 doi:10.20944/preprints202210.0420.v1

Article

A Concise Proof of Goldbach Conjecture

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Abstract: The Goldbach Conjecture, which is frequently termed as "1 + 1", has been a fascinating goal for many mathematicians over centuries. A Chinese mathematician, Dr. Jingrun Chen, proved 1 + 2, which is a great success and the best result so far achieved. Although there were several attempts proving the conjecture, these attempts are either tediously long, complicated, or logically imperfect, thus not widely accepted. Taking advantage of the periodicity of primes revealed recently, here the author provides a straight forward rigorous proof for the Conjecture.

Keywords: prime number; periodicity; Goldbach Conjecture; proof

1. Introduction

Goldbach Conjecture states "Every even number greater than 2 is the sum of two odd primes" (1 prime + 1 prime, frequently briefed as "1 + 1"). Although simple-appearing, the conjecture has been tantalizing mathematicians over centuries since 1742 [1-11]. Although the conjecture has been tested valid for all evens up to $4 \cdot 10^{18}$ [11] and several proofs were given for the conjecture [5, 7-10, 12, 13], these proofs are either tediously long, complicated, or logically imperfect or unclear. The general public expect a simple and comprehendible proof that does not lack rigorosity required for mathematics. Here the author aims to achieve this goal.

2. Definitions

Definition 1 The n_{th} prime is denoted as P_n.

Definition 2 A super product of prime P_n, denoted as X_n, is defined as the product of all primes smaller than P_n. Namely,

$$X_n = \prod_{i=1}^{n-1} P_i \qquad \text{(Wang, 2022[14])}$$

Table 1. List of the first ten super products of primes.

Super Product	Expression	Value
X_2	2	2
X_3	2 x 3	6
X_4	2 x 3 x 5	30
X_5	2 x 3 x 5 x 7	210
X_6	2 x 3 x 5 x 7 x 11	2,310
X_7	2 x 3 x 5 x 7 x 11 x 13	30,030
X_8	2 x 3 x 5 x 7 x 11 x 13 x 17	510,510
X_9	2 x 3 x 5 x 7 x 11 x 13 x 17 x 19	9,699,690
X_{10}	2 x 3 x 5 x 7 x 11 x 13 x 17 x 19 x 23	223,092,870
X_{11}	2 x 3 x 5 x 7 x 11 x 13 x 17 x 19 x 23 x 29	6,469,693,230

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3. Proof

3.1. Periodicity of primes

Although primes are notorious of their random occurrence, Dirichlet's theorem does predict the regular occurrence of certain primes in natural numbers. The theorem states that **there are infinitely many prime numbers in the collection of all numbers of the form *na* + *b***, in which the constants *a* and *b* are integers without a common divisor except 1 (namely, being relatively prime) and the variable *n* is any natural number. It is easy to see that **the numbers in the collection constitute an arithmetic progression (A.P.) with a common difference of *a***. This implication is clearly demonstrated in Figure 1 and 2.

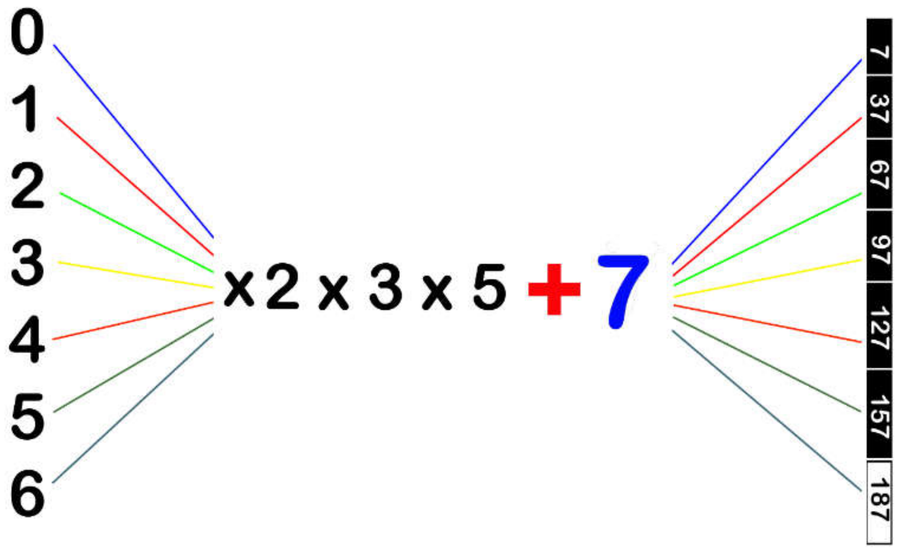

Figure 1. According to Dirichlet's theorem, the lack of a common factor > 1 shared between both sides of "+" implies potential primality for the sums on the right and a common difference of 30 between adjacent primes on the right, with one exception of 187.

