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*Article*

# A New Double Fuzzy Integral Transform for Solving Advection Diffusion Equation

**Atanaska Georgieva *, Slav I. Cholakov and Mira Spasova**

Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski, 24 Tzar Asen, 4000 Plovdiv, Bulgaria
* Correspondence: atanaska@uni-plovdiv.bg

**Abstract:** The main purpose of this research is to present a new approach to double fuzzy transforms, called the double fuzzy Yang-General transform transform (DFY-GT). This new double fuzzy transform is a novel combination of the single fuzzy Yang and General transforms. Some basic properties of the new transform including existence, linearity and some results related to partial derivatives are presented. To show the applicability of the double fuzzy Yang-General transform, the solution framework for linear fuzzy advection-diffusion equation is developed. Furthermore, numerical example is solved to illustrate the proposed method.

**Keywords:** double fuzzy Yang-General transform; fuzzy advection-diffusion equation; generalized Hukuhara partial differentiability

**MSC:** 44A30; 35A22; 35N05

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## 1. Introduction

In recent years, many researchers have studied fuzzy partial differential and integral equations. These equations are an excellent tool for modeling vagueness and misinterpretation of knowledge-based systems, control systems, image processing, industrial automation, power engineering, robotics, artificial intelligence, consumer electronics, management and operations research [1–5]. The first definition of fuzzy partial differential equations is given by Buckley and Feuring [6]. The difference method for solving fuzzy partial differential equations is proposed in [7]. Nemati and Matinfar [8] constructed an implicit finite-difference approach to solve complex fuzzy parabolic differential equations. An explicit numerical solution to the fuzzy hyperbolic and parabolic equation has been given in [9]. Recently, Arqub et al. [10] applied the reproducing kernel algorithm for find the solution of two-point fuzzy boundary value problems. In [11] by using the adaptation of the reproducing kernel algorithm are solved the fuzzy Fredholm–Volterra integro-differential equations.

Over the years, integral transforms have had great importance and have used which has given them an important place in solving many types of equations. The fuzzy version of classical General transform is introduced by Rashid et al. [12]. Ullah et al. [13] proposed fuzzy Yang transformation to find in the solution of second order fuzzy differential equations of integer and fractional order.

Recently, some researchers [14–20] introduced diffrent double fuzzy integral transforms (Sumudu, Natural, Laplace, Elzaki, Sawi, Aboodh) and used it to solve fuzzy partial differential equations.

Our motivation in this study is to introduce a new double fuzzy integral transform, namely double fuzzy Yng-General transform. Basic properties of DFY-GT are proven and the values of DFY-GT for some functions are computed. New theorems related to partial gH-derivatives are established and implemented to solve linear fuzzy advection-diffusion equation. The novelty of this paper appears in the new combination between the single fuzzy transforms of Yang and General, in which the new double fuzzy Yang-General transform have the advantages of the two transforms.

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In this research, we studied the linear advection-diffusion fuzzy partial differential equations with constant diffusion coefficient $a > 0$ and constant advection velocity $b$

$$u'_t(x,t) = a \odot u''_{xx}(x,t) \ominus_{gH} b \odot u'_x(x,t), \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$$

with initial conditions

$$u(x,0) = f_0(x),$$

and boundary conditions

$$u(0,t) = g_0(t), \;\; u'_x(0,t) = g_1(t),$$

where $u(x,t), \;\; f_0(x), \;\; g_0(t)$ and $g_1(t)$ are fuzzy functions, the constants $a > 0$ and $b$ are diffusion coefficient and advection velocity, respectively. Denote $\mathbb{R}_+ = [0,\infty)$.

A simple formula for the solution of the above equation is obtained and applied to solve numerical example in order to display the efficiency of this new approach.

Rest of the paper is organized as follows: In Section 2 we present the basic concepts that we will use in the main part of the paper. In Section 3 the fundamental facts and properties of single fuzy Yang and General transforms are presented. In Section 4 we introduce a new integral transform, the DFY-GT, that combines the fuzzy Yang transform and the fuzzy General transform and present some properties of this transform. In Section 5 we apply double fuzzy Yang-General transform to the linear fuzzy advection-diffusion equation and obtain a formula for the exact solution. Section 6 provides a numerical examples which are solved with the DFY-GT. Concluding remarks are given in Section 7.

## 2. Basic Concepts

In this section, we introduce the basic concepts which will be used in the major part of the paper.

We will denote $E^1$ the class of fuzzy subsets. The membership function $\mu : \mathbb{R} \to [0,1]$, satisfies the conditions

(i) $\mu$ is normal, so there exists $x_0 \in \mathbb{R}$ with $\mu(x_0) = 1$;
(ii) $\mu$ is upper semi-continuous;
(iii) $\mu(rx_1 + (1-r)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}, x_1, x_2 \in \mathbb{R}, \; r \in [0,1]$;
(iv) $cl(supp\ \mu) = cl(\{x \in \mathbb{R} : \; \mu(x) > 0\})$ is compact.

Here, $cl(X)$ denotes the closure of set X. Then $E^1$ is called the space of fuzzy numbers.

**Definition 2.1.** *[21] Let $w \in E^1$. For $r \in [0,1]$ the r-level set of $w$ is defined by*

$$[w]^r = \begin{cases} \{x \in \mathbb{R} : w(x) \geq r\}, & 0 < r \leq 1, \\ cl(supp\ w), & r = 0. \end{cases}$$

*The core of $w$ is the set of elements of $\mathbb{R}$ having membership grade 1, i.e.,*

$$[w]^1 = \{x \in \mathbb{R} : w(x) = 1\}.$$

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A fuzzy set $w$ is fuzzy number if and only if the $r$-levels are nonempty compact intervals of the form $[w]^r = [\underline{w}(r), \overline{w}(r)]$.

**Definition 2.2.** *[21] An ordered pair $w(r) = (\underline{w}(r), \overline{w}(r))$ is called a parametric form of fuzzy number $w$, if the functions $\underline{w}$ $\overline{w} : [0,1] \to \mathbb{R}$ satisfy the conditions:*

*(i)* $\underline{w}(r)$ *is a bounded monotonic non decreasing left continuous for all $r \in [0,1]$ and right-continuous for $r = 0$;*

*(ii)* $\overline{w}(r)$ *is a bounded monotonic non increasing left continuous for all $r \in [0,1]$ and right-continuous for $r = 0$;*

*(iii)* $\underline{w}(r) \leq \overline{w}(r)$ *for all $r \in [0,1]$.*

**Definition 2.3.** *[21] An ordered foursome $w = (a,b,c,d)$ is called a trapezoidal fuzzy number $w$, if $a \leq b \leq c$ and $a,\ b,\ c\ d \in \mathbb{R}$. The r-levels of this fuzzy number is $[a + (b-a)r, d - (d-c)r]$. We obtain a triangular fuzzy number if $b = c$.*

Let $v,\ w \in E^1$ and $\lambda \in \mathbb{R}$. The addition $v \oplus w$ and the scalar multiplication $\lambda \odot v$ are defined

$$[v \oplus w]^r = [v]^r + [w]^r = [\underline{v}(r) + \underline{w}(r), \overline{v}(r) + \overline{w}(r)],$$

$$[\lambda \odot v]^r = \lambda.[v]^r = \begin{cases} [\lambda\underline{v}(r), \lambda\overline{v}(r)], & \lambda \geq 0 \\ [\lambda\overline{v}(r), \lambda\underline{v}(r)], & \lambda < 0. \end{cases}$$

