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$C_3 \times Z_2$ Interference Geometry and a Light-Fermion Mass Cascade

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Abstract

The charged-lepton Koide relation $Q = (\sum m_\ell)/(\sum \sqrt{m_\ell})^2 = 2/3$ has held to one part in 10^5 for over four decades without an accepted derivation. The same rational value 2/3 is forced by C_3 phase-cancellation identities in three-beam interference, where three coherent sources at 120° separation produce hexagonal patterns with a fixed quadratic intensity ratio. Whether the shared 2/3 is a surface coincidence or reflects deeper structural overlap between interference geometry and the fermion mass spectrum is the question this paper investigates. We show that a companion identity (the F-identity) reduces to the classical Descartes circle theorem at $Q = 2/3$, and that integer-wavelength resonance conditions at $\sin\theta = 2/(3N)$ provide a geometric counterpart to Shulga's compact-cycle offset $\delta = 2/9$. A geometric-mean cascade motivated by hierarchies previously noted in the quark-mass literature suggests fixed closed-form targets for three light-quark masses (m_s, m_d, m_u) from a single input $\mu_\star = \sum m_\ell = 1883.1$ MeV, with internal self-consistency at 0.06% and deviations from PDG/FLAG reference values within current uncertainties. The algebraic overlap is not limited to the shared numerical value 2/3: the C_3 phase-cancellation algebra that forces $Q = 2/3$ in interference also fixes the companion identity and the Descartes connection, while the associated integer-wavelength resonance conditions place $\delta = 2/9$ in the same geometric framework. The construction does not propose a mechanism for fermion masses; we present these correspondences as empirical constraints on whatever dynamics produces the fermion spectrum, not as a theory of those dynamics. This work appears within the same month as three independent publications on the Koide relation (Section 10), suggesting the tools for understanding it may be maturing.

Keywords: Koide relation; light-quark masses; fermion mass spectrum; C_3 symmetry; three-beam interference; Descartes circle theorem; Berry phase; geometric mean cascade; Z_2 symmetry

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1. Introduction

Electron-positron pair production is one of three possible charged-lepton channels: a photon above threshold may produce e^+e^-, $\mu^+\mu^-$, or $\tau^+\tau^-$. It is the conversion of electromagnetic field energy into charged fermionic excitations whose masses are already encoded in the underlying quantum field theory; the threshold $2m_e$ represents the lowest energy scale at which charged-lepton pair excitations can be produced from the vacuum. Dumlu and Dunne [30,34] showed that pair production in time-dependent laser fields exhibits discrete resonance patterns in the momentum spectrum, with the production rate governed by interference between turning-point pairs, producing the same $\cos(2\alpha)$ structure as multi-slit diffraction. Akkermans and Dunne [38] extended this to Ramsey-fringe and time-domain multiple-slit interference directly from vacuum.

Three-beam laser interference provides a macroscopic parallel. Three coherent beams separated by 120° produce hexagonal intensity patterns [1,2]: a textbook result in interference lithography. The C_3 phase-cancellation identities underlying this pattern force the quadratic intensity ratio to $Q = 2/3$, independent of the phase offset (Section 2). This is the same rational value that defines the charged-lepton Koide relation [18,19], a mass-spectrum identity whose precision and persistence remain unexplained. That a mass formula shares its defining algebraic ratio with the mathematics

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governing pair production invites the question: is this a coincidence, or a diagnostic signature of common structure?

The Koide relation has been addressed via family gauge symmetry [7] and compact-cycle Berry phase [8]; its history and early extensions are reviewed in [37]. Koide-like relations also appear in the quark sector, though this is less widely known. Rivero [9] showed that the (s, c, b) triple satisfies $Q \approx 2/3$ at pole masses; under pure QCD, all quark mass ratios share the same anomalous dimension, so Q is approximately RG-invariant (broken only by small QED corrections). Rodejohann and Zhang [11] found an analogous relation for the (c, b, t) sector; because the top quark decays before hadronization ($\tau_t \sim 5 \times 10^{-25}$ s), it does not run under QCD, forcing the (c, b, t) Koide to hold at a specific scale rather than being scale-invariant. (The top is also anomalous in that its pole mass is perturbatively well-defined, free of the infrared renormalon ambiguity that afflicts lighter quarks, so any (c, b, t) comparison inherently mixes scheme-clean and scheme-ambiguous masses.) Rivero [10] recently obtained new Koide-type sum rules for down-type quarks. Gao and Li [35] showed that a Koide-like ratio for all six quarks is approximately RG-independent but persistently larger than 2/3 by $\sim$5%; Żenczykowski [36] explored doubly special parametrizations. The gap in this landscape is the light-quark sector: u, d, s are confined, cannot be isolated, and no common perturbative scale exists at which all three can be simultaneously evaluated. The present approach fills this gap not by claiming $Q = 2/3$ for (u, d, s) (the data do not support that) but by using the lepton-sector $C_3 \times Z_2$ identities to bridge to m_s via the F-identity, then cascading to m_d and m_u through geometric means. The light-quark masses are derived from lepton-sector identities; the construction does not require or claim that the (u, d, s) triple independently satisfies $Q = 2/3$.