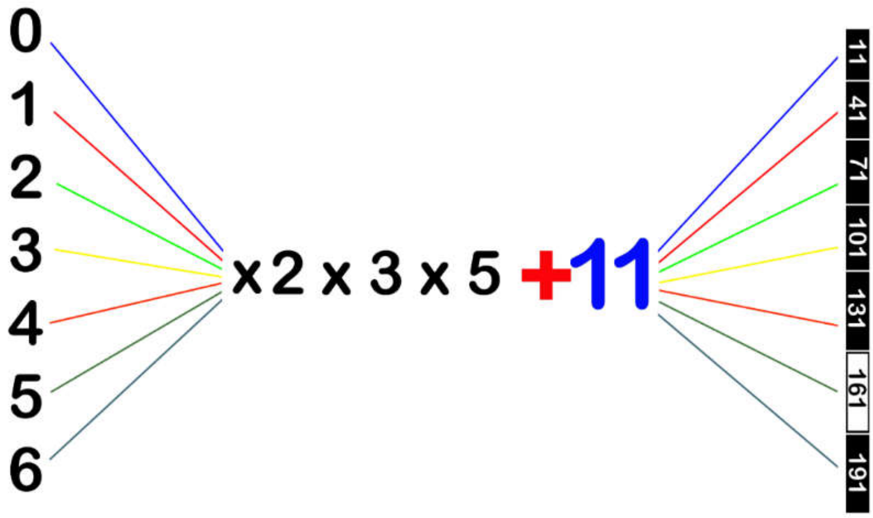

Figure 2. According to Dirichlet's theorem, the lack of a common factor > 1 shared between both sides of "+" implies potential primality for the sums on the right and a common difference of 30 between adjacent primes on the right, with one exception of 161.

The above cases of 7 and 11 in Figure 1 and 2 are not exceptional. Indeed, the same rule applies for other primes, including 13, 17, 19, 23, 29, and 31. After operations similar to those applied for 7 and 11 (Figure 1 and 2), it is easy to obtain other six similar 7-element arrays of numbers (prime or composite) corresponding to each prime in [13, 31], which

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are arithmetic progressions with a common difference of 30. Arranging these arithmetic progressions orderly on radii of circles, we can obtain the distribution of primes in [2, 211] (roughly the scope of X_5), as shown in Figure 3.

After similar operations, distribution of primes in [11, 2311] (roughly the scope of X_6) can be obtained, as shown in Figure 5.

3.2. Sums and differences of primes

It is noteworthy to analyze the features of primes in [3, 31] before we can make further inferences. If we collect all sums of arbitrary pairs of primes in [7, 31], it is obvious that such sums constitute **an arithmetic progression including all evens in [18, 52]** (Figure 4). In the meantime, if we collect all differences between arbitrary pairs of primes in [3, 31], the difference obviously also constitute **an arithmetic progression including all evens in [2, 28]** (Figure 4). These two outcomes are of crucial value for our following reasoning.

