (level 1: daily bags from stock) and then "*collections of those objects*" (level 2: multi-day plans from daily-bag types), with multiplicities recording the number of physically distinguishable realizations at each stage.*

Proposition 1 (Size of one power step)**.** *For any finite multiset* X*,*

$$\left|\mathbb{P}(X)\right| \;=\; \sum_{Y \in \mathcal{P}(X)} M_X(Y) \;=\; 2^{C(X)},$$

where $C(X) := \sum_{u \in U} m_X(u)$ *is the total (copy-counting) cardinality of* X*.*

Proof. By definition and Fubini-type factorization,

$$\sum_{Y \in \mathcal{P}(X)} M_X(Y) = \sum_{\{\, m_Y(u) : 0 \le m_Y(u) \le m_X(u) \}} \; \prod_{u \in \operatorname{supp}(X)} \binom{m_X(u)}{m_Y(u)}$$

$$= \prod_{u \in \operatorname{supp}(X)} \sum_{k=0}^{m_X(u)} \binom{m_X(u)}{k} = \prod_{u \in \operatorname{supp}(X)} 2^{m_X(u)} = 2^{\sum_u m_X(u)}.$$

□

Theorem 1 (Cardinality tower for the iterated power multiset)**.** *Let A be a finite multiset and set $C_0 := C(A) = \sum_u m_A(u)$. Define $C_{n+1} := 2^{C_n}$ for $n \geq 0$. Then, for every $n \geq 1$,*

$$\big|\mathbb{P}^n(A)\big| \;=\; C_n \;=\; \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n\ \text{times}}{}^{C_0}.$$

Proof. Induction on n. For $n = 1$, $|\mathbb{P}^1(A)| = |\mathbb{P}(A)| = 2^{C(A)} = C_1$ by Proposition 1. Assume $|\mathbb{P}^n(A)| = C_n$. Then

$$\big|\mathbb{P}^{n+1}(A)\big| \;=\; \big|\mathbb{P}(\,\mathbb{P}^n(A)\,)\big| \;=\; 2^{C(\,\mathbb{P}^n(A)\,)} \;=\; 2^{|\mathbb{P}^n(A)|} \;=\; 2^{C_n} \;=\; C_{n+1},$$

using Proposition 1 with $X = \mathbb{P}^n(A)$ and the identity $C(X) = |X|$ for multisets ($|X|$ sums multiplicities). $\square$

Theorem 2 (Unifying generalization of iterated powerset and power multiset)**.** *Let A be a finite multiset on U.*

(a) *(Reduction to power multiset) $\mathbb{P}^1(A) = \mathbb{P}(A)$ by Definition 9.*

(b) *(Reduction to iterated powerset on sets) If A is an ordinary set (i.e. $m_A(u) \in \{0,1\}$ for all u), then for all $n \geq 1$:*
  - *the support $S_A^{(n)} = \operatorname{supp}\big(\mathbb{P}^n(A)\big)$ is canonically equal to $\mathcal{P}^n(A)$;*
  - *every element of $\mathbb{P}^n(A)$ has multiplicity 1.*

Proof. (a) is immediate from the recursive definition.

(b) We prove by induction on n. For $n = 1$: since $m_A(u) \in \{0,1\}$, a submultiset $Y \subseteq A$ is the same data as a subset $Y \subseteq U$ with $Y \subseteq A$. Thus $\mathcal{P}(A) = \mathcal{P}(U \cap A) = \mathcal{P}(A)$ is the usual powerset of A. Moreover,

$$M_A^{(1)}(Y) \;=\; \prod_{u \in \operatorname{supp}(A)} \binom{m_A(u)}{m_Y(u)} \;=\; \prod_{u \in A} \binom{1}{\mathbf{1}_{u \in Y}} \;=\; 1,$$

because $\binom{1}{0} = \binom{1}{1} = 1$. Hence $\operatorname{supp}\big(\mathbb{P}^1(A)\big) = \mathcal{P}(A)$ and all multiplicities are 1.

Assume the claim holds for some $n \geq 1$. Then $\mathbb{P}^n(A)$ has all multiplicities equal to 1 and its support is $\mathcal{P}^n(A)$. A submultiset $Y \subseteq \mathbb{P}^n(A)$ is therefore nothing but an ordinary subset of $\mathcal{P}^n(A)$ (because each element of the base can be chosen at most once), and so

$$\mathcal{P}\big(\mathbb{P}^n(A)\big) \;=\; \mathcal{P}\big(\mathcal{P}^n(A)\big) \;=\; \mathcal{P}^{n+1}(A).$$

For the multiplicities, using Definition 9,

$$M_A^{(n+1)}(Y) \;=\; \prod_{Z \in S_A^{(n)}} \binom{M_A^{(n)}(Z)}{m_Y(Z)} \;=\; \prod_{Z \in \mathcal{P}^n(A)} \binom{1}{\mathbf{1}_{Z \in Y}} \;=\; 1.$$

Thus $\operatorname{supp}\big(\mathbb{P}^{n+1}(A)\big) = \mathcal{P}^{n+1}(A)$ and all multiplicities are 1. By induction, the statement holds for all n. $\square$

2.2. Signed Power Multiset

A Signed Power Multiset extends classical power multisets by allowing integer multiplicities, combining positive selections and negative recalls within submultisets.

Definition 10 (Signed submultiset set $\mathcal{P}_{\pm}(A)$)**.** *For A a signed multiset on U, write*

$$p_A(x) := \max\{m_A(x), 0\}, \qquad q_A(x) := \max\{-m_A(x), 0\},$$

so that $m_A = p_A - q_A$ *with* $p_A, q_A : U \to \mathbb{N}_0$ *and* $\operatorname{supp}(A) = \operatorname{supp}(p_A) \cup \operatorname{supp}(q_A)$. *The* set of signed submultisets of A *is*

$$\mathcal{P}_{\pm}(A) \;:=\; \Big\{ Y : U \to \mathbb{Z} \;\Big|\; Y \text{ has finite support and } -q_A(x) \;\le\; m_Y(x) \;\le\; p_A(x) \text{ for all } x \in U \Big\}.$$

(Equivalently, Y chooses up to $p_A(x)$ "positive copies" and up to $q_A(x)$ "negative copies" of each x.)

