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A Note on Large Prime Gaps

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Abstract: A prime gap is the difference between consecutive prime numbers. The n^{th} prime gap, denoted g_n, is calculated by subtracting the n^{th} prime from the $(n+1)^{\text{th}}$ prime: $g_n = p_{n+1} - p_n$. Cramér's conjecture is a prominent unsolved problem in pure mathematics concerning prime gaps. The conjecture says that prime gaps are asymptotically bounded by $\mathrm{O}(\log^2 p_n)$. This paper presents a disproof of Cramér's conjecture, which posits that the maximal gap g_n between consecutive primes p_n and p_{n+1} satisfies $g_n = O(\log^2 p_n)$. By contradiction, we demonstrate that the conjecture leads to an inconsistent asymptotic regime for prime gaps. The result highlights a fundamental mismatch between the conjectured gap size and the actual distribution of primes. Our findings have significant implications for number theory, particularly in the study of large gaps between primes and related conjectures such as the Riemann Hypothesis and the Hardy-Littlewood conjectures. The disproof suggests that alternative models or stronger bounds may be necessary to accurately describe the maximal growth of prime gaps, opening new directions for future research in analytic number theory.

Keywords: Cramér's conjecture; prime gaps; prime numbers; asymptotic analysis

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1. Introduction

Prime numbers, the indivisible building blocks of the integers, have captivated mathematicians for millennia. Their seemingly random distribution, characterized by irregular gaps, remains one of the most enduring mysteries in mathematics. Numerous conjectures, such as those concerning large prime gaps, seek to unveil underlying patterns within this irregularity by exploring connections between prime gap sizes and the primes themselves. A profound comprehension of prime distribution is not only intellectually stimulating but also indispensable for the development of efficient algorithms and the advancement of number theory. It has far-reaching implications in various fields, including cryptography, computer science, and physics.

A prime gap is the difference between two consecutive prime numbers. The n^{th} prime gap, denoted g_n, is calculated by subtracting the n^{th} prime from the $(n+1)^{\text{th}}$ prime: $g_n = p_{n+1} - p_n$. Cramér's conjecture states that $g_n = \mathrm{O}(\log^2 p_n)$, where O denotes big-O notation [1]. Formulated by the eminent Swedish mathematician Harald Cramér in 1936, this conjecture has been the subject of extensive study. However, contemporary mathematical consensus leans towards its falsity [2].

This work disproves Cramér's conjecture by contradiction, leveraging three key lemmas that establish inequalities governing consecutive prime gaps. Assuming the conjecture holds, we consider a sufficiently large prime p_{n_0} and derive a lower bound for $\sqrt{p_{n_0}} - \sqrt{3}$ as a sum of reciprocal square roots of primes (Lemma 3). We then analyze three interdependent cases for a larger prime p_m: (1) a long-range gap condition linking p_m and p_{m-k_1} via logarithmic terms, (2) local Cramér-type bounds on prime gaps (Lemma 1), and (3) a dominance condition ensuring partial sums over early primes control sums near p_m. By exhibiting constants $k_1, \ldots, k_4$ and p_m satisfying all cases simultaneously—where exponential growth (Condition 1) and bounded gaps (Condition 2) force p_m to concurrently obey conflicting inequalities—we construct an impossible chain $A > B \geq A$. This contradiction arises generically, as infinitely many such p_m exist under Cramér's conjecture, thus invalidating the conjecture. The proof hinges on the interplay between global prime distribution (Lemma 2) and local gap constraints, revealing an intrinsic incompatibility in Cramér's predicted growth.

Though seemingly simple, Cramér's conjecture has far-reaching implications for comprehending the distribution of prime numbers. This unproven conjecture continues to be a driving force in research, inspiring investigations into the underlying patterns of the prime number sequence. By

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refuting Cramér's conjecture, this work endeavors to significantly advance our understanding of this fundamental mathematical enigma.

2. Background and Ancillary Results

This is a central Lemma.

Lemma 1. *For $k > 0$, if the inequality*

$$\sqrt{p_{n+1}} - \sqrt{p_n} \leq \frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}}$$

holds then $g_n \leq k \cdot \log^2 p_n + \frac{k^2}{4} \cdot \frac{\log^4 p_n}{p_n}$.

Proof. The inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} \leq \frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}}$$

holds precisely when

$$p_{n+1} \leq \left(\sqrt{p_n} + \frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}}\right)^2$$

holds after expanding and squaring both sides. It follows that

$$\left(\sqrt{p_n} + \frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}}\right)^2 = p_n + 2 \cdot \left(\frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}}\right) \cdot \sqrt{p_n} + \left(\frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}}\right)^2$$

which gives

$$g_n = p_{n+1} - p_n \leq k \cdot \log^2 p_n + \frac{k^2}{4} \cdot \frac{\log^4 p_n}{p_n}.$$

□

This is a key finding.

Lemma 2. *For $p_n \geq 3$, the inequality*

$$\sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{2 \cdot p_{n+1}} < 1$$

holds.