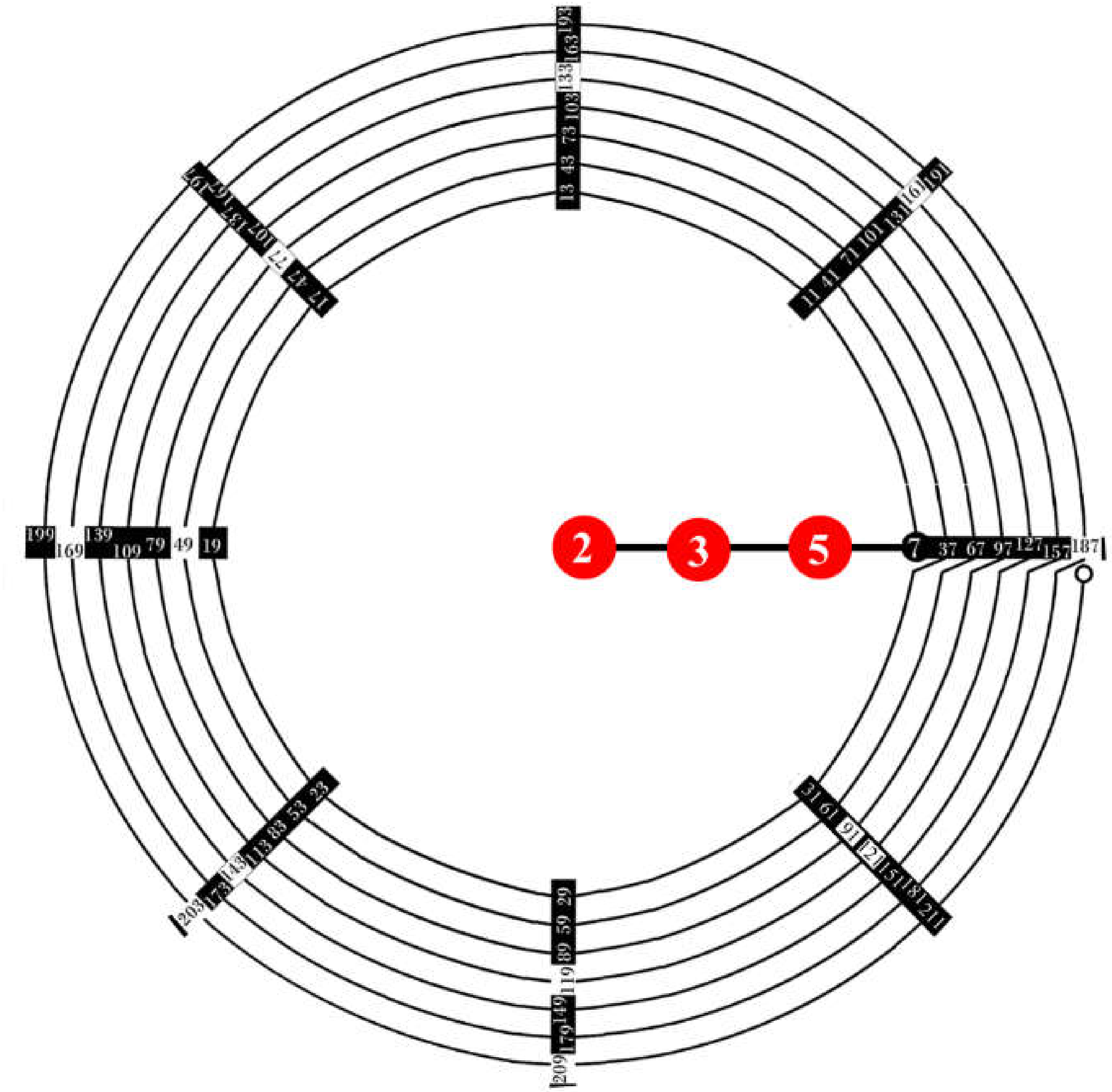

Figure 3. Combining the right sides of Figure 1, 2 as well as their counterparts for other primes (including 13, 17, 19, 23, 29, and 31, data not shown), the distribution of primes in [2, 211] is obtained.

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All primes in [7, 31] have their own 7-element series of primes that demonstrate clearly a periodicity of 30 (= 2 x 3 x 5 = X_4), with 12 exceptions. Modified from Wang [15].

First, all evens smaller than 18 are all sums of two primes, as
4 = 2 + 2
6 = 3 + 3
8 = 3 + 5
10 = 5 + 5 = 3 + 7
12 = 5 + 7
14 = 7 + 7
16 = 5 + 11 = 3 + 13

In addition to those shown in Figure 4, we can reach a conclusion that Goldbach Conjecture is true in [4, 52]. To be conservative and convenient, we can say that Goldbach Conjecture is true in [4, 30]. Namely, Goldbach Conjecture is true within the scope of X_4.