Definition 11 (Signed Power Multiset $\mathbb{P}_{\pm}(A)$)**.** *Let A be a signed multiset. The* Signed Power Multiset *of A is the signed multiset on the universe $\mathcal{P}_{\pm}(A)$ whose multiplicity function $M_A : \mathcal{P}_{\pm}(A) \to \mathbb{Z}$ is given pointwise, independently across $x \in U$, by the coefficient-extraction identity*

$$M_A(Y) \;:=\; \prod_{x \in \operatorname{supp}(A)} \Big[t^{\,m_Y(x)}\Big] \Big((1+t)^{\,p_A(x)} \, (1 - t^{-1})^{\,q_A(x)} \Big), \qquad Y \in \mathcal{P}_{\pm}(A).$$

Equivalently, for each x and each integer $r \in [-q_A(x),\, p_A(x)]$ the local *signed multiplicity is*

$$C_A(x;r) \;:=\; \sum_{\substack{0 \le i \le p_A(x) \\ 0 \le j \le q_A(x) \\ i - j = r}} (-1)^j \binom{p_A(x)}{i} \binom{q_A(x)}{j} \;=\; [t^r]\Big((1+t)^{\,p_A(x)} (1 - t^{-1})^{\,q_A(x)} \Big),$$

and $M_A(Y) = \prod_x C_A(x; m_Y(x))$.

Remark 2 (Well-definedness and support)**.** *Only finitely many x contribute nontrivially, since A (hence $\mathcal{P}_{\pm}(A)$) has finite support. Thus the product in $M_A(Y)$ and all sums/coefficient extractions are finite.*

Example 7 (Inventory reconciliation with additions (positive) and recalls (negative))**.** *A warehouse performs a one–shot reconciliation on three SKUs:*

$$A = \{Widget, Cable, Gadget\} \quad with \quad m_A(Widget) = 2, \;\; m_A(Cable) = 1, \;\; m_A(Gadget) = -1.$$

Here positive multiplicities are on–hand copies to allocate (ship/pack), while the negative multiplicity -1 for Gadget encodes a mandatory recall/removal of one copy.

Decompose $m_A = p_A - q_A$ by coordinates:

$$p_A(Widget) = 2,\; q_A(Widget) = 0; \quad p_A(Cable) = 1,\; q_A(Cable) = 0; \quad p_A(Gadget) = 0,\; q_A(Gadget) = 1.$$

A signed submultiset $Y \in \mathcal{P}_{\pm}(A)$ prescribes how many items to allocate (positive) or to recall (negative) at this step:

$$m_Y(Widget) \in \{0,1,2\}, \quad m_Y(Cable) \in \{0,1\}, \quad m_Y(Gadget) \in \{-1,0\}.$$

By Definition 11, the signed multiplicity factors coordinatewise via

$$C_A(x;r) = [t^r]\Big((1+t)^{p_A(x)}(1-t^{-1})^{q_A(x)}\Big), \qquad M_A(Y) = \prod_{x \in \{W,C,G\}} C_A(x; m_Y(x)),$$

and here the local coefficients are

$$\begin{aligned}
&\text{Widget:} \quad C_A(W;0) = \binom{2}{0} = 1, \quad C_A(W;1) = \binom{2}{1} = 2, \quad C_A(W;2) = \binom{2}{2} = 1,\\
&\text{Cable:} \qquad\qquad\qquad\qquad\quad C_A(C;0) = \binom{1}{0} = 1, \quad C_A(C;1) = \binom{1}{1} = 1,\\
&\text{Gadget:} \qquad C_A(G;0) = [t^0](1 - t^{-1}) = 1, \quad C_A(G;-1) = [t^{-1}](1 - t^{-1}) = -1.
\end{aligned}$$

Concrete signed plans Y and their multiplicities:

$$\begin{aligned}
&\text{(a) Allocate one Widget; recall one Gadget.} && m_Y(\mathrm{W}) = 1, \quad m_Y(\mathrm{C}) = 0, \quad m_Y(\mathrm{G}) = -1,\\
& && M_A(Y) = 2 \cdot 1 \cdot (-1) = -2.\\
&\text{(b) Allocate two Widgets and one Cable; no Gadget recall.} && m_Y(\mathrm{W}) = 2, \quad m_Y(\mathrm{C}) = 1, \quad m_Y(\mathrm{G}) = 0,\\
& && M_A(Y) = 1 \cdot 1 \cdot 1 = 1.\\
&\text{(c) Allocate one Cable; recall one Gadget.} && m_Y(\mathrm{W}) = 0, \quad m_Y(\mathrm{C}) = 1, \quad m_Y(\mathrm{G}) = -1,\\
& && M_A(Y) = 1 \cdot 1 \cdot (-1) = -1.\\
&\text{(d) Allocate one Widget and one Cable; no Gadget recall.} && m_Y(\mathrm{W}) = 1, \quad m_Y(\mathrm{C}) = 1, \quad m_Y(\mathrm{G}) = 0,\\
& && M_A(Y) = 2 \cdot 1 \cdot 1 = 2.
\end{aligned}$$

Theorem 3 (Reduction to the classical Power Multiset)**.** *If A is* unsigned *(i.e. $q_A \equiv 0$), then $\mathcal{P}_\pm(A)$ is the set of submultisets $\mathcal{P}(A)$ and*

$$\mathbb{P}_\pm(A) \;=\; \mathbb{P}(A)$$

(the classical power multiset). In particular, for $Y \subseteq A$ (as a multiset),

$$M_A(Y) \;=\; \prod_{x \in \mathrm{supp}(A)} \binom{m_A(x)}{m_Y(x)}.$$

Proof. If $q_A(x) = 0$ for all x, then $m_Y(x) \in [0,\, p_A(x)] = [0,\, m_A(x)]$ and

$$C_A(x;r) = [t^r](1+t)^{m_A(x)} = \binom{m_A(x)}{r}.$$

Thus $M_A(Y) = \prod_x \binom{m_A(x)}{m_Y(x)}$, which is exactly the classical power-multiset multiplicity. □

Theorem 4 (Purely negative case and sign twist)**.** *If A is* purely negative, *i.e. $p_A \equiv 0$ and $A = -B$ with B an unsigned multiset, then the allowed local exponents are $r \in [-m_B(x), 0]$ and*

$$C_A(x;r) \;=\; [t^r](1-t^{-1})^{m_B(x)} \;=\; [t^{r+m_B(x)}](t-1)^{m_B(x)} \;=\; (-1)^{-r}\binom{m_B(x)}{-r},$$

hence for $Y \in \mathcal{P}_\pm(A)$,

$$M_A(Y) \;=\; (-1)^{\sum_x -m_Y(x)} \prod_{x \in \mathrm{supp}(B)} \binom{m_B(x)}{-m_Y(x)}.$$

In words, $\mathbb{P}_\pm(-B)$ is the classical power multiset of B pulled back along $r \mapsto -r$ and multiplied by the global sign $(-1)^{\sum_x -m_Y(x)}$.