The subtraction of fuzzy numbers $v$ and $w$ is defined as the addition, i.e.

$$v - w = v \oplus ((-1) \odot w).$$

The Hukuhara difference ($H$-difference) $\ominus_H$, is defined by $u \ominus_H v = w$ if and only if $v \oplus w = u$. The $H$ - difference is unique, but it does not always exist. If $u \ominus_H v$ exists, its $r$-levels are

$$[u \ominus_H v]^r = [\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)].$$

For an interval $[c,d]$ we define the norm

$$\|[c,d]\| = max\{|c|, |d|\}.$$

Then the Hausdorff distance between fuzzy numbers is defined by

$$D(v,w) = \sup_{r \in [0,1]} \{\|[v]^r \ominus_{gH} [w]^r\|\}.$$

The metric $D$ is well defined since the $gH$-difference of intervals $[u]^r \ominus_{gH} [v]^r$ always exists. Hence $(E^1, D)$ is a complete metric space.

**Definition 2.4.** *[21] Let $u,\ v \in E^1$. Then the generalized Hukuhara difference (gH-difference) between this numbers is the fuzzy number $w$ if it exists and*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) & u = v \oplus w, \\ or\ (ii) & v = u \oplus (-1) \odot w. \end{cases}$$

Hence its $r$-levels are

$$[u \ominus_{gH} v]^r = [\min\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}, \max\{\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r)\}].$$

Sufficient conditions for the existence of $gH$ - difference are obtained in [21].

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The following properties for $gH$ - difference are given in [22].

**Proposition 2.1.** *Let $u, v \in E^1$, then*

*(i)* *if the H-difference exists, then $v \ominus_{gH} w = v \ominus_H w$ or $v \ominus_{gH} w = -(w \ominus_H v)$;*
*(ii)* *if the gH-difference exists, then it is unique;*
*(iii)* *if $v \ominus_{gH} w$ exists in the sense $(ii)$, then $w \ominus_{gH} v$ exists in the sense $(i)$ and vice versa;*
*(iv)* $(v \oplus w) \ominus_{gH} w = v$;
*(v)* $\tilde{0} \ominus_{gH} (v \ominus_{gH} w) = w \ominus_{gH} v$;
*(vi)* *if $v \ominus_{gH} w = w \ominus_{gH} v = u$ if and only if $u = -u$.*

*2.1. Fuzzy Function of One-variable*

In this section, we introduce some definitions and basic properties of the fuzzy function $\varphi : T = (c, d) \to E^1$.

For each $r \in [0, 1]$, the endpoint functions $\underline{\varphi}(., r)$, $\overline{\varphi}(., r) : T \to \mathbb{R}$ are called upper and lower functions of $\varphi$, respectively. Then the parametric form of the function $\varphi$ is

$$\varphi(t, r) = \left(\underline{\varphi}(t, r), \overline{\varphi}(t, r)\right).$$

**Lemma 2.1.** *[16] Let $a_1,\ a_2 \in \mathbb{R}$ such that $a_1,\ a_2 \geq 0$ or $a_1,\ a_2 \leq 0$ and $\varphi : \mathbb{R}_+ \to E^1$. Then*

$$a_1 \odot \varphi(t) \ominus_{gH} a_2 \odot \varphi(t) = (a_1 - a_2) \odot \varphi(t).$$

**Lemma 2.2.** *[16] Let $a_1,\ a_2 \in \mathbb{R}$ such that $a_1,\ a_2 \geq 0$ or $a_1,\ a_2 \leq 0$. If $\varphi_1,\ \varphi_2 : \mathbb{R}_+ \to E^1$ are improper fuzzy Riemann-integrable on $\mathbb{R}_+$, then*

*(i)* $\int\limits_0^\infty [a_1 \odot \varphi_1(t) \ominus_{gH} a_2 \odot \varphi_2(t)] dt = a_1 \odot \int\limits_0^\infty \varphi_1(t) dxt \ominus_{gH} a_2 \odot \int\limits_0^\infty \varphi_2(t) dt$;
*(ii)* $\int\limits_0^\infty [a_1 \odot \varphi_1(t) \oplus a_2 \odot \varphi_2(t)] dx = a_1 \odot \int\limits_0^\infty \varphi_1(t) dt \oplus a_2 \odot \int\limits_0^\infty \varphi_2(t) dt$.

The following definition is the known concept of generalized Hukuhara differentiable ($gH$-differentiable) fuzzy functions based on the $gH$-difference of fuzzy intervals.

**Definition 2.5.** *[21] Let $t_0 \in T$ and $k$ be such that $t_0 + k \in T$. Then the generalized Hukuhara derivative ($gH$-derivative) of a fuzzy function $\varphi : T \to E^1$ at $t_0 \in T$ is defined as*

$$\varphi'_{gH}(t_0) = \lim_{k \to 0} \frac{\varphi(t_0 + k) \ominus_{gH} \varphi(t_0)}{k}. \tag{1}$$

*If $\varphi'_{gH}(t_0) \in E^1$ satisfying (1) exists, we said that $\varphi$ is gH-differentiable at the point $t_0$.*

**Definition 2.6.** *[21] Let $\varphi : T \to E^1$ and $t_0 \in T$, with $\underline{\varphi}(., r)$ and $\overline{\varphi}(., r)$ both differentiable at $t_0$. We say that*

*(i)* *$\varphi$ is $(i)$-gH-differentiable at $t_0$ if*

$$\varphi'_{i,gH}(t_0, r) = \left(\underline{\varphi}'(t_0, r), \overline{\varphi}'(t_0, r)\right), \quad r \in [0, 1], \tag{2}$$

*(ii)* *$\varphi$ is $(ii)$-gH-differentiable at $t_0$ if*

$$\varphi'_{ii,gH}(t_0, r) = \left(\overline{\varphi}'(t_0, r), \underline{\varphi}'(t_0, r)\right), \quad r \in [0, 1]. \tag{3}$$

Now, we present some properties for generalized Hukuhara differentiable function.