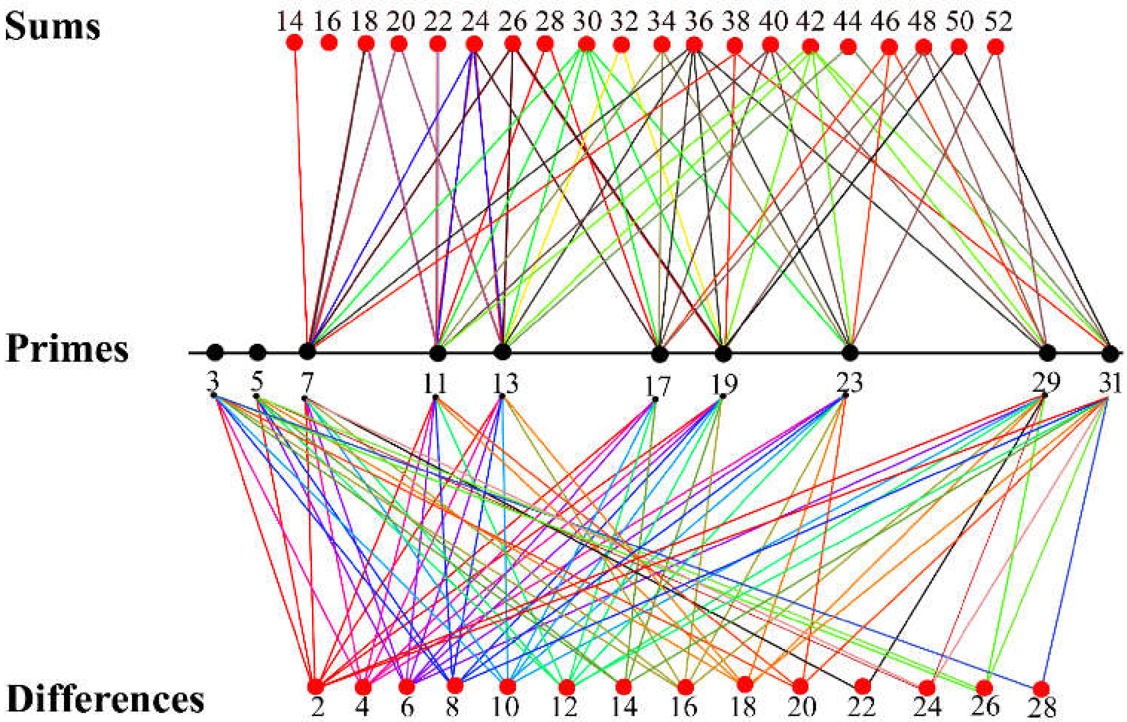

Figure 4. All differences among prime pairs in [3, 31] constitute an array of all evens in [2, 28], and all sums of prime pairs in [7, 31] constitute an array of all evens in [18, 52].

It is interesting observing that, for example, 37 + 11 = 48, and this sum can be increased by 30s to 78, 108, 138, and 168, respectively, by replacing 11 with any one of its peer primes on the same radius on the outer circles including 41, 71, 101, and 131, respectively. This is just one of many examples, as each of primes on the innermost circle in Figure 3 has its own similar peers on the same radius.

It is obvious that, as in Figure 3, the difference between two adjacent numbers (mostly primes) on the same radius is 30. If a number on such a radius is shifted to a position on adjacent outer circle, the value of the number increases by 30, correspondingly the sum related to this number increases by 30. Considering the sums of any prime pair in [7, 31] cover all evens in [18, 52] (Figure 4), shifting one number in a prime pair will increase the sums of the prime pair by 30. Applying this operation to all prime pairs, an array of such sums covering all evens in [48, 82] is obtained. Repeating this operation 5 more times, we will have five similar arrays covering all evens in [78, 112], [108, 142], [138, 172], [168, 202], [198, 232], respectively. Splicing these seven overlapping ranges, all evens in [18, 232] are proven to be sums of two primes. Namely, all evens in [18, 232] honor Goldbach Conjecture.

The above discussion concludes that Goldbach Conjecture is true up to 232. To be conservative and convenient, we can say that Goldbach Conjecture is true in [4, 210]. Namely, Goldbach Conjecture is true within the scope of a super product of primes, X_5.

3.3. Extending to the infinite

The above reasoning can be plausibly applied to greater scopes of X_n, where n is an arbitrary natural number.

As shown previously in [15], the distribution of primes demonstrates certain specific regularity within scopes defined by super products of primes X_n, and there are intrinsic pattern inheritable between adjacent scopes, namely, all numbers (except one) on all circles in a scope of X_n are the same numbers on the innermost circle of the next scope of X_{n+1}, and the number of circles in a scope equals to the initial number in the scope, P_n [15]. Through operation in 3.2, the valid scope of Goldbach Conjecture can be easily expanded from X_n to X_{n+1}. For example, Figure 3 demonstrates the distribution of primes in [7, 211] (which roughly matches the scope of super product of prime $X_5 = 210$) and includes 7 (= P_4) circles of numbers that starts with 7 (= P_4). Similarly, Figure 5 has all numbers on all circles in Figure 3, except 7, on its innermost circle and demonstrates the distribution of primes in [11, 2311] (which roughly matches the scope of super product of prime $X_6 = 2310$) and includes 11 (= P_5) circles of numbers that starts with 11.