Proof. For $p_A \equiv 0$, we have $(1+t)^{p_A(x)} \equiv 1$ and $(1-t^{-1})^{m_B(x)} = t^{-m_B(x)}(t-1)^{m_B(x)}$. Expanding $(t-1)^{m_B(x)} = \sum_{k=0}^{m_B(x)} \binom{m_B(x)}{k}(-1)^{m_B(x)-k}t^k$ and extracting the coefficient of $t^{r+m_B(x)}$ yields

$$C_A(x;r) = (-1)^{m_B(x)-(r+m_B(x))}\binom{m_B(x)}{r+m_B(x)} = (-1)^{-r}\binom{m_B(x)}{-r}.$$

Multiplying over x gives the stated formula. □

Theorem 5 (Multiplicativity on disjoint supports)**.** *Suppose A, B are signed multisets with* $\mathrm{supp}(A) \cap \mathrm{supp}(B) = \varnothing$*. Then there is a canonical bijection*

$$\mathcal{P}_\pm(A \oplus B) \;\cong\; \mathcal{P}_\pm(A) \times \mathcal{P}_\pm(B), \qquad Y \longleftrightarrow (Y|_{\mathrm{supp}(A)},\, Y|_{\mathrm{supp}(B)}),$$

under which

$$\mathbb{P}_\pm(A \oplus B) \;\cong\; \mathbb{P}_\pm(A)\,\widehat{\otimes}\,\mathbb{P}_\pm(B),$$

i.e. multiplicities factor: $M_{A\oplus B}(Y) = M_A(Y|_{\mathrm{supp}(A)}) \cdot M_B(Y|_{\mathrm{supp}(B)})$.

Proof. The bounds $-q_{A\oplus B} \le m_Y \le p_{A\oplus B}$ split coordinatewise on the disjoint union $\mathrm{supp}(A)\dot\cup\,\mathrm{supp}(B)$, so the stated bijection of universes holds. By Definition 11, the local factors $C_{A\oplus B}(x;\cdot)$ depend only on A (for $x \in \mathrm{supp}(A)$) or only on B (for $x \in \mathrm{supp}(B)$), hence $M_{A\oplus B}$ is a product of the two independent contributions. ☐

2.3. Signed Iterated Power Multiset

A Signed Iterated Power Multiset repeatedly applies signed power multiset construction, layering positive and negative multiplicities across hierarchical collection stages.

Definition 12 (Signed Iterated Power Multiset)**.** *Let* $A^{(0)} := A$*. For* $n \ge 0$ *define recursively*

$$A^{(n+1)} \;:=\; \mathbb{P}_\pm\big(A^{(n)}\big), \qquad \mathbb{P}^n_\pm(A) \;:=\; A^{(n)}.$$

Thus $\mathbb{P}^n_\pm(A)$ *is a signed multiset whose universe is the set of signed submultisets of* $\mathbb{P}^{n-1}_\pm(A)$*, and whose multiplicities are given by the Definition applied at level* $n-1$*. We write* $M^{(n)}_A$ *for the multiplicity function of* $\mathbb{P}^n_\pm(A)$ *and* $p^{(n)}_A, q^{(n)}_A$ *for its positive/negative parts.*

Example 8 (Two-stage operations plan with shipments (positive) and recalls (negative))**.** *A small logistics team has a one-shot stock-and-recall situation on three SKUs:*

$$A = \{Widget, Cable, Gadget\}, \qquad m_A(W) = 2,\;\; m_A(C) = 1,\;\; m_A(G) = -1.$$

Positive multiplicities encode available units to ship; the negative multiplicity for G encodes a mandatory recall of one unit.

Level 1 (signed daily action). *A level-1 element* $Y \in \mathbb{P}^1_\pm(A) = \mathbb{P}_\pm(A)$ *(Definition 12) chooses, for each SKU* x*, a signed amount* $m_Y(x)$ *with*

$$m_Y(W) \in \{0,1,2\}, \quad m_Y(C) \in \{0,1\}, \quad m_Y(G) \in \{-1,0\}.$$

Its signed multiplicity factors coordinatewise by Definition 11:

$$M^{(1)}_A(Y) = \prod_{x\in\{W,C,G\}} \Big[t^{m_Y(x)}\Big]\Big((1+t)^{p_A(x)}(1-t^{-1})^{q_A(x)}\Big),$$

where $p_A(W,C,G) = (2,1,0)$ *and* $q_A(W,C,G) = (0,0,1)$*. Thus the local coefficients are*

$$\begin{aligned}
&W\!: (1+t)^2 = 1+2t+t^2 \Rightarrow C_A(W;0) = 1,\; C_A(W;1) = 2,\; C_A(W;2) = 1;\\
&C\!: (1+t)^1 = 1+t \Rightarrow C_A(C;0) = 1,\; C_A(C;1) = 1;\\
&G\!: (1-t^{-1}) = 1-t^{-1} \Rightarrow C_A(G;0) = 1,\; C_A(G;-1) = -1.
\end{aligned}$$

Two concrete level-1 actions:

$$\begin{aligned}
&Y_+ : m_{Y_+}(W,C,G) = (1,1,0) &&\Rightarrow\quad M^{(1)}_A(Y_+) = 2\cdot 1\cdot 1 = +2 \quad (\textit{ship one } W \textit{ and one } C);\\
&Y_- : m_{Y_-}(W,C,G) = (1,0,-1) &&\Rightarrow\quad M^{(1)}_A(Y_-) = 2\cdot 1\cdot(-1) = -2 \quad (\textit{ship one } W \textit{ and recall one } G).
\end{aligned}$$

Level 2 (signed weekly plan). *A level-2 object $Z \in \mathbb{P}^2_\pm(A) = \mathbb{P}_\pm(\mathbb{P}_\pm(A))$ is a* signed *submultiset of level-1 actions. Bounds at level 2 use the positive/negative parts of level-1 multiplicities:*

$$p_A^{(1)}(Y) := \max\{M_A^{(1)}(Y), 0\}, \qquad q_A^{(1)}(Y) := \max\{-M_A^{(1)}(Y), 0\}.$$

From above, $p_A^{(1)}(Y_+) = 2$, $q_A^{(1)}(Y_+) = 0$ and $p_A^{(1)}(Y_-) = 0$, $q_A^{(1)}(Y_-) = 2$. Hence the admissible level-2 choices are

$$m_Z(Y_+) \in \{0, 1, 2\}, \qquad m_Z(Y_-) \in \{-2, -1, 0\}.$$

Plan A (one shipping day Y_+ and one audit day cancelling a recall Y_-): *take $m_Z(Y_+) = 1$ and $m_Z(Y_-) = -1$, with all other level-1 types unused. By Definition 11 applied at level 2,*

$$\begin{aligned} M_A^{(2)}(Z) &= \left[t^1\right](1+t)^{p_A^{(1)}(Y_+)}(1-t^{-1})^{q_A^{(1)}(Y_+)} \times \left[t^{-1}\right](1+t)^{p_A^{(1)}(Y_-)}(1-t^{-1})^{q_A^{(1)}(Y_-)} \\ &= \left[t^1\right](1+t)^2 \times \left[t^{-1}\right](1-t^{-1})^2 \;=\; \binom{2}{1} \times (-2) \;=\; -4. \end{aligned}$$

Interpretation: there are 4 concrete ways to realize this weekly plan when one keeps track of which *physical copies instantiate each daily action, but the plan carries a negative sign because it uses a "recall-type" action at level 1 (inclusion–exclusion weight).*

Plan B (two shipping days of type Y_+, no audit): *take $m_Z(Y_+) = 2$, $m_Z(Y_-) = 0$. Then*

$$M_A^{(2)}(Z) = \left[t^2\right](1+t)^2 \times \left[t^0\right](1-t^{-1})^2 = \binom{2}{2} \times 1 = +1.$$

This counts the unique way to pick two daily actions out of the two indistinguishable copies of Y_+.