Two of the mathematical tools used below are native to condensed matter: geometric-mean hierarchies in multi-phase media, and Berry phases / topological invariants on compact manifolds [4,5,12] (general background in [3]). Cross-domain applications of these tools to particle physics have been explored at book length by Volovik [6]. The C_3 phase-cancellation identity used here arises directly from the threefold geometry rather than from any of these references.

We emphasize what this paper does and does not claim. It does not propose that pair-production interference causes fermion masses or that the interference pattern is a dynamical mechanism. It observes that the algebraic identities forced by $C_3 \times Z_2$ symmetry in any physical realization (interference, crystallography, vortex systems) are the same identities the fermion mass spectrum satisfies. A snowflake is hexagonal because water's bonding geometry has sixfold character, not because hexagons cause ice; the pattern is diagnostic of the underlying structure. If the algebraic relations documented here survive further tests, they constrain the symmetry class of any future dynamical theory; if they do not, the $C_3 \times Z_2$ interference framework stands on its own mathematical terms. We report the correspondence and propose it merits investigation.

2. C_3 Phase Cancellation

Place $N = 3$ identical sources at 120° intervals on a circle. Three coherent beams at these angles produce an intensity pattern whose structure follows from two trigonometric identities that hold for any offset δ:

$$\sum_{k=0}^{2} \cos\theta_k = 0, \tag{1}$$

$$\sum_{k=0}^{2} \cos^2\theta_k = \tfrac{3}{2}, \tag{2}$$

where $\theta_k = \delta + 2\pi k/3$. The first identity encodes the C_3 phase cancellation underlying the hexagonal pattern; the second fixes the quadratic normalization.

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Writing the intensity at each sampling point in the cosine parametrization of Brannen [33] (with $\eta = 1/\sqrt{2}$; see also Foot [27] and Koide [19])

$$I_k = \frac{A}{6}\left(1 + \sqrt{2}\cos\theta_k\right)^2, \tag{3}$$

the total intensity is $\sum I_k = A$ (by Equations 1–2), and the ratio

$$Q \equiv \frac{\sum I_k}{(\sum \sqrt{I_k})^2} = \frac{2}{3} \tag{4}$$

is forced by the C_3 identities alone. The coefficient $\sqrt{2}$ is simultaneously fixed: requiring $\sum(1 + c\cos\theta_k)^2 = 6$ gives $c^2 = 2$. Neither Q nor $\sqrt{2}$ is fitted. The parametrization (3) can be viewed as an algebraic realization of the C_3 identities; whether it corresponds to a physical interference pattern in an internal space is an open question. Throughout, $\sqrt{I_k}$ denotes the positive branch; the C_3 identity is exact for signed roots, and the charged-lepton sector uses the branch where $1 + \sqrt{2}\cos\theta_k \geq 0$, which requires $\delta \in [-\pi/6, \pi/6] \pmod{2\pi/3}$; the value $\delta = 2/9 \approx 0.222$ satisfies this comfortably ($\pi/6 \approx 0.524$).

A Z_2 factor at each position distinguishes the two signs of $\sqrt{I_k}$, giving a sixfold $C_3 \times Z_2$ structure. Equivalently, writing $\sqrt{I_k} = x_0 + z_k$ with $\sum z_k = 0$, the condition $\text{avg}(z_k^2) = x_0^2$ forces $Q = 2/3$; this is the algebraic z-decomposition form Koide used in Phys. Rev. D [19] (see Foot [27] for the geometric interpretation).

3. Resonance Conditions

The hexagonal lattice constant from three-beam interference is [1]

$$a(\theta) = \frac{2\lambda}{3\sin\theta}. \tag{5}$$

Integer-wavelength resonances, $a = N\lambda$, occur at $\sin\theta = 2/(3N)$:

N	$\sin\theta$	a/λ	Interpretation
2	1/3	2	Z_2 sublattice doubling
3	2/9	3	Three wavelengths per cell
6	1/9	6	$C_3 \times Z_2$ fully resolved

The rational number 2/9 appears in three distinct roles: as $\sin\theta_3$ in the resonance table, as the Shulga orientation parameter δ in the charged-lepton spectrum, and as the compact-cycle Berry phase. The proposed connection is an equality of dimensionless rational values, not an identification of the physical incidence angle with the Koide phase.