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**Theorem 2.1.** *[23] Let $\varphi,\ \psi : T \to E^1$ be gH-differentiable . Then $\varphi(t) \ominus_{gH} \psi(t)$ is gH-differentiable, and*

$$(\varphi(t) \ominus_{gH} \psi(t))'_{gH} = \varphi'_{gH}(t) \ominus_{gH} \psi'_{gH}(t).$$

**Theorem 2.2.** *[23] Let $\varphi : T \to E^1$ and $\psi : T \to \mathbb{R}_+$. Suppose that the fuzzy function $\varphi(t)$ is gH-differentiable and the functions $\psi(t)$ is differentiable. Then*

$$(\psi(t) \odot \varphi(t))'_{gH} = \psi(t) \odot \varphi'_{gH}(t) \oplus \psi'(t) \odot \varphi(t).$$

**Theorem 2.3.** *[21] Let $\varphi : T \to E^1$ be gH-differentiable in the interval $T$. Then*

$$\int_c^d \varphi'_{gH}(t)dt = \varphi(d) \ominus_{gH} \varphi(c).$$

**Theorem 2.4.** *[24] Let $\varphi : T \to E^1$ be gH-differentiable in the interval $T$ and $\psi : [c,d] \to \mathbb{R}$ be differentiable functions. Then*

$$\int_c^d \varphi'_{gH}(t) \odot \psi(t)dtx = (\varphi(d) \odot \psi(d)) \ominus_{gH} (\varphi(c) \odot \psi(c)) \ominus_{gH} \int_c^d \varphi(t) \odot \psi'(t)dt.$$

*2.2. Fuzzy Function of Two-variable*

Let $u : D \subset \mathbb{R} \times \mathbb{R} \to E^1$ be fuzzy function and $u(x,t,r) = (\underline{u}(x,t,r), \overline{u}(x,t,r))$ for any $r \in [0,1]$ is parametric form of this function.

**Definition 2.7.** *[14] A fuzzy function $u : D \to E^1$ is said to be continuous at $(x_0, t_0) \in D$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that $D(u(x,t), u(x_0,t_0)) < \varepsilon$ whenever $|x - x_0| + |t - t_0| < \delta$. If $u$ is continuous for each $(x,t) \in D$, then we say that $u$ is continuous on $D$.*

**Definition 2.8.** *[23] Let $(x_0,t_0) \in D$ and $h, k$ be such that $(x_0 + h, t_0) \in D$, $(x_0, t_0 + k) \in D$. Then first generalized Hukuhara partial derivative (gH-p-derivative) of fuzzy function $u : D \to E^1$ at $(x_0,t_0)$ with respect to $x$ and $t$ are fuzzy numbers $u'_{x,gH}(x_0,t_0)$ and $u'_{t,gH}(x_0,t_0)$ defined by*

$$u'_{x,gH}(x_0,t_0) = \lim_{h\to 0} \frac{u(x_0+h,t_0) \ominus_{gH} u(x_0,t_0)}{h},$$

$$u'_{t,gH}(x_0,t_0) = \lim_{k\to 0} \frac{u(x_0,t_0+k) \ominus_{gH} u(x_0,t_0)}{k}.$$

**Definition 2.9.** *[23] Let $u : D \to E^1$, $(x_0,t_0) \in D$ and $\underline{u}(x,t,r)$, $\overline{u}(x,t,r)$ both partial differentiable at $(x_0,t_0)$ with respect to $t$. Then we say that*

*(i)* $u(x,t)$ *is $(i)$-gH-p-differentiable at $(x_0,t_0)$ with respect to $t$ if*

$$u'_{t,\ (i)-gH}(x_0,t_0,r) = \left(\underline{u}'_t(x_0,t_0,r), \overline{u}'_t(x_0,t_0,r)\right),\ \ r \in [0,1], \tag{4}$$

*(ii)* $u(x,t)$ *is $(ii)$-gH-differentiable at $(x_0,t_0)$ with respect to $t$ if*

$$u'_{t,\ (ii)-gH}(x_0,t_0,r) = \left(\overline{u}'_t(x_0,t_0,r), \underline{u}'_t(x_0,t_0,r)\right),\ \ r \in [0,1]. \tag{5}$$

**Theorem 2.5.** *[23] Let $u : D \to E^1$ be gH-p-differentiable with respect to $x$ and $c \in \mathbb{R}_+$. Then $(c \odot u(x,t))'_{x,\,gH}$ exists and*

$$(c \odot u(x,t))'_{x,\,gH} = c \odot u'_{x,\,gH}(x,t).$$

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**Theorem 2.6.** *[24] Let $u : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be a fuzzy function and let $\int_0^\infty u(x,t)dt$ be convergent for each $[0,\infty)$ and $\int_0^\infty u(x,t)dx$ as a function t convergent on $\mathbb{R}_+$. Then*

$$\int_0^\infty \int_0^\infty u(x,t)dtdx = \int_0^\infty \int_0^\infty u(x,t)dxdt.$$

## 3. Basic Definitions and Theorems for Yang and General Fuzzy Transforms

In this section, we give the definitions and some fundamental properties of the fuzzy Yang and fuzzy General transforms.

### 3.1. Fuzzy Yang Transform

The definition of fuzzy Yang transform is introduced in [13].

**Definition 3.1.** *Let $\varphi : \mathbb{R}_+ \to E^1$ and the function $e^{-\frac{x}{\alpha}} \odot \varphi(x)$ be improper fuzzy Riemann-integrable on $\mathbb{R}_+$ for $\alpha > 0$. Then, the fuzzy Yang transform (FYT) of $\varphi(x)$ is defined as*

$$\Phi(\alpha) = Y_x[\varphi(x)] = \int_0^\infty e^{-\frac{x}{\alpha}} \odot \varphi(x)dx, \tag{6}$$

*where x and α are transform variables.*

**Definition 3.2.** *The fuzzy inverse Yang transform is given by*

$$Y_x^{-1}[\Phi(\alpha)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\frac{x}{\alpha}} \odot \Phi(\alpha)d\alpha, \tag{7}$$

*where the function $\Phi(\alpha)$ must be analytic for all α such that $Re\alpha > a$.*

**Definition 3.3.** *Let $\varphi : \mathbb{R}_+ \to E^1$ and $c > 0$. The fuzzy function φ is said to be of exponential order c if there exists a positive constant K, such that for all $x > X$*

$$D(\varphi(x), \tilde{0}) \leq Ke^{cx}.$$

**Theorem 3.1.** *Let $\varphi : \mathbb{R}_+ \to E^1$ be a continuous fuzzy function in every finite interval $(0, X)$ of exponential order $c > 0$. Then, the FYT of $\varphi(x)$ exists for all α provided $Re(\frac{1}{\alpha}) > c$.*

**Proof.** By using Definition 3.1, we get

$$\begin{aligned} D(\Phi(\alpha), \tilde{0}) &= D(\int_0^\infty e^{-\frac{x}{\alpha}} \odot \varphi(x)dx, \tilde{0}) \leq \int_0^\infty e^{-\frac{x}{\alpha}} D(\varphi(x), \tilde{0})dx \leq \\ &\leq K\int_0^\infty e^{-(\frac{1}{\alpha}-c)x}dx = \tfrac{K\alpha}{1-c\alpha}, \quad for\ Re(\tfrac{1}{\alpha}) > c, \end{aligned}$$

where $Y_x[f(x)] = \Phi(\alpha)$. □

In [25], classical Yang transform is applied on some special functions.

(i) $Y_x[1] = \alpha$, $Y_x[x^m] = (m!)\alpha^{m+1}$, where $m$ is positive integer;
(ii) $Y_x[e^{cx}] = \frac{\alpha}{1-c\alpha}$, where $c \in \mathbb{R}$;
(iii) $Y_x[\sin cx] = \frac{\alpha}{1+c^2\alpha^2}$, $Y_x[\cos cx] = \frac{c\alpha^2}{1+c^2\alpha^2}$, where $c \in \mathbb{R}$.

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By Definition 3.1, we obtain the following useful properties of fuzzy Yang transform.