Takeaway. *The Signed* Iterated *Power Multiset models two-layer decision making:*

- *Level 1 chooses signed daily actions from a mixed stock/recall base; signs arise from recall coordinates.*
- *Level 2 chooses signed weekly plans from those daily actions; feasibility bounds are inherited from level-1 multiplicities, and new signs arise if the plan uses negatively weighted (recall-type) daily actions.*

All coefficients are computed explicitly from the generating factors $(1+t)^p$ and $(1-t^{-1})^q$ at each level, with finite products because the supports are finite.

Theorem 6 (Level-1 reduction)**.** *For every signed multiset A,*

$$\mathbb{P}^1_\pm(A) \;=\; \mathbb{P}_\pm(A).$$

Proof. By Definition 12, $A^{(1)} := \mathbb{P}_\pm(A)$. □

Theorem 7 (Unsigned case: agreement with the iterated (unsigned) power multiset)**.** *If A is* unsigned *(i.e. $q_A \equiv 0$), then for all $n \geq 1$,*

$$\mathbb{P}^n_\pm(A) \;=\; \mathbb{P}^n(A),$$

where $\mathbb{P}$ denotes the classical power-multiset operator on (unsigned) multisets.

Proof. We argue by induction on n. For $n = 1$, the Definition yields, when $q_A \equiv 0$,

$$M_A^{(1)}(Y) = \prod_{x \in \mathrm{supp}(A)} \left[t^{m_Y(x)}\right](1+t)^{m_A(x)} = \prod_{x \in \mathrm{supp}(A)} \binom{m_A(x)}{m_Y(x)},$$

which is the classical power-multiset multiplicity.

Assume the claim holds at level n. Then $\mathbb{P}^n_\pm(A) = \mathbb{P}^n(A)$ is unsigned (all multiplicities are nonnegative integers), so $q_A^{(n)} \equiv 0$. Applying the Definition at level n reduces again to the classical formula, hence $\mathbb{P}^{n+1}_\pm(A) = \mathbb{P}^{n+1}(A)$. □

Corollary 1 (Set case: agreement with the iterated powerset). *If A is an* ordinary set *(i.e. $m_A \in \{0,1\}$), then for all $n \geq 1$,*

$$\operatorname{supp}\big(\mathbb{P}_{\pm}^{n}(A)\big) \;=\; \mathcal{P}^{n}(A) \quad \textit{and} \quad M_A^{(n)} \equiv 1.$$

Proof. When A is a set, at $n = 1$ the only local coefficients are $[t^0](1+t) = 1$ and $[t^1](1+t) = 1$, so all multiplicities are 1 and the support equals $\mathcal{P}(A)$. Inductively, every level remains a $0/1$-valued multiset on its support, hence the same reasoning applies and yields multiplicity 1 and support $\mathcal{P}^n(A)$. ☐

Lemma 1 (Sum of local coefficients). *For $p, q \in \mathbb{N}_0$,*

$$\sum_{r=-q}^{p} \Big[t^r\Big]\Big((1+t)^p(1-t^{-1})^q\Big) \;=\; \big((1+1)^p(1-1)^q\big) \;=\; \begin{cases} 2^p, & q = 0, \\ 0, & q > 0. \end{cases}$$

Proof. The left-hand side is the sum of the coefficients of the finite Laurent polynomial $F_{p,q}(t) := (1+t)^p(1-t^{-1})^q$, which equals $F_{p,q}(1)$. Evaluating at $t = 1$ gives $(2)^p(0)^q$, i.e. 2^p if $q = 0$ and 0 otherwise. ☐

Theorem 8 (Total signed size of one step). *For any signed multiset X,*

$$\#\mathbb{P}_{\pm}(X) \;=\; \prod_{z\in \operatorname{supp}(X)} \sum_{r=-q_X(z)}^{p_X(z)} \Big[t^r\Big]\Big((1+t)^{p_X(z)}(1-t^{-1})^{q_X(z)}\Big) \;=\; \begin{cases} 2^{\sum_z p_X(z)}, & q_X \equiv 0, \\ 0, & \textit{otherwise}. \end{cases}$$

In particular, if X is unsigned then $\#\mathbb{P}_{\pm}(X) = 2^{\#X}$, while if X has any negative multiplicity then $\#\mathbb{P}_{\pm}(X) = 0$.

Proof. By the Definition,

$$\#\mathbb{P}_{\pm}(X) = \sum_{Y\in \mathcal{P}_{\pm}(X)} M_X(Y) = \prod_{z\in \operatorname{supp}(X)} \sum_{r=-q_X(z)}^{p_X(z)} \Big[t^r\Big]\Big((1+t)^{p_X(z)}(1-t^{-1})^{q_X(z)}\Big),$$

where we exchanged sum and product using the independence of choices across coordinates. Apply Lemma 1 to each factor. When $q_X \equiv 0$, we have $\sum_z p_X(z) = \sum_z m_X(z) = \#X$. ☐

Theorem 9 (Total signed size along the iteration). *Let A be a signed multiset and set $T_0 := \#A$ and $T_{n+1} := 2^{T_n}$.*

- *If A is unsigned, then for all $n \geq 1$, $\#\mathbb{P}_{\pm}^{n}(A) = T_n$.*
- *If A has a negative entry (i.e. $q_A \not\equiv 0$), then for all $n \geq 1$, $\#\mathbb{P}_{\pm}^{n}(A) = 0$.*

Proof. If A is unsigned, Theorem 7 reduces the iteration to the classical (unsigned) one, and Theorem 8 yields $\#\mathbb{P}^1(A) = 2^{\#A} = T_1$; induction using the same theorem at each level gives $\#\mathbb{P}^n(A) = T_n$.

If A has $q_A \not\equiv 0$, apply Theorem 8 at $X = A$ to get $\#\mathbb{P}_{\pm}(A) = 0$. For every $n \geq 2$, apply the same theorem to $X = \mathbb{P}_{\pm}^{n-1}(A)$; regardless of the internal sign pattern, the product representation again includes at least one factor with $q > 0$ (indeed, already at $n = 1$ there exist negative coefficients), hence each level has total signed size 0. ☐

Theorem 10 (Multiplicativity over disjoint supports, all depths). *If A, B are signed multisets with* $\operatorname{supp}(A) \cap \operatorname{supp}(B) = \varnothing$, *then for every $n \geq 1$ there is a canonical bijection*

$$\mathcal{P}_{\pm}\big(\mathbb{P}_{\pm}^{n-1}(A \oplus B)\big) \;\cong\; \mathcal{P}_{\pm}\big(\mathbb{P}_{\pm}^{n-1}(A)\big) \;\times\; \mathcal{P}_{\pm}\big(\mathbb{P}_{\pm}^{n-1}(B)\big),$$

under which

$$\mathbb{P}^n_{\pm}(A \oplus B) \;\cong\; \mathbb{P}^n_{\pm}(A) \,\widehat{\otimes}\, \mathbb{P}^n_{\pm}(B),$$

i.e. multiplicities factor coordinatewise.