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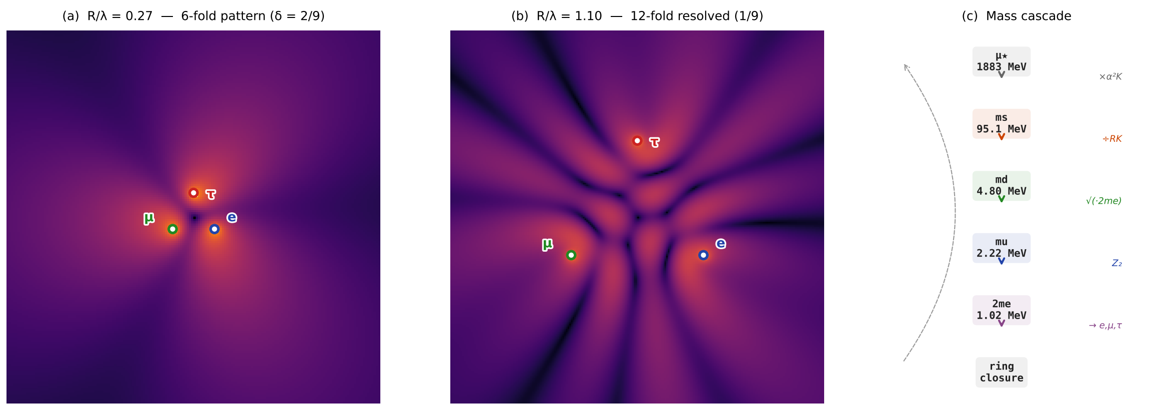

Figure 1. (a) Steady-state intensity $\langle\psi^2\rangle$ for three cylindrical-wave sources at 120° on a ring of radius $R/\lambda = 0.27$, showing the C_3 interference pattern with sixfold visual symmetry. (b) At $R/\lambda = 1.10$, additional fringes resolve between the primary arms, consistent with the $N = 6$ row of the resonance table (1/9 regime). Peak counts follow the sequence $3, 6, 9, 12, 15, 18, \ldots$ as R/λ increases, adding three peaks per step in the numerical examples studied here. Each resolved interference ring contributes one peak per C_3 source. (c) The geometric-mean cascade from $\mu_\star$ to $2m_e$, with ring closure.

At $N = 3$, three standing-wave antinodes fit inside one lattice cell. Numerical simulation of three cylindrical-wave point sources (2D Helmholtz equation, equal amplitude $A = 2/3$, wavelength $\lambda = 82$ grid units, exponential damping $e^{-\gamma r}$ with $\gamma = 0.004$) on a ring of radius R illustrates this: at $R/\lambda \approx 0.27$, six intensity peaks appear at $2\pi/6$ spacing. Doubling R resolves the sublattice structure, doubling the peak count and halving the sector width to 1/9. Simulation code is available from the corresponding author.

Shulga [8] independently derived $\delta = 2/9$ as a Berry phase on a compact family cycle, from integrating out higher Fourier harmonics on an internal circle. The resonance interpretation proposed here reads the same rational value as a boundary-condition analogue: the cycle accommodates three wavelengths, suggesting three selected modes. Shulga's compact-cycle calculation remains the derivation of $\delta = 2/9$; the resonance picture is a geometric correspondence. All prior Berry-phase-fixes-an-observable examples in condensed matter (TKNN, Zak, Jauregui; see Xiao *et al.* [12] for a review) give integer or π-rational values; a non-integer rational like 2/9 is, to our knowledge, without CM precedent outside Shulga's construction.

3.1. Quark-Sector Offsets

Żenczykowski [13] reported the low-energy values

$$\delta_U = \frac{2}{27}, \qquad \delta_D = \frac{4}{27}, \tag{6}$$

parametrized in his treatment via weak hypercharge Y. In the present interference picture, the same numerical offsets correspond to $\delta_U = Q/9$ and $\delta_D = 2Q/9$ with $Q = 2/3$, so the charge ratio reads $\delta_D/\delta_U = 2$. The factor 1/9 corresponds naturally to the $N = 6$ row, where the full $C_3 \times Z_2$ structure is resolved.

4. Effective Medium: Geometric Means

Geometric-mean hierarchies are independently established in the quark-mass literature: Ng [31] showed that the empirical relations $m_s^2 \approx m_d \cdot m_b$ and $m_c^2 \approx m_u \cdot m_t$ organize the quark spectrum. The present construction substitutes lepton-derived scales $(\mu_\star, 2m_e)$ for the heavy-quark boundaries:

$$m_s^2 = \mu_\star \cdot m_d, \tag{7}$$

$$m_u^2 = m_d \cdot 2m_e, \tag{8}$$

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treating adjacent scales as boundary values of a geometric-mean interpolation. The ansatz is adopted here as the simplest form consistent with the Z_2 structure; it is an empirical observation, not a derived result. The pair-production threshold $2m_e$ enters as the lightest scale at which charged fermions can be produced.