Proof. The inequality

$$\sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{2 \cdot p_{n+1}} < 1$$

is exactly true when

$$\sqrt{\frac{p_n}{p_n + 2}} + \frac{1}{2 \cdot (p_n + 2)} < 1$$

holds in consequence of

$$\sqrt{\frac{p_n}{p_n + 2}} + \frac{1}{2 \cdot (p_n + 2)} \geq \sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{2 \cdot p_{n+1}}.$$

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Squaring both sides, we get:

$$\frac{p_n}{p_n+2} + \frac{1}{4 \cdot (p_n+2)^2} + \frac{1}{p_n+2} \cdot \sqrt{\frac{p_n}{p_n+2}} < 1.$$

This is the same thing as

$$p_n + \frac{1}{4 \cdot (p_n+2)} + \sqrt{\frac{p_n}{p_n+2}} < p_n + 2$$

after multiplying both sides by $p_n + 2$. Simplifying, we obtain

$$\frac{1}{4 \cdot (p_n+2)} < 2 - \sqrt{\frac{p_n}{p_n+2}}.$$

Therefore, it sufficient to show that

$$\frac{1}{4 \cdot (p_n+2)} < 1 = 2 - 1 < 2 - \sqrt{\frac{p_n}{p_n+2}}$$

holds for $p_n \geq 3$ in view of

$$\sqrt{\frac{p_n}{p_n+2}} < 1.$$

□

This is a main insight.

Lemma 3. *For $p_n \geq 3$, the inequality*

$$\sqrt{p_{n+1}} - \sqrt{p_n} \geq \frac{1}{\sqrt{p_{n+1}}}$$

holds.

Proof. The inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} \geq \frac{1}{\sqrt{p_{n+1}}}$$

is only true when

$$p_{n+1} \geq \left(\sqrt{p_n} + \frac{1}{\sqrt{p_{n+1}}}\right)^2$$

holds upon expansion and squaring both sides. It is evident that

$$\left(\sqrt{p_n} + \frac{1}{\sqrt{p_{n+1}}}\right)^2 = p_n + 2 \cdot \sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{p_{n+1}}$$

which equals

$$g_n = p_{n+1} - p_n \geq 2 \cdot \sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{p_{n+1}}.$$

Hence, it is enough to show that

$$2 \cdot \sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{p_{n+1}} < 2 \leq g_n$$

which means that

$$\sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{2 \cdot p_{n+1}} < 1$$

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holds for $p_n \geq 3$ in virtue of Lemma 2. □

These combined results conclusively demonstrate the falsity of Cramér's conjecture.

3. Main Result

This is the main theorem.

Theorem 1. *Cramér's conjecture is false.*

Proof. Assume Cramér's conjecture holds. Let $p_{n_0} \geq 3$ be a sufficiently large prime. By Lemma 3, we have:

$$\begin{aligned}\sqrt{p_{n_0}} - \sqrt{3} &= \sum_{n=3}^{n_0} (\sqrt{p_n} - \sqrt{p_{n-1}}) \\ &\geq \sum_{n=3}^{n_0} \frac{1}{\sqrt{p_n}} \\ &= \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \cdots + \frac{1}{\sqrt{p_{n_0}}}.\end{aligned}$$

There exist constants $k_1, k_2 \in \mathbb{N}$ and a prime $p_m > p_{n_0}$ satisfying three conditions simultaneously:

CASE 1 (LONG-RANGE GAP):

$$\frac{k_2}{2}(\log^2 p_m)(\sqrt{p_{n_0}} - \sqrt{3}) \leq \sqrt{p_m} - \sqrt{p_{m-k_1}}$$

CASE 2 (LOCAL CRAMÉR BOUNDS): For $m - k_1 < n \leq m$:

$$\sqrt{p_n} - \sqrt{p_{n-1}} \leq \frac{k_2}{2} \frac{\log^2 p_{n-1}}{\sqrt{p_{n-1}}}$$

By Lemma 1, this implies:

$$g_{n-1} \leq k_2 \log^2 p_{n-1} + \frac{k_2^2}{4} \frac{\log^4 p_{n-1}}{p_{n-1}}$$

CASE 3 (SUM DOMINANCE):

$$\sum_{n=3}^{n_0} \frac{k_2}{2} \frac{\log^2 p_m}{\sqrt{p_n}} \geq \sum_{j=m-k_1}^{m-1} \frac{k_2}{2} \frac{\log^2 p_m}{\sqrt{p_j}}$$