Proof. For $n = 1$ this is the multiplicativity of the Definition (local factors depend on disjoint coordinates). Assuming the claim at depth $n - 1$, the universe at depth n is the signed-submultiset set of the depth-$(n - 1)$ base, which splits on the disjoint union; multiplicities at depth n are given by a product of local coefficient-extraction terms, which again separate across the two blocks. Hence the factorization (and the bijection of universes) persist for all n. ☐

2.4. Multi–Iterated Powerset

A multi–iterated powerset repeatedly applies the powerset operator in blocks, producing layered collections of subsets, generalizing the n-th powerset.

Definition 13 (Multi–iterated powerset $\mathcal{P}^{\boldsymbol{a}}$ with a block vector)**.** *Let* $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)$ *be a finite vector with entries* $a_i \in \mathbb{N}_0$*. The* multi–iterated powerset *of* X *with block exponents* $\boldsymbol{a}$ *is*

$$\mathcal{P}^{\boldsymbol{a}}(X) \;:=\; \underbrace{\mathcal{P}^{a_k}\Big(\mathcal{P}^{a_{k-1}}\big(\cdots\mathcal{P}^{a_2}(\mathcal{P}^{a_1}(X))\cdots\big)\Big)}_{\text{apply blocks from left to right}}.$$

We also write $|\boldsymbol{a}| := \sum_{i=1}^{k} a_i$ *for the* total height.

Example 9 (Weekly meal planning as a two-block multi–iteration $\mathcal{P}^{(1,1)}$)**.** *Let the base set be the available* dishes

$$X = \{Pasta,\ Curry,\ Salad\} \quad (|X| = 3).$$

Applying one powerset produces all admissible day menus*:*

$$\mathcal{P}^1(X) = \mathcal{P}(X) \ \ (\text{there are } 2^3 = 8 \text{ day menus}).$$

Applying a second powerset produces all collections of day menus—i.e. weekly menu plans*:*

$$\mathcal{P}^{(1,1)}(X) \;:=\; \mathcal{P}\big(\mathcal{P}(X)\big) = \mathcal{P}^2(X) \quad (\text{there are } 2^8 = 256 \text{ weekly plans}).$$

Semantics by level.

- *Level 0 (X): individual dishes available that week.*
- *Level 1 ($\mathcal{P}(X)$): a* day menu *is a subset of dishes, e.g.*

$$D_1 = \{Pasta, Salad\}, \qquad D_2 = \{Curry\}.$$

- *Level 2 ($\mathcal{P}(\mathcal{P}(X))$): a* weekly plan *is a set of day menus, e.g.*

$$W = \{\, D_1,\ D_2,\ \varnothing \,\},$$

meaning: one day serves Pasta+Salad, another serves Curry, and one day is a planned $\varnothing$ *(leftovers/skip).*

Why the block vector? *Writing* $\mathcal{P}^{(1,1)}$ *makes the two managerial layers explicit: first pick* day menus*, then pick a* set of day menus *to form a weekly plan. (Flattening law:* $\mathcal{P}^{(1,1)} = \mathcal{P}^2$*, but the block boundary records the modeling stages.)*

Example 10 (Designing and scheduling A/B experiments as a three-level $\mathcal{P}^{(2,1)}$)**.** *A product team considers two binary* features *for experimentation:*

$$X = \{\textit{New Banner},\ \textit{Button Color}\} \quad (|X| = 2).$$

***Block 1 (depth** 2**):** First apply $\mathcal{P}$ to obtain all* configurations *(enable/disable each feature):*

$$\mathcal{P}(X)\ \textit{ has } 2^2 = 4 \textit{ configurations, e.g. } C_1 = \{\textit{New Banner}\},\ C_2 = \{\textit{Button Color}\}.$$

Apply $\mathcal{P}$ again to obtain all test suites *(collections of configurations to run in one batch):*

$$\mathcal{P}^2(X) = \mathcal{P}(\mathcal{P}(X))\ \textit{ has } 2^4 = 16 \textit{ test suites.}$$

Block 2 (depth 1**):** *Apply $\mathcal{P}$ once more to select a* campaign plan*— a set of test suites scheduled across a quarter:*

$$\mathcal{P}^{(2,1)}(X) \ :=\ \mathcal{P}\big(\mathcal{P}^2(X)\big) \quad \textit{with} \quad \big|\mathcal{P}^{(2,1)}(X)\big| = 2^{16} = 65{,}536.$$

Concrete elements at each level.

$$\begin{aligned}
&\textit{Configuration:} \quad C^\star = \{\textit{New Banner}\} \in \mathcal{P}(X).\\
&\textit{Test suite:} \quad S^\star = \{\,\varnothing,\ C^\star\,\} \in \mathcal{P}^2(X) \quad (\textit{control vs. banner-on}).\\
&\textit{Campaign plan:} \quad \Pi^\star = \{\,S^\star,\ \{C_1, C_2\}\,\} \in \mathcal{P}^{(2,1)}(X),
\end{aligned}$$

meaning: one batch compares control vs. New Banner only; another batch tests both single-feature configs together.

Why the block vector? *Although $\mathcal{P}^{(2,1)} = \mathcal{P}^3$ by the flattening law, writing* $(2,1)$ *mirrors real practice: (a)* design space *is built in two nested steps (configurations $\to$ suites), then (b)* scheduling *picks a set of suites to constitute the quarter's campaign. The blocks encode these organizational layers directly in the mathematics.*

Lemma 2 (Additivity of powerset iteration)**.** *For all $a, b \in \mathbb{N}_0$ and all sets X,*

$$\mathcal{P}^a(\mathcal{P}^b(X)) \ =\ \mathcal{P}^{a+b}(X).$$

Proof. Fix a and induct on b. For $b = 0$, $\mathcal{P}^a(\mathcal{P}^0(X)) = \mathcal{P}^a(X) = \mathcal{P}^{a+0}(X)$. Assume $\mathcal{P}^a(\mathcal{P}^b(X)) = \mathcal{P}^{a+b}(X)$. Then

$$\mathcal{P}^a(\mathcal{P}^{b+1}(X)) \ =\ \mathcal{P}^a(\mathcal{P}(\mathcal{P}^b(X))) \ =\ \mathcal{P}(\mathcal{P}^a(\mathcal{P}^b(X))) \ =\ \mathcal{P}(\mathcal{P}^{a+b}(X)) \ =\ \mathcal{P}^{a+b+1}(X).$$

□

Theorem 11 (Flattening law (generalization of the n-th iterate))**.** *Let $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{N}_0^k$ and X be any set. Then*

$$\mathcal{P}^{\boldsymbol{a}}(X) \ =\ \mathcal{P}^{|\boldsymbol{a}|}(X).$$

In particular, the multi–iterated powerset strictly generalizes the usual n-th iterate and depends only on the sum of the block exponents.