5. A Companion Identity

The C_3 phase cancellation and the geometric-mean cascade suggest a natural companion quantity. Given any three intensities I_1, I_2, I_3, define

$$F \equiv \sum_k \sqrt{I_k} - \sqrt{\sum_k I_k}. \tag{9}$$

The companion F^2 uses only addition, square roots, and subtraction.

The Descartes circle theorem [14,15] gives a companion via curvatures $\kappa_k = \sqrt{I_k}$: $\kappa_{\text{Soddy}} = \sum \kappa_k - 2\sqrt{\sum_{i<j} \kappa_i \kappa_j}$. Writing $E = \sum \kappa_k$, $P = \sum \kappa_k^2$, $C = \sum_{i<j} \kappa_i \kappa_j$, the Soddy formula subtracts $2\sqrt{C}$; the F-identity subtracts $\sqrt{P}$. They agree when $4C = P$:

$$4C = P \;\Leftrightarrow\; Q = \frac{P}{E^2} = \frac{2}{3}. \tag{10}$$

This is proved in Section 2: the C_3 cancellation forces $Q = 2/3$ identically. The companion identity therefore reduces to the Descartes circle theorem as a consequence of the interference geometry.

The identification $\kappa = \sqrt{m}$ as a curvature variable was noted by Kocik [16]. In condensed matter, $m^* \propto (\partial^2 E/\partial k^2)^{-1}$ is standard [3]; Satija [17] showed that the integer Apollonian gasket maps onto the Hofstadter butterfly energy spectrum, with curvatures encoding quantum Hall conductivities; this establishes a precedent for Descartes-type circle geometry appearing as a physical energy spectrum.

From $Q = 2/3$:

$$F^2 = \left(\tfrac{5}{2} - \sqrt{6}\right) A = \alpha_K^2\, A, \tag{11}$$

where $\alpha_K \equiv \sqrt{3/2} - 1$. The cascade ratio

$$R_K \equiv \frac{1}{\alpha_K^2} = \frac{2}{5 - 2\sqrt{6}} = 19.798 \tag{12}$$

is unitless and independent of δ.

6. Correspondence With Charged-Lepton Masses

The Koide relation [18,19] for charged-lepton pole masses,

$$Q = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3}, \tag{13}$$

is approximately RG-invariant [20]: the charged-lepton pole masses are scheme-independent observables, and under $\overline{\text{MS}}$ running all three acquire the same multiplicative QED factor at one loop, preserving Q exactly; residual two-loop threshold effects are $O(10^{-5})$. Setting $A = \mu_\star \equiv m_e + m_\mu + m_\tau = 1883.1$ MeV and $\delta = 2/9$ in Eq. 3 reproduces m_τ, m_μ, and m_e to four significant figures.

The companion evaluates to

$$F^2(e, \mu, \tau) = 95.12 \text{ MeV}. \tag{14}$$

All comparisons use PDG 2024 / FLAG 2024 values as a frozen reference convention. Since light-quark masses are scheme- and scale-dependent parameters rather than direct observables, the comparison below is convention-dependent; F^2, computed from charged-lepton pole masses, is compared with $\overline{\text{MS}}$ running masses at the same numerical scale $\mu_\star$. The stability of this comparison across conventions is audited in Appendix A. We note that $\mu_\star$ is not selected to optimize the match; it is the sum of

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charged-lepton pole masses, a scheme-independent physical observable that enters the construction before any quark-sector comparison is made. Comparison with PDG 2024 [21], run to $\mu_\star$ via four-loop $\overline{\mathrm{MS}}$ (RunDec 3 [22], $\alpha_s(M_Z) = 0.1180$, $N_f = 3$):

$$m_s^{\overline{\mathrm{MS}}}(\mu_\star) = 95.0 \pm 0.8 \text{ MeV}, \quad \Delta = +0.15\sigma. \tag{15}$$

The companion of the charged-lepton intensity pattern lies on the strange-quark $\overline{\mathrm{MS}}$ mass at $\mu_\star$ within current uncertainties. The comparison scale is not arbitrary: the residual $|F^2 - m_s^{\overline{\mathrm{MS}}}(\mu)|$ under four-loop QCD evolution is minimized at $\mu_\times = 1877$ MeV, within 0.33% of $\mu_\star$; conversely, $m_s^{\overline{\mathrm{MS}}}(\mu_\star) = F^2$ to $+0.15\sigma$ (Fig. 2). The two directions agree to better precision than either input is known. This coincidence is empirical; the interference model does not predict *which* particle the companion should match. Among QCD mass scales below 1 GeV, only m_s falls within the 1σ band of F^2; the pion decay constant $f_\pi = 92.1$ MeV lies 3.2% below, and no meson, baryon, or other quark mass falls within 5%.