It is conceivable (but not demonstrable here) that ensuing scopes will cover increasingly greater ranges of primes, for example, [13, 30030], [17, 510510], [19, 9699690] ……..corresponding to the scopes of X_6, X_7, X_8, …… respectively (Table 1). Obviously, there is no upper limit for such scopes and the applicable scope of the above generalization.

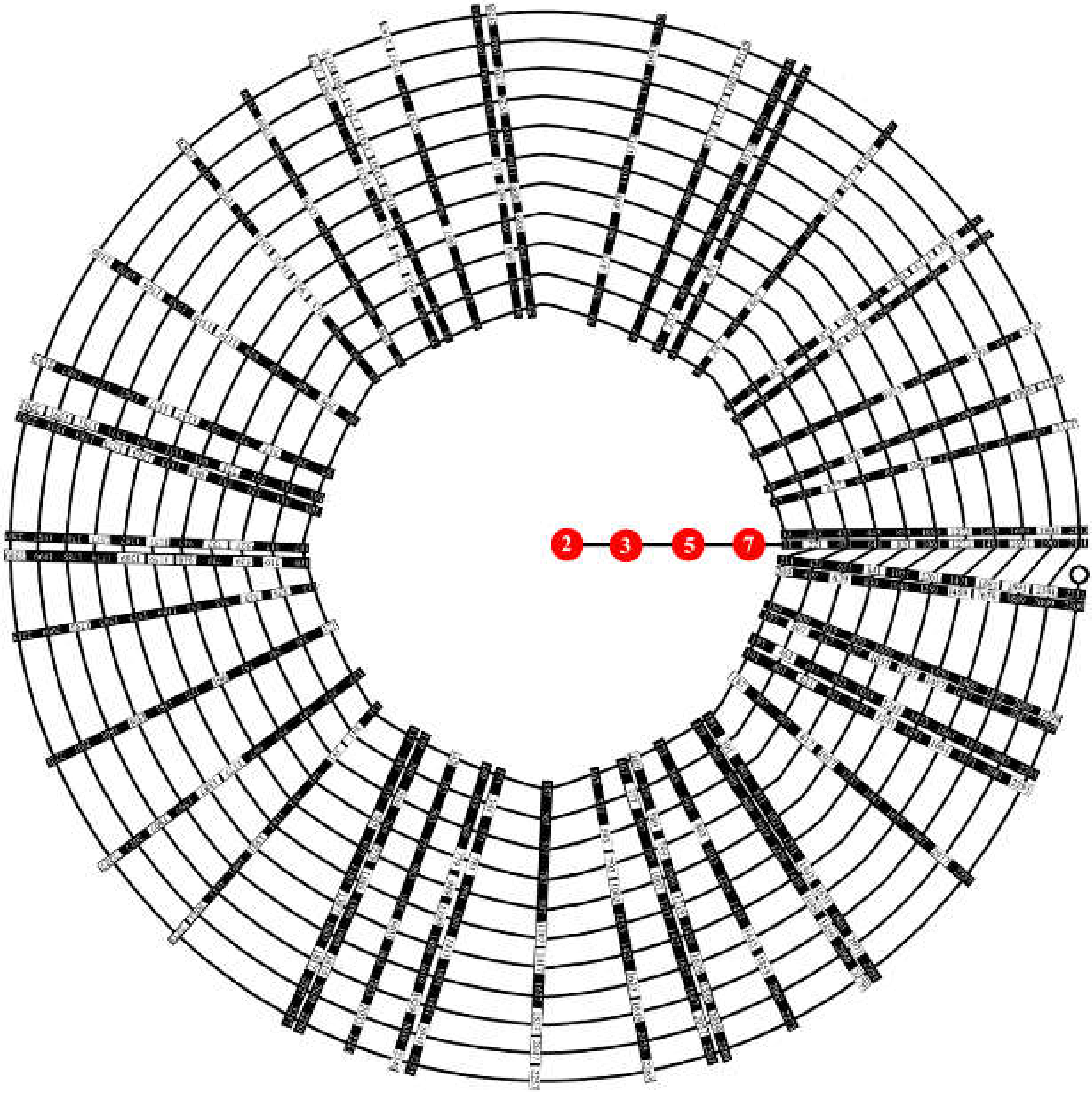

Figure 5. After similar operation as in Figure 1-3, this figure is obtained. Each prime in [11, 211] has its own 11-element series of primes that demonstrates clearly a periodicity of 210 (= 2 x 3 x 5 x 7 = X_5), with 188 exceptions. Note that all numbers in Figure 3 (except 7) are on the innermost circle here. Modified from Wang [15].