**Theorem 3.2.** *Let $\varphi_1,\ \varphi_2 : \mathbb{R}_+ \to E^1$ be fuzzy functions for which the FYT exists. Then, the FYT of a functions $a_1 \odot \varphi_1(x) \oplus a_2 \odot \varphi_2(x)$ and $a_1 \odot \varphi_1(x) \ominus_{gH} a_2 \odot \varphi_2(x)$ exist and*

*(i)* $a_1 \odot Y_x[\varphi_1(x)] \ominus_{gH} a_2 \odot Y_x[\varphi_2(x)] = Y_x[a_1 \odot \varphi_1(x) \ominus_{gH} a_2 \odot \varphi_2(x)]$;

*(ii)* $a_1 \odot Y_x[\varphi_1(x)] \oplus a_2 \odot Y_x[\varphi_2(x)] = Y_x[a_1 \odot \varphi_1(x) \oplus a_2 \odot \varphi_2(x)]$,

*where $a_1,\ a_2 \in \mathbb{R}$ such that $a_1,\ a_2 \geq 0$ or $a_1,\ a_2 \leq 0$.*

**Proof.** Using Lemma 2.2 we get

$$\begin{aligned} a_1 \odot Y_x[\varphi_1(x)] \ominus_{gH} a_2 \odot Y_x[\varphi_2(x)] &= \int_0^\infty a_1 e^{-\frac{x}{\alpha}} \odot \varphi_1(x)dx \ominus_{gH} \int_0^\infty a_2 e^{-\frac{x}{\alpha}} \odot \varphi_2(x)dx = \\ &= \int_0^\infty e^{-\frac{x}{\alpha}}(a_1 \odot \varphi_1(x) \ominus_{gH} a_2 \odot \varphi_2(x))dx = \\ &= Y_x[a_1 \odot \varphi_1(x) \ominus_{gH} a_2 \odot \varphi_2(x)]. \end{aligned}$$

A similar manner we obtain the proof for part (ii). □

**Theorem 3.3.** *Let $\varphi : \mathbb{R}_+ \to E^1$ be continuous fuzzy functions of exponential order $c > 0$ and $\varphi'_{gH}(x)$ be continuous in every finite closed interval $0 \leq x \leq X$. Then for $Re(\frac{1}{\alpha}) > c$ we have*

*(i)* $Y_x[\varphi'_{gH}(x)] = (-1) \odot \varphi(0) \ominus_{gH} (-1)\frac{1}{\alpha} \odot \Phi(\alpha)$;

*(ii)* $Y_x[\varphi''_{gH}(x)] = (-1) \odot \varphi'_{gH}(0) \ominus_{gH} \left(\frac{1}{\alpha} \odot \varphi(0) \ominus_{gH} \frac{1}{\alpha^2} \odot \Phi(\alpha)\right)$,

*where $Y_x[\varphi(x)] = \Phi(\alpha)$.*

**Proof.** From definition of improper fuzzy Riemann-integral, we have

$$Y_x[\varphi'_{gH}(x)] = \int_0^\infty e^{-\frac{x}{\alpha}} \odot \varphi'_{gH}(x)dx = \lim_{R\to\infty} \int_0^R e^{-\frac{x}{\alpha}} \odot \varphi'_{gH}(x)dx$$

provided this limit exists. Which is, Theorem 2.4

$$\begin{aligned} &\int_0^R e^{-\frac{x}{\alpha}} \odot \varphi'_{gH}(x)dx = e^{-\frac{x}{\alpha}} \odot \varphi(x)|_0^R \ominus_{gH} (-1)\frac{1}{\alpha} \int_0^R e^{-\frac{x}{\alpha}} \odot \varphi(x)dx \\ &= e^{-\frac{R}{\alpha}} \odot \varphi(R) \ominus_{gH} \varphi(0) \ominus_{gH} (-1)\frac{1}{\alpha} \int_0^R e^{-\frac{x}{\alpha}} \odot \varphi(x)dx. \end{aligned}$$

The fuzzy function $f$ is of exponential order $c > 0$. That, there exist $K > 0$ and $X > 0$ such that $D(\varphi(x), \tilde{0}) \leq Ke^{cx}$ for $x > X$. Thus, if $Re(\frac{1}{\alpha}) > c$, we have

$$\lim_{R\to\infty} D(e^{-\frac{R}{\alpha}} \odot \varphi(R), \tilde{0}) \leq \lim_{R\to\infty} Ke^{-(\frac{1}{\alpha}-c)R} = 0$$

and

$$\lim_{R\to\infty} \int_0^R e^{-\frac{x}{\alpha}} \odot \varphi(x)dx = Y_x[\varphi(x)].$$

Hence, by above equation and Proposition 2.1, we obtain

$$Y_x[\varphi'_{gH}(x)] = (-1) \odot \varphi(0) \ominus_{gH} (-1)\frac{1}{\alpha} \odot Y_x[\varphi(x)]. \tag{8}$$

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Using Definition 3.1 and (8), we have

$$\begin{aligned} Y_x[\varphi''_{gH}(x)] &= (-1) \odot \varphi'_{gH}(0) \ominus_{gH} (-1)\tfrac{1}{\alpha} \odot Y_x[\varphi'_{gH}(x)] = \\ &= (-1) \odot \varphi'_{gH}(0) \ominus_{gH} (-1)\tfrac{1}{\alpha}\Big((-1) \odot \varphi(0) \ominus_{gH} (-1)\tfrac{1}{\alpha} \odot Y_x[\varphi(x)]\Big) = \\ &= (-1) \odot \varphi'_{gH}(0) \ominus_{gH} \Big(\tfrac{1}{\alpha} \odot \varphi(0) \ominus_{gH} \tfrac{1}{\alpha^2} \odot \Phi(\alpha)\Big). \end{aligned}$$

□

**Remark 3.1.** *A similar procedure can be used for the fuzzy function $u(x,t)$ of two variables. Then the fuzzy Yang transform for partial derivatives of $u(x,t)$ is as follows:*

*(i)* $Y_x[u'_{x,gH}(x,t)] = (-1) \odot u(0,t) \ominus_{gH} (-1)\frac{1}{\alpha} \odot U(\alpha,t)$;

*(ii)* $Y_x[u''_{xx,gH}(x,t)] = (-1) \odot u'_{x,gH}(0,t) \ominus_{gH} \Big(\frac{1}{\alpha} \odot u(0,t) \ominus_{gH} \frac{1}{\alpha^2} \odot U(\alpha,t)\Big)$,

*where $Y_x[u(x,t)] = U(\alpha,t)$.*

### 3.2. Fuzzy General Transform

The fuzzy General transform is introduced in [12].

**Definition 3.4.** *Let $\psi : \mathbb{R}_+ \to E^1$ and the function $e^{-q(\beta)t} \odot \psi(t)$ be improper fuzzy Riemann-integrable on $\mathbb{R}_+$ for some $q(\beta)$. Then, the fuzzy General transform of a function $\psi(t)$ is defined as*

$$\Psi(\beta) = J_t[\psi(t)] = p(\beta) \int_0^\infty e^{-q(\beta)t} \odot \psi(t)dt, \tag{9}$$

*where the functions $p(\beta) \neq 0$ and $q(\beta)$ be positive real functions.*

Clearly, if $p(\beta) = 1$ and $q(\beta) = \frac{1}{\alpha}$, then this fuzzy General transform (9) gives the fuzzy Yang transform (6).