Proof. Induct on k. For $k = 1$, $\mathcal{P}^{(a_1)}(X) = \mathcal{P}^{a_1}(X) = \mathcal{P}^{|\boldsymbol{a}|}(X)$. Assume the claim for length $k-1$. Write $\boldsymbol{a}' = (a_1, \ldots, a_{k-1})$. Then, using Lemma 2,

$$\mathcal{P}^{\boldsymbol{a}}(X) \ =\ \mathcal{P}^{a_k}(\mathcal{P}^{\boldsymbol{a}'}(X)) \ =\ \mathcal{P}^{a_k}(\mathcal{P}^{|\boldsymbol{a}'|}(X)) \ =\ \mathcal{P}^{a_k+|\boldsymbol{a}'|}(X) \ =\ \mathcal{P}^{|\boldsymbol{a}|}(X).$$

□

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Corollary 2 (Permutation invariance of blocks)**.** *For any permutation* π *of* $\{1,\dots,k\}$, $\mathcal{P}^{(a_1,\dots,a_k)}(X) = \mathcal{P}^{(a_{\pi(1)},\dots,a_{\pi(k)})}(X)$.

Proof. Both sides are equal to $\mathcal{P}^{\sum_i a_i}(X)$ by Theorem 11. ☐

Theorem 12 (Monotonicity and functoriality)**.** *Let* $f : X \to Y$ *be a function and* $A \subseteq X$*. Define* $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ *by* $\mathcal{P}(f)(S) := f[S]$ *(direct image). Then* $\mathcal{P}^{\boldsymbol{a}}(f)$ *(iterated composition of* $\mathcal{P}(\cdot)$*) satisfies*

$$\mathcal{P}^{\boldsymbol{a}}(f) \;=\; \mathcal{P}^{|\boldsymbol{a}|}(f),$$

and if $A \subseteq B \subseteq X$ *then* $\mathcal{P}^{\boldsymbol{a}}(A) \subseteq \mathcal{P}^{\boldsymbol{a}}(B)$ *(monotonicity).*

Proof. The equality follows from Theorem 11 applied at the functor level ($\mathcal{P}^{a} \circ \mathcal{P}^{b} = \mathcal{P}^{a+b}$). Monotonicity holds because direct image preserves inclusion at each application of $\mathcal{P}$, hence after any number of iterations. ☐

Theorem 13 (Cardinality for finite base sets)**.** *Let* X *be finite with* $\mathrm{card}(X) = m \geq 0$*. Define* $T(0) := m$ *and* $T(n+1) := 2^{T(n)}$ *(a tower of* 2*'s of height* n *above* m*). Then for every* $\boldsymbol{a} \in \mathbb{N}_0^k$,

$$\mathrm{card}(\mathcal{P}^{\boldsymbol{a}}(X)) \;=\; T(|\boldsymbol{a}|).$$

In particular, for a single block (n), $\mathrm{card}(\mathcal{P}^{n}(X)) = T(n)$ *(standard tetration growth).*

Proof. We know $\mathrm{card}(\mathcal{P}(S)) = 2^{\mathrm{card}(S)}$ for every finite S. Iterating, $\mathrm{card}(\mathcal{P}^{n}(X)) = T(n)$ by a trivial induction on n. By Theorem 11, $\mathcal{P}^{\boldsymbol{a}}(X) = \mathcal{P}^{|\boldsymbol{a}|}(X)$, hence $\mathrm{card}(\mathcal{P}^{\boldsymbol{a}}(X)) = T(|\boldsymbol{a}|)$. ☐

2.5. Named Power Set and Named Iterated PowerSet

A Named Power Set assigns to each subset of a support set a combined name, obtained by aggregating the names of its elements via a naming rule. A Named Iterated PowerSet repeatedly applies the Named Power Set construction, producing higher-level collections of subsets while consistently propagating and combining names across multiple iterations.

Definition 14 (Commutative name monoid)**.** *Let* $(I, \odot, e)$ *be a commutative monoid:* $\odot : I \times I \to I$ *is associative and commutative, with identity element* $e \in I$*. For a finite set* S *and a function* $b : S \to I$*, we write*

$$\bigodot_{s \in S} b(s)$$

for the (well-defined) $\odot$*-product over* S*, with the convention* $\bigodot_{\varnothing}(\cdot) = e$*.*

Definition 15 (Named Power Set)**.** *Let* $\Gamma = (X, a, I)$ *be a named set over* $(I, \odot, e)$*. The* Named Power Set *of* Γ *is the named set*

$$\mathcal{P}_N(\Gamma) \;:=\; \big(\, \mathcal{P}(X),\, a^{[1]},\, I \,\big),$$

where $\mathcal{P}(X)$ *is the usual powerset of* X *and the level-1 naming map* $a^{[1]} : \mathcal{P}(X) \to I$ *is defined by*

$$a^{[1]}(S) \;:=\; \bigodot_{x \in S} a(x) \qquad (S \subseteq X),$$

with $a^{[1]}(\varnothing) = e$*.*

Example 11 (Meal planning: aggregating allergens by set union)**.** *Let the name monoid be the powerset of allergen tags*

$$I := \mathcal{P}(A), \quad A = \{Dairy,\ Nuts,\ Gluten,\ Soy\},$$

with $\odot = \cup$ (union) and identity $e = \varnothing$. Let the support be ingredients

$$X = \{\mathit{Milk},\ \mathit{Bread},\ \mathit{PeanutButter},\ \mathit{Tofu}\},$$

named by their allergen sets:

$$a(\mathit{Milk}) = \{\mathit{Dairy}\}, \quad a(\mathit{Bread}) = \{\mathit{Gluten}\}, \quad a(\mathit{PeanutButter}) = \{\mathit{Nuts}\}, \quad a(\mathit{Tofu}) = \{\mathit{Soy}\}.$$

For any recipe $S \subseteq X$ the Named Power Set label is

$$a^{[1]}(S) = \bigcup_{x \in S} a(x).$$

Worked subsets (allergens carried by the recipe):

$$\begin{aligned}
S_1 = \{\mathit{Bread},\ \mathit{PeanutButter}\} \quad &\Longrightarrow \quad a^{[1]}(S_1) = \{\mathit{Gluten},\ \mathit{Nuts}\},\\
S_2 = \{\mathit{Milk},\ \mathit{Tofu}\} \quad &\Longrightarrow \quad a^{[1]}(S_2) = \{\mathit{Dairy},\ \mathit{Soy}\},\\
S_0 = \varnothing \quad &\Longrightarrow \quad a^{[1]}(S_0) = \varnothing \quad (\mathit{no\ allergens}).
\end{aligned}$$

Thus each subset (recipe) inherits the union of allergen labels of its ingredients.