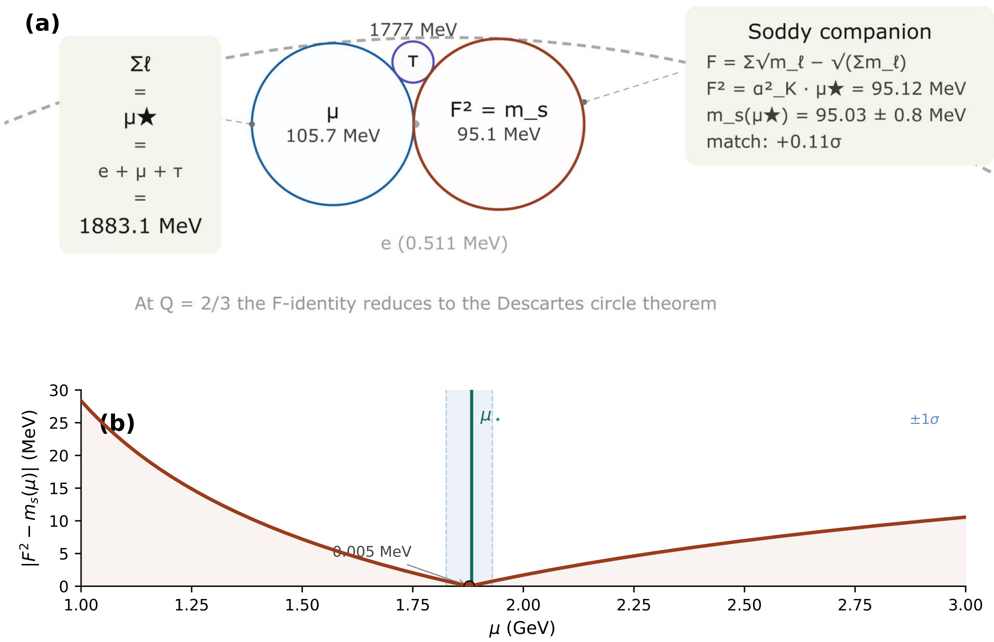

Figure 2. (a) Descartes circle geometry for the Koide–Soddy relation. Three lepton circles with curvatures $\kappa = \sqrt{m}$ determine a fourth tangent circle (the Soddy companion) whose squared curvature $F^2 = \alpha_K^2\,\mu_\star = 95.12$ MeV matches m_s. The electron circle (dashed arc) encloses the configuration; at this scale its radius is $\sim 50\times$ larger than μ or m_s, so only a segment is visible. (b) The residual $|F^2 - m_s^{\overline{\mathrm{MS}}}(\mu)|$ under four-loop QCD evolution (RunDec 3, $\alpha_s(M_Z) = 0.1180$, $N_f = 3$). The minimum (0.005 MeV) occurs at $\mu_\times = 1877$ MeV; the vertical line marks $\mu_\star = \Sigma m_\ell = 1883.1$ MeV. The shaded band is the $\pm 1\sigma$ region on m_s. Both $\mu_\times$ and $\mu_\star$ fall well within the band.

7. Light-Quark Cascade

The geometric-mean cascade (Equations 7–8) gives:

$$\begin{aligned} m_d &= \alpha_K^4\,\mu_\star = 4.804 \text{ MeV}, \\ m_u &= \alpha_K^2\sqrt{2m_e\,\mu_\star} = 2.216 \text{ MeV}. \end{aligned} \tag{16}$$

PDG 2024 gives $m_u(\mu_\star) = 2.20 \pm 0.07$ MeV ($+0.23\sigma$). QED corrections for charge-2/3 quarks reach $\sim 1\%$ (Appendix A), giving ± 0.02 MeV systematic. The relation selects $m_u = 2.216$ MeV. The quoted ± 0.02 MeV scale reflects the estimated size of the missing QED correction (QED is not applied; see Appendix A), not an experimental uncertainty. The $+0.23\sigma$ deviation is computed in a QCD-only scheme against PDG values.