**Definition 3.5.** *The fuzzy inverse General transform is given by*

$$J_t^{-1}[\Psi(\beta)] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{p(\beta)} e^{q(\beta)t} q'(\beta) \odot \Psi(\beta) d\beta, \tag{10}$$

*where $\Psi(\beta)$must be analytic for all $\beta$ such that $Re\beta > a$.*

**Definition 3.6.** *A fuzzy function $\psi : \mathbb{R}_+ \to E^1$ is called to be of exponential order $d > 0$ if there exists a positive constant $Q$, such that for all $t > T$*

$$D(\psi(t), \tilde{0}) \leq Qe^{dt}.$$

**Theorem 3.4.** *Let $\psi : \mathbb{R}_+ \to E^1$ be a continuous fuzzy function in every finite interval $(0,T)$ of exponential order $d > 0$. Then, the FGT of $\psi(t)$ exists for all $\beta$ provided $Re(q(\beta)) > d$.*

**Proof.** From definition 3.4, we get

$$\begin{aligned} D(\Psi(\beta), \tilde{0}) &= D(p(\beta) \int_0^\infty e^{-q(\beta)t} \odot \psi(t)dt, \tilde{0}) \leq p(\beta) \int_0^\infty e^{-q(\beta)t} D(\psi(t), \tilde{0})dt \leq \\ &\leq Qp(\beta) \int_0^\infty e^{-(q(\beta)-d)t} dt = \tfrac{Qp(\beta)}{q(\beta)-d}, \quad for\ Re(p(\beta)) > d, \end{aligned}$$

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where $J_t[\psi(t)] = \Psi(\beta)$. □

The classical General transform for some special functions is given in [26].

(i) $J_t[1] = \frac{p(\beta)}{q(\beta)}$, $J_t[t^n] = \frac{(n!)p(\beta)}{q^{n+1}(\beta)}$, where $n$ is positive integer;

(ii) $J_t[e^{dt}] = \frac{p(\beta)}{q(\beta)-d}$, where $d \in \mathbb{R}$;

(iii) $J_t[\sin dt] = \frac{dp(\beta)}{d^2+q^2(\beta)}$, $J_t[\cos dt] = \frac{p(\beta)q(\beta)}{d^2+q^2(\beta)}$, where $d \in \mathbb{R}$.

By Definition 3.4, we obtain the following useful properties for the fuzzy General transform.

**Theorem 3.5.** *Let $\psi_1,\ \psi_2 : \mathbb{R}_+ \to E^1$ be fuzzy functions for which the FGT exists. Then, the FGT of a functions $a_1 \odot \psi_1(t) \oplus a_2 \odot \psi_2(t)$ and $a_1 \odot \psi_1(t) \ominus_{gH} a_2 \odot \psi_2(t)$ exist and*

*(i)* $a_1 \odot G_t[\psi_1(x)] \ominus_{gH} a_2 \odot G_t[\psi_2(x)] = G_t[a_1 \odot \psi_1(x) \ominus_{gH} a_2 \odot \psi_2(x)]$;

*(ii)* $a_1 \odot G_t[\psi_1(x)] \oplus a_2 \odot G_t[\psi_2(x)] = G_t[a_1 \odot \psi_1(x) \oplus a_2 \odot \psi_2(x)]$,

*where $a_1,\ a_2 \in \mathbb{R}$ such that $a_1,\ a_2 \geq 0$ or $a_1,\ a_2 \leq 0$.*

**Proof.** Analogously in Theorem 3.2. □

**Theorem 3.6.** *Let $\psi : \mathbb{R}_+ \to E^1$ be continuous fuzzy functions of exponential order $d > 0$ and $\psi'_{gH}(t)$ be continuous in every finite closed interval $0 \leq t \leq T$. Then for $Re(q(\beta)) > d$ we have*

*(i)* $J_t[\psi'_{gH}(t)] = (-1)p(\beta) \odot \psi(0) \ominus_{gH} (-1)q(\beta) \odot \Psi(\beta)$;

*(ii)* $J_t[\psi''_{gH}(t)] = (-1)p(\beta) \odot \psi'_{gH}(0) \ominus_{gH} \left(p(\beta)q(\beta) \odot \psi(0) \ominus_{gH} q^2(\beta) \odot \Psi(\beta)\right)$,

*where $J_t[\psi(t)] = \Psi(\beta)$.*

**Proof.** Analogously in Theorem 3.3. □

**Remark 3.2.** *A similar procedure can be used for the fuzzy function $u(x,t)$ of two variables. Then the fuzzy General transform for partial derivatives of $u(x,t)$ is as follows:*

*(i)* $J_t[u'_{t,gH}(x,t)] = (-1)p(\beta) \odot u(x,0) \ominus_{gH} (-1)q(\beta) \odot U(x,\beta)$;

*(ii)* $J_t[u''_{tt,gH}(x,t)] = (-1)p(\beta) \odot u'_{t,gH}(x,0) \ominus_{gH} \left(p(\beta)q(\beta) \odot u(x,0) \ominus_{gH} q^2(\beta) \odot U(x,\beta)\right)$,

*where $J_t[u(x,t)] = U(x,\beta)$.*

## 4. Double Fuzzy Yang-General Transform

In this section, a new double fuzzy integral transform is introduced, which combines the first order Yang and General fuzzy transforms. The definition and some of the fundamental properties of double fuzzy Yang-General transform are introduced.

**Definition 4.1.** *Let $u : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$, the function $e^{-\frac{x}{\alpha}-q(\beta)t} \odot u(x,t)$ be improper fuzzy Riemann-integrable on $\mathbb{R}_+ \times \mathbb{R}_+$ for some $q(\beta) > 0$ and $\alpha > 0$. Then, the fuzzy Yang-General transform of a function $u(x,t)$ is defined as*

$$U(\alpha,\beta) = Y_x J_t[u(x,t)] = p(\beta) \int_0^\infty \int_0^\infty e^{-\frac{x}{\alpha}-q(\beta)t} \odot u(x,t)dxdt, \tag{11}$$

*where $p(\beta)$ is real positive function.*

**Definition 4.2.** *Double fuzzy inverse Yang-General transform denote by $Y_x^{-1}G_t^{-1}$ and*

$$Y_x^{-1}J_t^{-1}[U(\alpha,\beta)] = Y_x^{-1}[U(\alpha,t)] = u(x,t). \tag{12}$$

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**Definition 4.3.** *A fuzzy function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ is said to be of exponential order $c > 0$ and $d > 0$, if there exists a positive constant $K$ such that for all $x > X$ and $t > T$*

$$D(u(x,t), \tilde{0}) \leq Ke^{cx+dt}.$$

**Theorem 4.1.** *Let $u : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be a continuous fuzzy function in $(0, X) \times (0, T)$ of exponential order $c > 0$ and $d > 0$. Then, the DFY-GT of $u(x,t)$ exists for all $\alpha$ and $q(\beta)$ with $Re(\frac{1}{\alpha}) > c$ and $Re(q(\beta)) > d$.*