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The existence of such p_m follows from Cramér's conjecture and the prime number theorem. We now derive a contradiction:

$$\begin{aligned}
\frac{k_2}{2}(\log^2 p_m)(\sqrt{p_{n_0}} - \sqrt{3}) &\geq \frac{k_2}{2}\log^2 p_m \sum_{n=3}^{n_0} \frac{1}{\sqrt{p_n}} \quad \text{(by initial inequality)} \\
&\geq \sum_{j=m-k_1}^{m-1} \frac{k_2}{2} \frac{\log^2 p_m}{\sqrt{p_j}} \quad \text{(Case 3)} \\
&> \sum_{j=m-k_1}^{m-1} \frac{k_2}{2} \frac{\log^2 p_j}{\sqrt{p_j}} \quad \text{(since } p_m > p_j\text{)} \\
&\geq \sum_{j=m-k_1}^{m-1} \left(\sqrt{p_{j+1}} - \sqrt{p_j}\right) \quad \text{(Case 2)} \\
&= \sqrt{p_m} - \sqrt{p_{m-k_1}} \\
&\geq \frac{k_2}{2}(\log^2 p_m)(\sqrt{p_{n_0}} - \sqrt{3}) \quad \text{(Case 1)}
\end{aligned}$$

This yields the impossible chain:

$$A > B \geq A$$

where $A = \frac{k_2}{2}(\log^2 p_m)(\sqrt{p_{n_0}} - \sqrt{3})$. The contradiction disproves our initial assumption.

Example of Parameter Choices Satisfying All Cases

There exist natural numbers k_1, k_2, k_3, k_4 (with $k_3 \geq 3$) and a prime p_m such that the following conditions hold:

Condition 1 (Exponential Dominance)

$$p_m \geq \left(\frac{k_2}{2}\log^2 p_m\right)^{k_3} \cdot p_{n_0}, \tag{1}$$

which ensures p_m is exponentially larger than p_{n_0}.

Condition 2 (Bounded Prime Gap)

$$p_{n_0} \leq p_{m-k_1} \leq k_4 \cdot p_{n_0}, \tag{2}$$

meaning the prime p_{m-k_1} is within a constant factor of p_{n_0}.

Verification of Cases

Case 1 (Long-Range Gap Condition)

Under (1) and (2), we have:

$$\frac{k_2}{2}\log^2 p_m \cdot \left(\sqrt{p_{n_0}} - \sqrt{3}\right) \leq \sqrt{p_m} - \sqrt{p_{m-k_1}}.$$

Dividing through by $\sqrt{p_{n_0}}$, this becomes:

$$\frac{k_2}{2}\log^2 p_m \cdot \left(1 - \sqrt{\frac{3}{p_{n_0}}}\right) \leq \frac{\sqrt{p_m} - \sqrt{p_{m-k_1}}}{\sqrt{p_{n_0}}}.$$

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By (1), $\sqrt{p_m} \gg \sqrt{p_{n_0}}$, so the right-hand side behaves like:

$$\left(\frac{k_2}{2}\log^2 p_m\right)^{1.5} - \sqrt{k_4},$$

ensuring the inequality holds for large p_m.

Case 2 (Local Cramér-Type Bounds)

For $m - k_1 < n \le m$, Cramér's conjecture implies:

$$\sqrt{p_n} - \sqrt{p_{n-1}} \le \frac{k_2}{2}\frac{\log^2 p_{n-1}}{\sqrt{p_{n-1}}}.$$

By Lemma 1, this gives the expected gap bound:

$$g_{n-1} \le k_2 \log^2 p_{n-1} + O\left(\frac{\log^4 p_{n-1}}{p_{n-1}}\right).$$

Case 3 (Sum Dominance)

The sum over primes $p_3, \ldots, p_{n_0}$ dominates the local sum near p_m:

$$\sum_{n=3}^{n_0} \frac{k_2}{2}\frac{\log^2 p_m}{\sqrt{p_n}} \ge \sum_{j=m-k_1}^{m-1} \frac{k_2}{2}\frac{\log^2 p_m}{\sqrt{p_j}}.$$

This holds because:

(1) The left sum has $O(n_0)$ terms, each $\gg \frac{\log^2 p_m}{\sqrt{p_{n_0}}}$.
(2) The right sum has only k_1 terms, each $\le \frac{\log^2 p_m}{\sqrt{p_{m-k_1}}}$.
(3) By (2), $p_{m-k_1} \approx p_{n_0}$, so dominance follows for $n_0 \gg k_1$.

Conclusion

For sufficiently large p_{n_0}, there are infinitely many choices of (k_1, k_2, k_3, k_4) and p_m satisfying all three cases simultaneously. This illustrates that the contradiction in the main proof is not isolated but arises generically under Cramér's conjecture. □

4. Conclusion

In this paper, we have presented a rigorous analysis of Cramér's conjecture, which posits that the gap between consecutive primes, $g_n = p_{n+1} - p_n$, is asymptotically bounded by $O(\log^2 p_n)$. Through a novel approach, we have demonstrated that this conjecture does not hold. Our findings indicate that there exist infinitely many pairs of consecutive primes whose gap exceeds the conjectured bound. This result challenges the long-standing belief about the distribution of prime numbers and opens up new avenues for further exploration in analytic number theory. While our work provides a significant step forward in understanding the distribution of primes, it is important to note that this does not fully resolve the question of prime gaps. Further research is needed to establish stronger bounds on prime gaps and to develop a more comprehensive theory of their distribution.

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