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The prediction chain forms a loop:

$$\mu_\star \xrightarrow{\alpha_K^2} m_s \xrightarrow{\div R_K} m_d \xrightarrow{\sqrt{\cdot 2m_e}} m_u \to 2m_e \xrightarrow{Z_2} e \xrightarrow{\text{ring}} \mu_\star. \tag{17}$$

The light-quark sum, pair-normalized, follows from the cascade: $(m_u + m_d + m_s)/(2m_e) = \alpha_K^2 R(1 + \alpha_K^2 + R^{-1/2})$ where $R \equiv \mu_\star/(2m_e) = 1842.6$. This evaluates to 99.94, a self-consistency check rather than an independent prediction:

$$\frac{m_u + m_d + m_s}{2m_e} \approx 100. \tag{18}$$

FLAG 2024 [24] values run to $\mu_\star$ give 99.8 ± 0.6. This empirical closure provides a fourth equation for three unknowns; equations 7–8 plus the closure alone give $m_s = 95.17$ MeV versus $F^2 = 95.12$ MeV. Agreement: 0.058%.

The ratio m_s/m_d is RG-invariant under QCD [23]. FLAG 2024 [24] reports $m_s/m_d = 20.1 \pm 0.3$, consistent with $R_K = 19.80$ at 1.0σ.

8. Cross-Checks

The $F^2 \leftrightarrow m_s$ coincidence was subjected to pre-registered Monte Carlo analysis [25]: under a log-uniform prior for Koide-compatible lepton spectra, the conditional hit fraction is $\sim$0.34% (Clopper–Pearson 95% CI $[0.24\%, 0.48\%]$; range 0.25–0.59% across three priors), with pre-registered predictions archived at [26]. Figure 3 displays the constraint graphically: $F^2(\delta)$ is approximately flat near 95 MeV for small δ, then rises steeply; the value $\delta = 2/9$, fixed by the lepton masses, sits at the elbow where F^2 begins to depart the m_s band. The present paper extends the chain to four masses through parameter-free closed-form links (Table 1); no free parameter buffers a discrepancy at any step, so a single failure at the bridge propagates through the entire cascade. The observed residual at the bridge point is $F^2 - m_s(\mu_\star) = +0.12$ MeV $= +0.15\,\sigma_{m_s}$. The cascade outputs are fixed-point predictions against which future lattice determinations will converge or diverge.

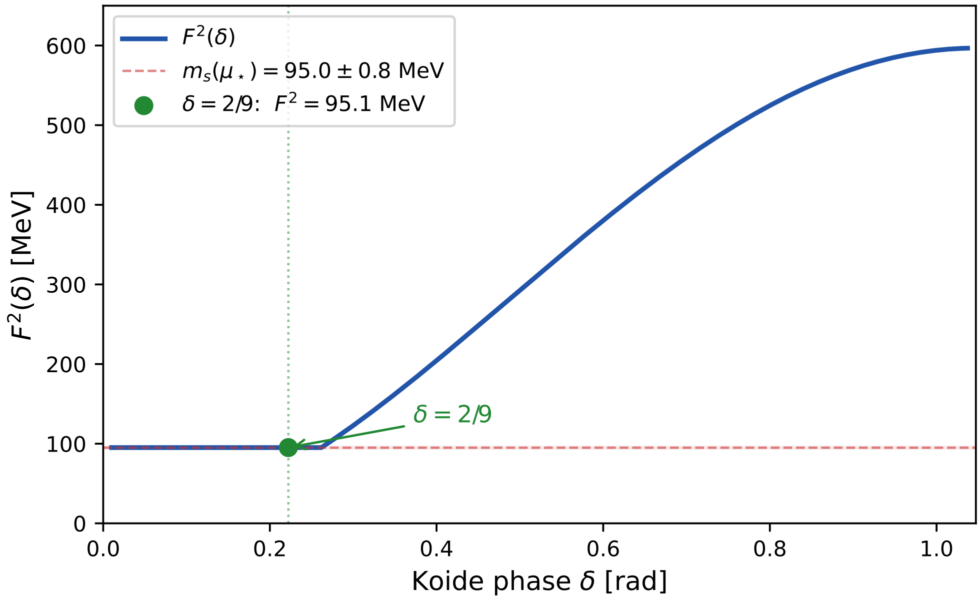

Figure 3. F^2 as a function of the Koide phase δ over the fundamental domain $[0, \pi/3)$. The dashed red line marks $m_s(\mu_\star) = 95.0 \pm 0.8$ MeV. F^2 intersects the m_s band at $\delta = 2/9$ (green dot), the value independently determined by the charged-lepton masses. For $\delta \gtrsim 0.3$, F^2 diverges from m_s rapidly, illustrating that the match is not generic.

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9. Frozen-Input Outputs And Comparisons

Table 1. Six masses from closed-form expressions. Input: $\mu_\star = m_e + m_\mu + m_\tau$. The lepton masses (τ, μ, e) are reproduced by construction via $\mu_\star$ and δ; they are inputs, not predictions. The quark masses (s, d, u) are the testable outputs.