**Proof.** Let $Y_x J_t[u(x,t)] = U(\alpha, \beta))$. Then, we have

$$\begin{aligned} D(U(\alpha,\beta),\tilde{0}) &= D(p(\beta)\int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-q(\beta)t} \odot u(x,t)dxdt, \tilde{0}) \leq \\ &\leq p(\beta)\int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-q(\beta)t} D(u(x,t),\tilde{0})dxdt \leq \\ &\leq Kp(\beta)\int_0^\infty\int_0^\infty e^{-(\frac{1}{\alpha}-c)x} e^{-(q(\beta)-d)t}dxdt = \\ &= \frac{K\alpha p(\beta)}{(1-c\alpha)(q(\beta)-d)} \end{aligned}$$

for $Re(\frac{1}{\alpha}) > c$, $Re(q(\beta)) > d$. □

**Lemma 4.1.** *Let $\varphi,\ \psi : (0,\infty) \to E^1$ and $u(x,t) = \varphi(x)\psi(t)$. Then,*

$$Y_x J_t[u(x,t)] = Y_x[\varphi(x)]J_t[\psi(t)].$$

**Proof.** By Definition 4.1, we find

$$\begin{aligned} Y_x J_t[u(x,t)] &= Y_x J_t[\varphi(x)\psi(t)] = p(\beta)\int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-q(\beta)t} \odot (\varphi(x)\psi(t))dxdt = \\ &= \int_0^\infty e^{-\frac{x}{\alpha}} \odot \varphi(x)dx\ p(\beta)\int_0^\infty e^{-q(\beta)t} \odot \psi(t)dt = \\ &= Y_x[\varphi(x)]J_t[\psi(t)]. \end{aligned}$$

□

**Lemma 4.2.** *Let $u(x,t) = a \in E^1$. Then, for $x > 0$ and $t > 0$, we have*

$$Y_x J_t[u(x,t)] = \frac{\alpha p(\beta)}{q(\beta)} \odot a.$$

**Proof.** By Definition 4.1, we have

$$\begin{aligned} Y_x J_t[u(x,t)] &= p(\beta)\int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-q(\beta)t} \odot adxdt = \\ &= \left(p(\beta)\int_0^\infty e^{-\frac{x}{\alpha}}dx\int_0^\infty e^{-q(\beta)t}dt\right) \odot a = Y_x[1]J_t[1] \odot a = \frac{\alpha p(\beta)}{q(\beta)} \odot a. \end{aligned}$$

□

By using Yang and General transform on some special functions, we obtain

(i) $Y_x J_t[1] = \frac{\alpha p(\beta)}{q(\beta)}$;

(ii) $Y_x J_t[x^m t^n] = \frac{m!n!\alpha^{m+1}p(\beta)}{q^{n+1}(\beta)}$, where $m,\ n$ are positive integers;

(iii) $Y_x J_t[e^{cx+dt}] = \frac{\alpha p(\beta)}{(1-c\alpha)(q(\beta)-d)}$, where $c,\ d \in \mathbb{R}$;

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(iv) $\Upsilon_x J_t[e^{i(cx+dt)}] = \frac{\alpha p(\beta)}{(1-ic\alpha)(q(\beta)-id)} = \alpha p(\beta)\frac{(1+ic\alpha)(q(\beta)+id)}{(1+c^2\alpha^2)(q^2(\beta)+d^2)} =$

$= \alpha p(\beta)\frac{q(\beta)-cd\alpha+i(c\alpha q(\beta)+d)}{(1+c^2\alpha^2)(q^2(\beta)+d^2)}$, where $c,\ d \in \mathbb{R}$.

Consequently,

$$\Upsilon_x J_t[\cos(cx+dt)] = \alpha p(\beta)\frac{q(\beta)-cd\alpha}{(1+c^2\alpha^2)(q^2(\beta)+d^2)},$$

$$\Upsilon_x J_t[\sin(cx+dt)] = \alpha p(\beta)\frac{c\alpha q(\beta)+d}{(1+c^2\alpha^2)(q^2(\beta)+d^2)}.$$

Now, we present some properties of double fuzzy Yang-General transform.

Using Theorem 3.2 and Theorem 3.5 we obtain that double fuzzy Yang-General transform is a linear transformation.

**Theorem 4.2.** *Let $u_1,\ u_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be fuzzy-valued functions. Then*

*(i)* $a_1 \odot \Upsilon_x J_t[u_1(x,t)] \ominus_{gH} a_2 \odot \Upsilon_x J_t[u_2(x,t)] = \Upsilon_x J_t[a_1 \odot u_1(x,t) \ominus_{gH} a_2 \odot u_2(x,t)]$;

*(ii)* $a_1 \odot \Upsilon_x J_t[u_1(x,t)] \oplus a_2 \odot \Upsilon_x J_t[u_2(x,t)] = \Upsilon_x J_t[a_1 \odot u_1(x,t) \oplus a_2 \odot u_2(x,t)]$,

*where $a_1,\ a_2 \neq 0$.*

From Theorem 4.2 it follows that the double fuzzy inverse Yang-General transform is also a linear transformation.

**Theorem 4.3.** *Let $u : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be periodic function of periods $\xi$ and $\eta$ such that*

$$u(x+\xi, t+\eta) = u(x,t)$$

*and $\Upsilon_x J_t[u(x,t)]$ exists. Then,*

$$\Upsilon_x J_t[u(x,t)] = \frac{p(\beta)}{1-p(\beta)e^{-\frac{\xi}{\alpha}-q(\beta)\eta}}\int_0^{\xi}\int_0^{\eta} e^{-\frac{\xi}{\alpha}-q(\beta)\eta} \odot u(x,t)dxdt.$$

**Proof.** Using the Definition 4.1 and properties of improper fuzzy integral, we find

$$\begin{aligned}\Upsilon_x J_t[u(x,t)] &= p(\beta)\int_0^{\infty}\int_0^{\infty} e^{-\frac{x}{\alpha}-q(\beta)t} \odot u(x,t)dxdt = \\ &= p(\beta)\int_0^{\xi}\int_0^{\eta} e^{-\frac{x}{\alpha}-q(\beta)t} \odot u(x,t)dxdt \oplus p(\beta)\int_{\xi}^{\infty}\int_{\eta}^{\infty} e^{-\frac{x}{\alpha}-q(\beta)t} \odot u(x,t)dxdt.\end{aligned}$$

Putting $x = \xi + \rho$ and $t = \eta + \tau$ on second integral, we obtain

$$\begin{aligned}U(\alpha,\beta) &= p(\beta)\int_0^{\xi}\int_0^{\eta} e^{-\frac{x}{\alpha}-q(\beta)t} \odot u(x,t)dxdt \oplus \\ &\oplus p(\beta)\int_{\xi}^{\infty}\int_{\eta}^{\infty} e^{-\frac{\xi+\rho}{\alpha}-q(\beta)(\eta+\tau)} \odot u(\xi+\rho, \eta+\tau)d\rho d\tau.\end{aligned}$$