The scopes demonstrated here and in Wang (2021) [15] are just the first few of such super products of primes corresponding to first few of primes. These scopes and intrinsic pattern underlying primes within the scopes governs the deployment of primes in natural numbers. As shown in Figure 6, each red triangle represents a divide in the sequence of primes. All primes to the left of the divide are used to generate a super products of a prime, while all those to the right can be added to the super product to generate greater primes (or their candidates). As there are there infinite number of such divides, there are infinite number of super products of primes, which can reach the infinite (Table 1).

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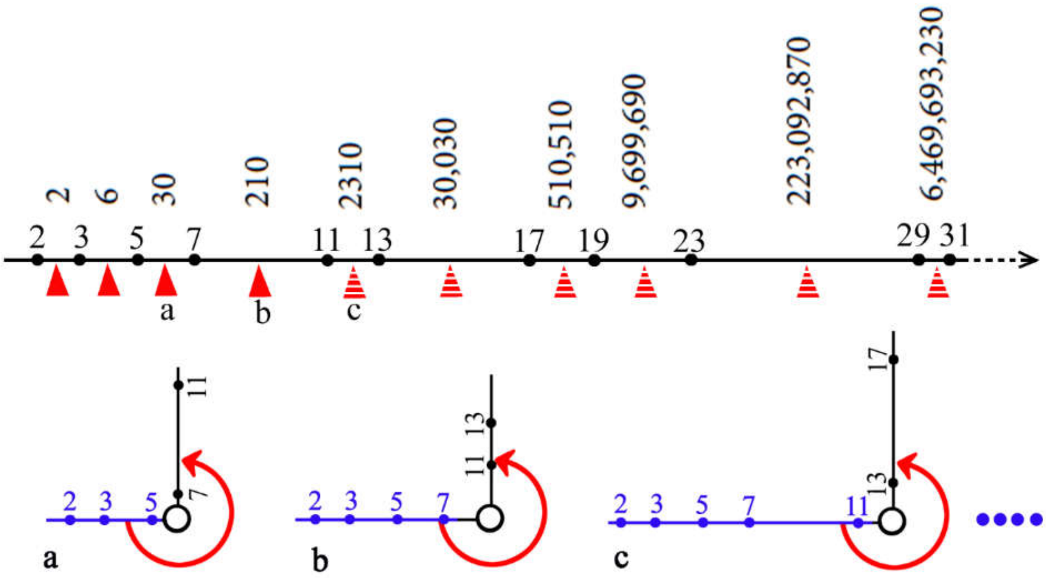

Figure 6. All primes on an axis may be divided into two segments by a divide (red triangles, solid or broken). To generate all primes in the scope of X_n (top), all prime smaller than the divide are used to generate a super product while those greater than the divide (and their products) in [P_n, $X_n + 1$] can be added to the super product X_n to give rise to all greater primes in scope [$X_n + 2$, $X_{n+1} + 1$], as shown in Figure 3 and 5. Corresponding to each divide, there is a scope of primes equal to X_n, as shown in Figure 1-5 in Wang[15] and Figure 3 and 5 in this paper. *a* and *b* correspond to the prime distribution in scopes demonstrated in Figure 3 and 5, respectively. Since there is an one-to-one correlation between scopes and primes, and the scopes increase as rapidly by P_n multiple (Table 1), further presentations similar to those in Figure 3 and 5 cover too many numbers to be shown here.

As Goldbach Conjecture holds for all scopes of X_n, which can be infinitely great, we can say that Goldbach Conjecture holds into the infinite.

This completes the proof of the Goldbach Conjecture, with some conservation due to exceptions that are dealt with below.

3.4. Skipping lacunae

3.4.1. Limited number and influence of exceptions

The above reasoning would be perfect if there were no exceptions in each scope of X_n. However, the fact is, as seen in Figure 3 and 5, there are exceptions to the above demonstrated scopes. These exceptions undermine the robustness of the above reasoning.

The occurrence of these exceptions is due to the influence of primes in [P_n, sqrt(X_{n+1})]. It is fortunate that the greater a prime is, the less its influence on primality of a number is. For example, 11 determines that only one number out of every 11 natural numbers can be divided exactly by 11 and thus cannot be a prime, while 101 determines that only one out of every 101 natural numbers can be divided exactly by 101 and thus cannot be a prime. For the scope shown in Figure 3, the number of exception caused by 7, 11, and 13 is 56 x (1/7 + 1/11 + 1/13) = 17.4. After minus the repeatedly counted cases (3), the estimation is 14.4, slightly greater than the actual number of exceptions 12. For the scope shown in Figure 5, the number of exception caused by 11, 13, 17, 19, 23, 29, 31, 37, 42, 43, and 47 is 528 x (1/7 + 1/11 + 1/13 + 1/17 + 1/19 + 1/23 + 1/29 + 1/31 + 1/37 + 1/41 + 1/43 + 1/47) = 256.3. After minus the repeatedly counted cases (55), the estimation is 201.3, slightly greater than the actual number of exceptions 188.