	Closed-form	Pred.	PDG/FLAG	Dev.
τ	$\frac{\mu_\star}{6}(1+\sqrt{2}\cos\frac{2}{9})^2$	1776.93	1776.93	+0.00%
μ	$\frac{\mu_\star}{6}(1+\sqrt{2}\cos(\frac{2}{9}+\frac{4\pi}{3}))^2$	105.66	105.66	+0.00%
e	$\frac{\mu_\star}{6}(1+\sqrt{2}\cos(\frac{2}{9}+\frac{2\pi}{3}))^2$	0.5110	0.5110	−0.00%
s	$\alpha_K^2 \mu_\star$	95.12	95.0±0.8	$+0.15\sigma$
d	$\alpha_K^4 \mu_\star$	4.804	$4.75^{+0.49}_{-0.17}$	$+0.10\sigma$
u	$\alpha_K^2 \sqrt{2 m_e \mu_\star}$	2.216	$2.20^{+0.07}_{-0.05}$	$+0.23\sigma$

The outputs are chained: m_s determines m_d, which determines m_u. No free parameters buffer a discrepancy at any link.

Predictions were archived at Zenodo [26] prior to PDG 2026; any drift relative to updated PDG/FLAG values will be reported without retrofitting. The current agreements are suggestive but not yet decisive: the u and d quark masses carry large uncertainties (PDG quotes $\sim 3\%$ for m_u, $\sim 4\%$ for m_d), and the cascade values fall well within these bands. The true test is convergence: as lattice QCD+QED determinations sharpen over the next several years, the cascade outputs are fixed point values that will either be confirmed or excluded. Only that convergence, not the present-day overlap, can discriminate the construction from coincidence. Concretely, the strange-quark mass is the most sensitive discriminator: FLAG error bars on m_s have been shrinking at approximately 15% per review cycle, and a 1% drift in the central value combined with two further cycles of improvement (projected ~ 2030) would produce a $> 2\sigma$ exclusion. For m_u, the inclusion of QED corrections in future lattice determinations is expected to shift the central value at the $\sim 1\%$ level; the cascade output $m_u = 2.216$ MeV is a fixed target against which that shift will be measured.

10. Discussion

The central result is a diagnosis, not a mechanism. The charged-lepton and light-quark mass spectra satisfy a specific set of algebraic identities: the Koide ratio $Q = 2/3$, the Descartes companion $F^2 = \alpha_K^2 \mu_\star$, and the geometric-mean cascade $m_s^2 = \mu_\star \cdot m_d$. These are not generic relations that many mass spectra would satisfy. The Monte Carlo analysis (Section 8) tests the $F^2 \leftrightarrow m_s$ coincidence alone and reports a conditional hit fraction of $\sim 0.34\%$ (log-uniform prior; 0.25–0.59% across three priors); the full cascade chains three additional masses through parameter-free closed-form links, each of which must independently land within current uncertainties. These identities coincide with the characteristic algebraic constraints of $C_3 \times Z_2$ systems, forced by phase cancellation in any physical realization with threefold rotational symmetry and a twofold sign structure.

This identification constrains, but does not construct, the underlying theory. The Standard Model accommodates fermion masses through Yukawa couplings that are free parameters; the present work observes that six of those parameters are matched by closed-form geometric relations involving a single measured scale $\mu_\star$ and the dimensionless constant $\alpha_K = \sqrt{3/2} - 1$ fixed by the C_3 identities. For the charged leptons and light quarks, the construction effectively reduces six independent Yukawa couplings to one input, constraining the Yukawa sector rather than competing with it. The geometric-mean eigenvalue structure ($m_s^2 = m_d \cdot m_b$ in Ng's formulation [31], or $m_s^2 = \mu_\star \cdot m_d$ here) is mathematically equivalent to the texture-zero mass matrices introduced by Fritzsch [29]; that two independent routes (matrix ansätze and $C_3 \times Z_2$ identities) arrive at the same eigenvalue relations suggests both may be symptoms of a common algebraic cause.

The diagnostic value is specific: any dynamical theory that produces the observed fermion mass spectrum must, as a necessary consequence, reproduce the $C_3 \times Z_2$ identity structure documented here. This is a constraint on the solution space, not a solution. Whether the fermion spectrum ultimately

 doi:10.20944/preprints202605.1389.v2

reflects an internal compact space with threefold symmetry, or whether the algebraic correspondence has a different origin, remains open.