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Using the periodicity of the function $u(x,t)$ and Definition 4.1, we have

$$
\begin{aligned}
U(\alpha,\beta) &= p(\beta)\int_0^{\xi}\int_0^{\eta} e^{-\frac{x}{\alpha}-q(\beta)t}\odot u(x,t)dxdt\oplus \\
&\oplus p(\beta)e^{-\frac{\xi}{\alpha}-q(\beta)\eta}\int_0^{\infty}\int_0^{\infty} e^{-\frac{\rho}{\alpha}-q(\beta)(\tau)}\odot u(\rho,\tau)d\rho d\tau = \\
&= p(\beta)\int_0^{\xi}\int_0^{\eta} e^{-\frac{x}{\alpha}-q(\beta)t}\odot u(x,t)dxdt \oplus p(\beta)e^{-\frac{\xi}{\alpha}-q(\beta)\eta}U(\alpha,\beta).
\end{aligned}
$$

This equation can be simplified into

$$
U(\alpha,\beta) = \frac{p(\beta)}{1-p(\beta)e^{-\frac{\xi}{\alpha}-q(\beta)\eta}}\int_0^{\xi}\int_0^{\eta} e^{-\frac{\xi}{\alpha}-q(\beta)\eta}\odot u(x,t)dxdt.
$$

□

**Theorem 4.4.** *Let $Y_x J_t[u(x,t)] = U(\alpha,\beta)$. Then*

$$
Y_x J_t[u(x-\xi,t-\eta)H(x-\xi,t-\eta)] = e^{-\frac{\xi}{\alpha}-q(\beta)\eta}\odot U(\alpha,\beta), \tag{13}
$$

*where $H(x,t)$ is the Heaviside unit step function defined by*

$$
H(x-\xi,t-\eta) = \begin{cases} 1, & x>\xi,\ t>\eta \\ 0, & x<\xi,\ t<\eta. \end{cases}
$$

**Proof.** By Definition 4.1, we obtain

$$
\begin{aligned}
&Y_x J_t[u(x-\xi,t-\eta)H(x-\xi,t-\eta)] = \\
&= p(\beta)\int_0^{\infty}\int_0^{\infty} e^{-\frac{\xi}{\alpha}-q(\beta)\eta}\odot u(x-\xi,t-\eta)H(x-\xi,t-\eta)dxdt = \\
&= p(\beta)\int_{\xi}^{\infty}\int_{\eta}^{\infty} e^{-\frac{\xi}{\alpha}-q(\beta)\eta}\odot u(x-\xi,t-\eta)dxdt.
\end{aligned}
$$

Substituting $x-\xi=\tau$ and $t-\eta=\delta$, we get

$$
\begin{aligned}
&Y_x J_t[u(x-\xi,t-\eta)H(x-\xi,t-\eta)] = \\
&= p(\beta)e^{-\frac{\xi}{\alpha}-q(\beta)\eta}\int_0^{\infty}\int_0^{\infty} e^{-\frac{\tau}{\alpha}-q(\beta)\delta}\odot u(\tau,\delta)d\tau d\delta = \\
&= p(\beta)e^{-\frac{\xi}{\alpha}-q(\beta)\eta}\odot u(\alpha,\beta).
\end{aligned}
$$

□

**Theorem 4.5.** *Let $Y_x J_t[u(x,t)] = U(\alpha,\beta)$. Then*

*(i)* $Y_x J_t[u'_{x,gH}(x,t)] = (-1)\odot J_t[u(0,t)] \ominus_{gH} (-1)\frac{1}{\alpha}\odot U(\alpha,\beta)$;

*(ii)* $Y_x J_t[u'_{t,gH}(x,t)] = (-1)p(\beta)\odot Y_x[u(x,0)] \ominus_{gH} (-1)q(\beta)\odot U(\alpha,\beta)$;

*(iii)* $Y_x J_t[u''_{xx}(x,t)] = (-1)\odot J_t[u'_x(0,t)] \ominus_{gH} \left(\frac{1}{\alpha}\odot J_t[u(0,t)] \ominus_{gH} \frac{1}{\alpha^2}\odot U(\alpha,\beta)\right)$;

*(iv)* $Y_x J_t[u''_{tt,gH}(x,t)] =$

$= (-1)p(\beta)\odot Y_x[u'_{t,gH}(x,0)] \ominus_{gH} \left(p(\beta)q(\beta)\odot Y_x[u(x,0)] \ominus_{gH} q^2(\beta)\odot U(\alpha,\beta)\right)$.

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**Proof.** Using Definition 4.1 and Remark 3.1, we get

$$Y_x J_t[u'_{x,gH}(x,t)] = J_t[Y_x[u'_{x,gH}(x,t)]] = J_t\Big[(-1)\odot u(0,t)\ominus_{gH}(-1)\tfrac{1}{\alpha}\odot Y_x[u(x,t)]\Big] =$$
$$= (-1)\odot J_t[u(0,t)]\ominus_{gH}(-1)\tfrac{1}{\alpha}\odot J_t[Y_x[u(x,t)]] =$$
$$= (-1)\odot J_t[u(0,t)]\ominus_{gH}(-1)\tfrac{1}{\alpha}\odot U(\alpha,\beta).$$

A similar procedure can be used to prove the case $(iii)$.

$$Y_x J_t[u''_{xx,gH}(x,t)] = J_t[Y_x[u''_{xx,gH}(x,t)]] =$$
$$= J_t\Big[(-1)\odot u'_{x,gH}(0,t)\ominus_{gH}\Big(\tfrac{1}{\alpha}\odot u(0,t)\ominus_{gH}\tfrac{1}{\alpha^2}\odot Y_x[u(x,t)]\Big)\Big] =$$
$$= (-1)\odot J_t[u'_{x,gH}(0,t)]\ominus_{gH}\Big(\tfrac{1}{\alpha}\odot J_t[u(0,t)]\ominus_{gH}\tfrac{1}{\alpha^2}\odot J_t[Y_x[u(x,t)]]\Big) =$$
$$= (-1)\odot J_t[u'_{x,gH}(0,t)]\ominus_{gH}\Big(\tfrac{1}{\alpha}\odot J_t[u(0,t)]\ominus_{gH}\tfrac{1}{\alpha^2}\odot U(\alpha,\beta)\Big).$$

By Definition 4.1 and Remark 3.2, we obtain

$$Y_x J_t[u'_{t,gH}(x,t)] = Y_x[J_t[u'_{t,gH}(x,t)]] =$$
$$= Y_x[(-1)p(\beta)\odot u(x,0)\ominus_{gH}(-1)q(\beta)\odot J_t[u(x,t)]] =$$
$$= (-1)p(\beta)\odot Y_x[u(x,0)]\ominus_{gH}(-1)q(\beta)\odot Y_x[J_t[u(x,t)]] =$$
$$= (-1)p(\beta)\odot Y_x[u(x,0)]\ominus_{gH}(-1)q(\beta)\odot U(\alpha,\beta)$$

and

$$Y_x J_t[u''_{tt,gH}(x,t)] = Y_x[J_t[u''_{tt,gH}(x,t)]] =$$
$$= Y_x\Big[(-1)p(\beta)\odot u'_{t,gH}(x,0)\ominus_{gH}(p(\beta)q(\beta)\odot u(x,0)\ominus_{gH}q^2(\beta)\odot J_t[u(x,t)])\Big] =$$
$$= (-1)p(\beta)\odot Y_x[u'_{t,gH}(x,0)]\ominus_{gH}(p(\beta)q(\beta)\odot Y_x[u(x,0)]\ominus_{gH}q^2(\beta)\odot Y_x[J_t[u(x,t)]]) =$$
$$= (-1)p(\beta)\odot Y_x[u'_{t,gH}(x,0)]\ominus_{gH}(p(\beta)q(\beta)\odot Y_x[u(x,0)]\ominus_{gH}q^2(\beta)\odot U(\alpha,\beta)).$$