Limited number of exceptions restrict their influence on the robustness of our above reasoning.

3.4.2. Remedying the influence of exceptions

Although with limited influence, the negative influence of these exceptions have to be remedied before declaring a full success.

3.4.2.1 Vertical shifting

Vertical shifting designates replacing an exception number with a prime number on the same radius (as in Figure 3 and 5). For example, 119 is an exception in Figure 3. The existence of this exception influences any two-prime-sum related to 119. Since 186 = 119 + 67, therefore 186 may not be a sum of two primes. To prove the Goldbach Conjecture, we have to choose a prime to replace the composite 119. This replacing will alter the value of 119 (increase or decrease) and related sums. To offset this change, a counter operation (decrease or increase) must be applied on another prime in the pair (67). Namely,

$186 = 119 + 67 = (119 + a) + (67 – a)$

Finding ONE a is crucial to safeguard the validity of Goldbach Conjecture. Fortunately, there are at least 5 alternatives to choose from, and X_4 (= 30) as well as its multiples are good choices. Following five solutions can be obtained through using different values of a.

186 = 119 + 67 = (119 + 30) + (67 – 30) = 149 + 37
186 = 119 + 67 = (119 + 60) + (67 – 60) = 179 + 7
186 = 119 + 67 = (119 - 30) + (67 + 30) = 89 + 107
186 = 119 + 67 = (119 - 60) + (67 + 60) = 59 + 127
186 = 119 + 67 = (119 - 90) + (67 + 90) = 29 + 157

To generalize, in any scope of X_{n+1}, X_n appears to be an ideal difference to shift to remedy the influence caused by an exception in total ($P_n – 1$) ways theoretically. Although the actual number of the ways may still be discounted by random occurring exceptions, as long as there is ONE solution left workable, the validity of Goldbach Conjecture is ensured.

3.4.2.2 Horizontal shifting

We still use 186 = 119 + 67 as an example.

As shown in Figure 4, although there are only 8 primes in [7, 31], the differences among them demonstrate certain regularity, and a common difference is frequently shared among some prime pairs. For example, 29 – 23 = 23 – 17 = 19 – 13 = 17 – 11 = 13 – 7 = 11 – 5 = 6. Please note that this value equals to X_3.

Since 119 and 67 belong to the array starting with 29 and 7, respectively, in Figure 3, we may try to find a remediation applying the strategy similar to that in vertical shifting, namely, synchronously increase and decrease two addends, respectively, by 6 or its multiples. Following this rule, three out of five solutions are valid (of prime pairs).

186 = 119 + 67 = (119 - 6) + (67 + 6) = 113 + 73
186 = 119 + 67 = (119 - 12) + (67 + 12) = 107 + 79
186 = 119 + 67 = (119 - 18) + (67 + 18) = 101 + <u>85</u>
186 = 119 + 67 = (119 - 24) + (67 + 24) = <u>95</u> + <u>91</u>
186 = 119 + 67 = (119 - 30) + (67 + 30) = 89 + 97

And differences other than multiples of 6 between two primes in [7, 31] may also help finding more solutions. Following four solutions can be obtained this way.

186 = 119 + 67 = (119 - 16) + (67 + 16) = 103 + 83
186 = 119 + 67 = (119 - 22) + (67 + 22) = 97 + 89
186 = 119 + 67 = (119 + 8) + (67 - 8) = 127 + 59
186 = 119 + 67 = (119 + 20) + (67 - 20) = 139 + 47

It is obvious that for one exception (119), there are 7 ways to remedy its negative influence of one exception 119. This is far more than enough to annihilate the negative influence of an exception, as only ONE solution is required to prove the validity of Goldbach Conjecture. As this operation searches solutions along the same circle, it is termed horizontal shifting.

3.4.2.3 All combinations

If we arbitrary combine the above shiftings in 3.4.2.1 and 3.4.2.2, since both vertical and horizontal shiftings generate several solutions independently, combination of them can amplify the number of solutions to remedy the negative influence of such an exception.