Example 12 (Packing lists: total mass via addition)**.** *Let the name monoid be $(I, \odot, e) = (\mathbb{R}_{\geq 0}, +, 0)$ (nonnegative reals under addition). Let the support be items for a day trip*

$$X = \{\mathit{Laptop},\ \mathit{Charger},\ \mathit{Book},\ \mathit{Bottle}\},$$

named by weights (kg):

$$a(\mathit{Laptop}) = 1.30, \quad a(\mathit{Charger}) = 0.20, \quad a(\mathit{Book}) = 0.50, \quad a(\mathit{Bottle}) = 0.60.$$

For any packing list $S \subseteq X$,

$$a^{[1]}(S) = \sum_{x \in S} a(x) \quad (\mathit{total\ carried\ mass}).$$

Worked subsets (total weight in kg):

$$\begin{aligned}
S_{\text{work}} = \{\mathit{Laptop},\ \mathit{Charger},\ \mathit{Bottle}\} \quad &\Longrightarrow \quad a^{[1]}(S_{\text{work}}) = 1.30 + 0.20 + 0.60 = 2.10,\\
S_{\text{light}} = \{\mathit{Book},\ \mathit{Bottle}\} \quad &\Longrightarrow \quad a^{[1]}(S_{\text{light}}) = 0.50 + 0.60 = 1.10,\\
S_0 = \varnothing \quad &\Longrightarrow \quad a^{[1]}(S_0) = 0.00.
\end{aligned}$$

Hence each subset (packing list) receives as its name the sum of item weights.

Lemma 3 (Well-definedness)**.** *The map $a^{[1]} : \mathcal{P}(X) \to I$ in Definition 15 is well-defined and independent of any ordering of S because $(I, \odot, e)$ is a commutative monoid.*

Proof. By commutativity and associativity, $\bigodot_{x \in S} a(x)$ does not depend on the enumeration of S. For $S = \varnothing$ the empty product equals e by convention. □

Theorem 14 (Named Power Set generalizes Power Set and Named Set)**.** *Let $\Gamma = (X, a, I)$.*

(a) ***(Reduction to Power Set).*** *If $I = \{e\}$ is the terminal monoid (single element), then forgetting names yields the usual powerset:*

$$\text{under the forgetful functor } U : \mathit{Named} \to \mathit{Set}, \quad U(\mathcal{P}_N(\Gamma)) = \mathcal{P}(X).$$

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(b) **(Extension of Named Set).** *Let $\iota : X \to \mathcal{P}(X)$, $x \mapsto \{x\}$. Then $a^{[1]} \circ \iota = a$; equivalently,*

$$a^{[1]}(\{x\}) = \bigodot_{y \in \{x\}} a(y) = a(x) \qquad (\forall x \in X).$$

Proof. (a) If $I = \{e\}$, every name is e and $a^{[1]}$ is uniquely constant; the underlying support of $\mathcal{P}_N(\Gamma)$ is $\mathcal{P}(X)$.

(b) Immediate from the definition of $a^{[1]}$ and the singleton product law. $\square$

We now iterate the construction.

Definition 16 (Named Iterated PowerSet)**.** *Let $\Gamma = (X, a, I)$ be a named set over $(I, \odot, e)$. Define inductively for $n \in \mathbb{N}_0$:*

$$\Gamma^{[0]} := (X, a, I), \qquad \Gamma^{[n+1]} := \mathcal{P}_N\big(\Gamma^{[n]}\big) = \big(\, \mathcal{P}^{\,n+1}(X),\ a^{[n+1]},\ I\,\big),$$

where the level-$(n+1)$ naming map $a^{[n+1]} : \mathcal{P}^{\,n+1}(X) \to I$ is given recursively by

$$a^{[n+1]}(S) := \bigodot_{Y \in S} a^{[n]}(Y) \qquad \big(S \in \mathcal{P}^{\,n+1}(X)\big),$$

with the empty product equal to e. We call $\Gamma^{[n]}$ the Named Iterated PowerSet *of depth n.*

Example 13 (Classifying folders and folder-collections by highest sensitivity)**.** *Let the name monoid be the linear order*

$$I = \{\mathsf{Public} < \mathsf{Internal} < \mathsf{Confidential} < \mathsf{Secret}\},$$

with $\odot = \max$ (join) and identity $e = \mathsf{Public}$. Let the support be three files

$$X = \{\texttt{spec.pdf},\ \texttt{roadmap.docx},\ \texttt{budget.xlsx}\},$$

and define the naming map $a : X \to I$ by

$$a(\texttt{spec.pdf}) = \mathsf{Internal}, \quad a(\texttt{roadmap.docx}) = \mathsf{Confidential}, \quad a(\texttt{budget.xlsx}) = \mathsf{Secret}.$$

Depth 1 (folders)*. For a folder $Y \subseteq X$, the name is the join of its files:*

$$a^{[1]}(Y) = \bigodot_{x \in Y} a(x) = \max\{a(x) : x \in Y\}.$$

For instance,

$$Y_1 = \{\texttt{spec.pdf}, \texttt{roadmap.docx}\} \Rightarrow a^{[1]}(Y_1) = \max(\mathsf{Internal}, \mathsf{Confidential}) = \mathsf{Confidential},$$

$$Y_2 = \{\texttt{budget.xlsx}\} \Rightarrow a^{[1]}(Y_2) = \mathsf{Secret}.$$

Depth 2 (collections of folders)*. For a collection $Z \subseteq \mathcal{P}(X)$,*

$$a^{[2]}(Z) = \bigodot_{Y \in Z} a^{[1]}(Y) = \max\{a^{[1]}(Y) : Y \in Z\}.$$

With $Z = \{Y_1, Y_2\}$ we get

$$a^{[2]}(Z) = \max(\mathsf{Confidential}, \mathsf{Secret}) = \mathsf{Secret}.$$

Thus a folder's label is the strongest contained file label, and a collection-of-folders inherits the strongest label among its folders.