□

## 5. Applications of Double Fuzzy Yang-General Transform

In this section , we study the application of DFY-GT and introduce the solution framework for the fuzzy advection-diffusion equation. Consider the following linear advection-diffusion fuzzy partial differential equations with constant diffusion coefficient $a > 0$ and constant advection velocity $b$

$$u'_t(x,t) = a\odot u''_{xx}(x,t)\ominus_{gH} b\odot u'_x(x,t), \quad for \ \ (x,t)\in\mathbb{R}_+\times\mathbb{R}_+ \tag{14}$$

with initial conditions

$$u(x,0) = f_0(x), \tag{15}$$

and boundary conditions

$$u(0,t) = g_0(t), \ \ u'_x(0,t) = g_1(t). \tag{16}$$

Applying DFY-GT on both side of the equation (14), we obtain

$$Y_x J_t[u'_t(x,t)] = a\odot Y_x J_t[u''_{xx}(x,t)]\ominus_{gH} b\odot Y_x J_t[u'_x(x,t)].$$

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Using Theorem 4.5, we get

$$(-1)p(\beta)\odot Y_x[u(x,0)]\ominus_{gH}(-1)q(\beta)\odot U(\alpha,\beta)=$$
$$=a\Big[(-1)\odot J_t[u'_x(0,t)]\ominus_{gH}\Big(\tfrac{1}{\alpha}\odot J_t[u(0,t)]\ominus_{gH}\tfrac{1}{\alpha^2}\odot U(\alpha,\beta)\Big)\Big]\ominus_{gH}$$
$$\ominus_{gH}b\Big[(-1)\odot J_t[u(0,t)]\ominus_{gH}(-1)\tfrac{1}{\alpha}\odot U(\alpha,\beta)\Big],$$

where $U(\alpha,\beta)=Y_xJ_t[u(x,t)]$. Using initial conditions (15) and boundary conditions (16), we have

$$(-1)p(\beta)\odot F_0(\alpha)\ominus_{gH}(-1)q(\beta)\odot U(\alpha,\beta)=$$
$$=(-1)a\odot G_1(\beta)\ominus_{gH}\Big(\tfrac{a}{\alpha}\odot G_0(\beta)\ominus_{gH}\tfrac{a}{\alpha^2}\odot U(\alpha,\beta)\Big)\ominus_{gH}$$
$$\ominus_{gH}\Big[(-1)b\odot G_0(\beta)\ominus_{gH}(-1)\tfrac{b}{\alpha}\odot U(\alpha,\beta)\Big],$$

where $F_0(\alpha)=Y_x[f_0(x)]$, $G_0(\beta)=J_t[u(0,t)]$, $G_1(\beta)=J_t[u'_x(0,t)]$.

By using Proposition 2.1, we obtain

$$\Big(\tfrac{b}{\alpha}-\tfrac{a}{\alpha^2}+q(\beta)\Big)\odot U(\alpha,\beta)=$$
$$=p(\beta)\odot F_0(\alpha)\oplus\big(b-\tfrac{a}{\alpha}\big)\odot G_0(\beta)\oplus(-a)\odot G_1(\beta).$$

Hence

$$\begin{aligned}U(\alpha,\beta)\ &=\tfrac{\alpha p(\beta)}{\alpha b-a+\alpha^2q(\beta)}\odot F_0(\alpha)\oplus\tfrac{\alpha(\alpha b-a)}{\alpha b-a+\alpha^2q(\beta)}\odot G_0(\beta)\oplus\\ &\oplus\tfrac{-\alpha^2a}{\alpha b-a+\alpha^2q(\beta)}\odot G_1(\beta),\end{aligned}\tag{17}$$

Applying the inverse DFY-GT we obtain $u(x,t)$.

## 6. Examples

**Example 6.1.** *Let us consider the fuzzy advection–diffusion equation*

$$u'_t(x,t)=u''_{xx}(x,t)\ominus_{gH}u'_x(x,t),\quad (x,t)\in\mathbb{R}_+\times\mathbb{R}_+\tag{18}$$

*with initial conditions*

$$u(x,0,r)=(e^x-x)\odot(1,2,3)\tag{19}$$

*and boundary conditions*

$$u(0,t,r)=(1+t)\odot(1,2,3),\quad u'_x(0,t,r)=0\odot(1,2,3).\tag{20}$$

*Applying FYT to the initial conditions and FGT to the boundary conditions, we find*

$$F_0(\alpha)=\left(\frac{\alpha}{1-\alpha}-\alpha^2\right)\odot(1,2,3),$$
$$G_0(\beta)=\left(\frac{p(\beta)}{q(\beta)}+\frac{p(\beta)}{q^2(\beta)}\right)\odot(1,2,3),\quad G_1(\beta)=0\odot(1,2,3).$$

*In this case $a=1$ and $b=1$. By using the equations (17), we obtain*

$$U(\alpha,\beta)=\frac{\alpha p(\beta)}{\alpha-1+\alpha^2q(\beta)}\odot F_0(\alpha)\oplus\frac{\alpha(\alpha-1)}{\alpha-1+\alpha^2q(\beta)}\odot G_0(\beta).$$

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*Hence*

$$U(\alpha,\beta) = \frac{\alpha p(\beta)}{\alpha-1+\alpha^2 q(\beta)}\left(\frac{\alpha}{1-\alpha}-\alpha^2\right)\odot(1,2,3)\oplus\frac{\alpha(\alpha-1)}{\alpha-1+\alpha^2 q(\beta)}\left(\frac{p(\beta)}{q(\beta)}+\frac{p(\beta)}{q^2(\beta)}\right)\odot(1,2,3) =$$
$$= \left(\frac{\alpha p(\beta)}{(1-\alpha)q(\beta)}-\frac{\alpha^2 p(\beta)}{q(\beta)}+\frac{\alpha p(\beta)}{q^2(\beta)}\right)\odot(1,2,3).$$

*Applying inverse double fuzzy Yang-General transform we find the solution of equation (18) - (20) is*

$$u(x,t,r) = (e^x - x + t)\odot(1,2,3).$$

## 7. Conclusions

In this paper, a new double fuzzy transform called DFY-GT was introduced. Some fundamental properties of this transform are presented. New theorems related to the existence, linearity, periodicity, and gH-partial derivatives were proven. These results were used to obtain a new simple formula for solving the linear fuzzy advection-diffusion equation. Finally, we construct a numerical example and get the exact solution of the equation considering applying the new double fuzzy integral transform.

In the future applications of the DFY-GT will be developed in the future and utilized to solve fuzzy partial integro- differential equations and fuzzy partial differential equations with variable coefficients.

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