Again, we still use 186 = 119 + 67 as an example. There are two alternative ways to make 186,

186 = 150 + 36

186 = 210 – 24

First, we start from the first way, namely, 186 = 150 + 136. Looking at Figure 3 and 4, there are three alternative ways to make a sum of 36, namely, 36 = 7 + 29 = 13 + 23 = 17 + 19, if only all primes in [7, 31] are considered. To reach to the sum of 186, we can add or subtract 30 or its multiples to either one of the 6 primes in the 3 prime pairs. We use the first prime pair, 7 + 29, as an example.

186 = 150 + 36

= 150 + (7 + 29)

= 157 + 29 = 127 + 59 = 97 + 89 = 67 + 119 = 37 + 149 = 7 + 179

Although there is one flawed pair due to exception of 119 (which itself is the cause of this problem), there are 5 valid prime pairs with sums of 186. Namely, there are 5 ways to fix the problem caused by one exception of 119.

Similarly, we may obtain the following solutions for 13 + 23.

186 = 150 + 36

= 150 + (13 + 23)

= 163 + 23 = 133 + 53 = 103 + 83 = 73 + 113 = 43 + 143 = 13 + 173

There are 4 valid solutions of 6 alternatives.

And we may obtain the following solutions for 17 + 19

186 = 150 + 36

= 150 + (17 + 19)

= 167 + 19 = 137 + 49 = 107 + 79 = 77 + 109 = 47 + 139 = 17 + 169

There are 3 valid solutions of 6 alternatives.

In total, dissecting 36 in three ways, we obtain 12 solutions out of 18 alternative solutions for the problem caused by the exception of 119.

Second, we start from the second way, namely, 186 = 210 - 24. Looking at Figure 3 and 4, there are two alternative ways to make a difference of 24, namely, 24 = 31 – 7 = 29 – 5. Therefore

186 = 210 – 24

= 210 – (31 – 7) = 210 – (29 – 5)

= 179 + 7 = 181 + 5

Again, **there are 2 valid prime pairs with sums of 186** (only one prime pair, 181 + 5, is novel, as 179 + 7 has occurred above).

In summary, to solve the problem caused by the exception (119), there are 13 valid prime pairs ready to fill the lacunae. The number of valid solutions (of prime pairs) overwhelms that of problems, despite decimation due to greater primes in [P_n, sqrt(X_n)] (= {7, 11, 13}).

It is noteworthy that the number of alternative solutions for problems caused by an exception is correlated with the value of P_n. As shown in Figure 3 and 5, in the scope of P_{n+1}, there are P_n – 1 alternative ways to generate new potential prime pairs to solve the problem caused by an exception. The number of solutions is further amplified by the number of ways to make the sum of remainder even (which is 36 in the above case). Since the number of potential solutions grows with P_n and the minimal probability of primes decreases with the inverse of square root of P_n (as shown in Formula 1 in [5]), their product (number of valid prime pairs) increases with P_n in general, which promises increasing number of prime pairs with a sum equal to a specific even in greater scopes. As long as the number of such prime pairs is greater than 0, Goldbach Conjecture holds.

In short, through vertical shifting, horizontal shifting, and their combinations, overwhelming number of solutions can be found for problems caused by one exception, in scope of X_5. Compared with the scope of X_6 in Figure 5, which have more numbers on the same circle and more circles, there must be much more solutions for an exception than in the scope of X_5 (as shown in Figure 3). It is logical to expect increasingly greater number of solutions for an exception in the scope of X_{n+1} than in the scope of X_n, as the number of solutions is closely hinged with P_n, which keeps on growing. Although decimated by greater primes (which have increasingly less influencing power on primality of numbers), the existence of enough solutions to remedy negative effect caused by exceptions guarantee that, for each even, there are at least one pairs of primes, which have their sum equal to the even. Namely, Goldbach Conjecture holds.

4. Conclusions

Goldbach Conjecture is a lasting challenge due to a lack of pattern governing the distribution of primes. However, this situation is highly ameliorated as the periodicity underlying primes has been revealed recently. A fact that smaller primes, although sparse, may give rise two-prime-sums covering consecutive evens in a certain range is of fundamental importance. Taking advantage of the cascade relation from one scope to the next, the initial consecutive even array of two-prime-sums can be amplified into the infinite step by step. Troubles caused by exceptions in the periodicity of primes can be overwhelmed by much greater number of alternative solutions. Taking all together, the author proves that Goldbach Conjecture is true.

Acknowledgments: This research was supported by the Strategic Priority Research Program (B) of Chinese Academy of Sciences (Grant No. XDB26000000).

Conflicts of Interest: The author declares no conflicts of interests regarding the publication of this paper.

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