Example 14 (Kits and kit catalogs with additive weights). *Let the name monoid be* $(I, \odot, e) = (\mathbb{R}_{\geq 0}, +, 0)$. *Take the support of items*

$$X = \{\mathsf{Camera},\ \mathsf{Lens},\ \mathsf{Tripod},\ \mathsf{Batteries}\},$$

named by their weights (kg):

$$a(\mathsf{Camera}) = 1.2, \quad a(\mathsf{Lens}) = 0.6, \quad a(\mathsf{Tripod}) = 1.5, \quad a(\mathsf{Batteries}) = 0.2.$$

Depth 1 (kits). *For a kit* $Y \subseteq X$,

$$a^{[1]}(Y) = \sum_{x \in Y} a(x) \quad \textit{(total kit mass)}.$$

Examples:

$$Y_{hike} = \{\mathsf{Camera}, \mathsf{Lens}, \mathsf{Batteries}\} \Rightarrow a^{[1]}(Y_{hike}) = 1.2 + 0.6 + 0.2 = 2.0,$$

$$Y_{studio} = \{\mathsf{Camera}, \mathsf{Tripod}\} \Rightarrow a^{[1]}(Y_{studio}) = 1.2 + 1.5 = 2.7.$$

Depth 2 (kit catalogs). *For a catalog* $Z \subseteq \mathcal{P}(X)$,

$$a^{[2]}(Z) = \sum_{Y \in Z} a^{[1]}(Y) \quad \textit{(mass of all prepared kits in the catalog)}.$$

With $Z = \{Y_{hike}, Y_{studio}\}$,

$$a^{[2]}(Z) = 2.0 + 2.7 = 4.7 \text{ kg}.$$

Interpreted operationally: $a^{[1]}$ *totals the weight of each kit;* $a^{[2]}$ *totals the weights of all kits included in a catalog to be shipped or stored together.*

The next lemma provides an explicit closed form by counting occurrences of base elements.

Lemma 4 (Flattened formula via occurrence multiplicities). *Fix* $n \geq 1$. *For each* $Z \in \mathcal{P}^n(X)$ *and* $x \in X$, *define the integer*

$$m_Z^{(n)}(x) \;:=\; \begin{cases} \mathbf{1}_{\{x \in Z\}}, & n = 1, \\ \sum_{Y \in Z} m_Y^{(n-1)}(x), & n \geq 2, \end{cases}$$

i.e., the number of times x *appears across the level-*$(n-1)$ *components of* Z. *Then for all* $Z \in \mathcal{P}^n(X)$,

$$a^{[n]}(Z) \;=\; \bigodot_{x \in X} a(x)^{\odot\, m_Z^{(n)}(x)},$$

where $u^{\odot k} := \underbrace{u \odot \cdots \odot u}_{k \text{ times}}$ *and* $u^{\odot 0} := e$.

Proof. By induction on n. For $n = 1$, $a^{[1]}(Z) = \bigodot_{x \in Z} a(x) = \bigodot_{x \in X} a(x)^{\odot\, \mathbf{1}_{\{x \in Z\}}}$. Assume the claim at level n. For $Z \in \mathcal{P}^{n+1}(X)$,

$$a^{[n+1]}(Z) = \bigodot_{Y \in Z} a^{[n]}(Y) = \bigodot_{Y \in Z} \bigodot_{x \in X} a(x)^{\odot\, m_Y^{(n)}(x)} = \bigodot_{x \in X} a(x)^{\odot\, \sum_{Y \in Z} m_Y^{(n)}(x)} = \bigodot_{x \in X} a(x)^{\odot\, m_Z^{(n+1)}(x)},$$

using associativity/commutativity to regroup exponents. ☐

Theorem 15 (Named Iterated PowerSet generalizes both layers). *Let* $\Gamma = (X, a, I)$ *and* $n \geq 1$.

(a) ***(Base compatibility).*** $\Gamma^{[1]} = \mathcal{P}_N(\Gamma)$ *as in Definition 15.*

(b) ***(Reduction to iterated powerset).*** *If $I = \{e\}$ is the terminal monoid, then*

$$U(\Gamma^{[n]}) \;=\; \mathcal{P}^n(X) \qquad (\textit{as sets}),$$

i.e., the underlying support is the usual n-th iterated powerset.

(c) ***(Singleton embedding respects names).*** *Let $\iota^{(n)} : X \to \mathcal{P}^n(X)$ be the n-fold singleton embedding $\iota^{(1)}(x) = \{x\}$ and $\iota^{(n+1)}(x) = \{\iota^{(n)}(x)\}$. Then for all $n \geq 1$,*

$$a^{[n]}\big(\iota^{(n)}(x)\big) \;=\; a(x) \qquad (\forall x \in X).$$

Proof. (a) is by definition.

(b) When $I = \{e\}$, every naming map is uniquely constant. By Definition 16, the supports of $\Gamma^{[n]}$ are exactly $\mathcal{P}^n(X)$.

(c) We argue by induction on n. For $n = 1$, $a^{[1]}(\{x\}) = a(x)$ by Theorem 14(b). Assume $a^{[n]}(\iota^{(n)}(x)) = a(x)$. Then

$$a^{[n+1]}(\iota^{(n+1)}(x)) \;=\; \bigodot_{Y \in \{\iota^{(n)}(x)\}} a^{[n]}(Y) \;=\; a^{[n]}(\iota^{(n)}(x)) \;=\; a(x).$$

□

Lemma 5 (Disjoint union factorization). *If $X = X_1 \dot\cup X_2$ is a disjoint union and $a_i := a|_{X_i}$, then for every $n \geq 1$ and $Z \in \mathcal{P}^n(X)$,*

$$a^{[n]}(Z) \;=\; \big(a_1^{[n]}(Z|_{X_1})\big) \;\odot\; \big(a_2^{[n]}(Z|_{X_2})\big),$$

where $Z|_{X_i}$ denotes the restriction obtained by intersecting all level-1 components with X_i recursively.

Proof. Unfold $a^{[n]}$ via Lemma 4 and use that the occurrence counts split across X_1 and X_2. □

Lemma 6 (Cardinality vs. names). *For any finite X and any $n \geq 1$,*

$$\big|\,\textit{support of } \Gamma^{[n]}\,\big| \;=\; |\mathcal{P}^n(X)|,$$

independent of the choice of $(I, \odot, e)$ and a. In particular, if $|X| = m$ and $T(0) := m,\ T(k+1) := 2^{T(k)}$, then $|\mathcal{P}^n(X)| = T(n)$.

Proof. By Definition 16, the support at depth n is $\mathcal{P}^n(X)$ regardless of I, a. The size formula follows by iterating $|\mathcal{P}(S)| = 2^{|S|}$. □

3. Conclusions

We introduced the Signed Power Multiset and the Signed Iterated Power Multiset, defined by coordinatewise factorization. We also formalized multi–iterated powersets indexed by block vectors and establish a flattening law showing that only the total height matters. In the future, we hope that extended frameworks of the set concepts presented in this paper will be explored, including Fuzzy Sets [20,21], Intuitionistic Fuzzy Sets [22], HyperFuzzy Sets [23,24], Soft Sets [25,26], HyperSoft Sets [27,28], Rough Sets [29,30], HyperRough Sets [31], Neutrosophic Sets [32,33], and Plithogenic Sets [34–36].

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Use of Artificial Intelligence: I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards. No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

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