A structural limitation of the present comparison is that F^2 derives from charged-lepton pole masses (scheme-independent observables) while the light-quark masses it is compared against are scheme- and scale-dependent parameters. This scheme mixing is not an artefact of the construction; it is inherent to any comparison involving confined quarks, whose masses can only be defined within a renormalization convention. We have disclosed the conventions used, audited the comparison across PDG and FLAG reference values (Appendix A), and shown that the match is specific to $\mu_\star$ rather than generically compatible. As lattice QCD+QED determinations converge toward percent-level precision over the coming years, the cascade outputs provide fixed numerical targets that will either be confirmed or excluded without any change to the lepton-sector input.

We note that this work appears within weeks of three independent publications exploring related aspects of the Koide relation: Rivero [10] obtains approximately scale-invariant sum rules for down-type quarks in PLB, Shulga [8] derives $\delta = 2/9$ as a Berry phase on a compact family cycle, and Hübner [28] proves a minimization theorem for the Koide ratio and introduces an effective-participant interpretation $N_{\text{eff}} \equiv 1/Q$. Four groups arriving at geometric or algebraic readings of the same relation in the same month may indicate that the tools for understanding it are maturing. We caution that the preprints (including the present work) are not yet independently peer-reviewed; Rivero's PLB paper has passed review. The convergence is noted as context for the timeliness of the investigation, not as mutual corroboration.

AI usage disclosure. AI tools were used for programming assistance (Monte Carlo implementation, LaTeX formatting) and document editing only. All mathematical derivations, physical arguments, numerical predictions, and scientific claims originate from the author or from prior published work.

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Appendix A. Mass Running Procedure

Light quark masses are compared at $\mu_\star = 1883.1$ MeV. PDG 2024 reference values at 2 GeV [21]: $m_s^{\overline{\text{MS}}}(2) = 93.4 \pm 0.8$ MeV, $m_d^{\overline{\text{MS}}}(2) = 4.67^{+0.48}_{-0.17}$ MeV, $m_u^{\overline{\text{MS}}}(2) = 2.16^{+0.07}_{-0.05}$ MeV.

The scheme- and scale-independence of light-quark mass ratios was established by Gasser and Leutwyler [32]; running from 2 GeV to $\mu_\star$ uses four-loop QCD $\overline{\text{MS}}$ evolution (RunDec 3 [22], version 3.1) with $\alpha_s(M_Z) = 0.1180 \pm 0.0009$, $N_f = 3$ active flavors, charm threshold at $m_c^{\overline{\text{MS}}}(m_c) = 1.273$ GeV. No flavor thresholds are crossed in the interval $[\mu_\star, 2\text{ GeV}]$.

QED corrections are not applied. For charge-1/3 quarks (d, s) these are sub-per-mille. For charge-2/3 quarks (u), QED effects can reach $\sim 1\%$ and represent the dominant systematic uncertainty in the m_u comparison.

The ratio m_s/m_d is RG-invariant for same-charge quarks under QCD. FLAG 2024 [24] reports $m_s/m_{ud} = 27.23 \pm 0.10$ and $m_u/m_d = 0.474 \pm 0.020$ for $N_f = 2+1+1$, giving $m_s/m_d = 20.1 \pm 0.3$.

Numerical verification is available from the corresponding author.

Appendix A.1. Cross-Scheme Comparison

Table A1 shows cascade predictions against PDG and FLAG (QCD-only lattice) reference values at two scales: the cascade scale $\mu_\star$ and the conventional reference scale 2 GeV. The match is specific to $\mu_\star$; at 2 GeV, m_s deviates by $> 2\sigma$ in both PDG and FLAG, confirming that the cascade output is scale-locked rather than generically compatible.

Table A1. Cascade predictions vs. PDG/FLAG reference values across schemes and scales. FLAG values are QCD-only (no QED); the m_u entry carries a $\sim$1% systematic from the missing QED correction (†).

	Cascade	Ref. value	σ	Δ/σ
PDG 2024, $\overline{MS}$ at $\mu_\star$				
m_s	95.12	95.0 ± 0.8		$+0.15$
m_d	4.804	4.75 ± 0.17		$+0.32$
m_u	2.216	2.20 ± 0.07		$+0.23$
FLAG 2024 (QCD only), $\overline{MS}$ at $\mu_\star$				
m_s	95.12	95.0 ± 0.7		$+0.11$
m_d	4.804	4.78 ± 0.14		$+0.17$
$m_u{}^\dagger$	2.216	2.18 ± 0.08		$+0.48$
PDG 2024, $\overline{MS}$ at 2 GeV (wrong scale)				
m_s	95.12	93.4 ± 0.8		$+2.1$
m_d	4.804	4.67 ± 0.17		$+0.8$
m_u	2.216	2.16 ± 0.07		$+0.8$
Scale-invariant ratio				
m_s/m_d	19.80	20.1 ± 0.3		-1.0

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