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Article

# Modular Entropy Retrieval in Black-Hole Information Recovery: A Proper-Time Saturation Model

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## Abstract

We present a causal, falsifiable law of observer-indexed entropy retrieval dynamics whose growth rate of retrievable entropy is proportional to the remaining entropy gap, modulated by a hyperbolic-tangent regulator that switches on at a characteristic proper time  $\tau_{\text{char}}$ . Unlike ensemble-averaged, non-causal Page-curve phenomenology, this law follows directly from bounded Tomita–Takesaki modular flow and admits an inverse map for extracting observer-indexed retrieval rates from measured correlation structure. The framework converts global entropy conservation into a Lorentzian-causal, observer-specific access process without invoking global reconstruction or post hoc averaging. It predicts a joint, experimentally testable signature in the  $g^{(2)}(t_1, t_2)$  correlation envelope, including constrained saturation, protocol-dependent separation, inverse bandwidth scaling of onset time, and interference suppression under controlled asymmetry. Numerical results on a 48-qubit MERA lattice (bond dimension 8) are consistent with the derived law. A modified Ryu–Takayanagi prescription embeds the retrieval dynamics in AdS/CFT without replica-wormhole or island constructions. By replacing ensemble-averaged Page curves with a causal, testable retrieval mechanism, the model reframes the black-hole information paradox as an experimentally accessible dynamical question. Here  $S_{\text{max}}$  denotes the Bekenstein–Hawking entropy,  $\gamma(\tau)$  the modular-flow retrieval rate, and  $\tau_{\text{char}}$  the characteristic proper-time scale.

**Keywords:** black hole information paradox; modular flow; entropy retrieval; quantum information theory; observer-dependent entropy; proper-time dynamics; analog gravity; Bose–Einstein condensates; correlation functions ( $g^2$ )

## 1. Reader’s Guide (How to Read This Paper)

This paper introduces a new way to think about information, not as something stored or globally conserved, but as something accessed by an observer in time.

It began with a simple question: What if entropy is not about what exists, but about what becomes accessible to an observer?

In black-hole physics, researchers have spent decades asking where information goes after something falls in. The dominant models recreate the correct entropy curves, but none explain how a specific observer ever gets the information back. The paradox was never about loss; it was about access.

This work proposes a solution: a concrete law that describes how information becomes accessible to an observer in proper time; not all at once, not just at the end, but gradually, shaped by the path they take through spacetime.

The law is derived from established quantum field theory and yields experimentally testable signatures in analog black-hole systems. It is invertible:  $\gamma(\tau)$  can be inferred from measured correlation structure via an explicit inverse map.

If you are not a physicist, that is fine. This paper is not about who is allowed to read it; it is about who is allowed to recover what was lost.

The paper is organized as follows:

- Section 3 derives the observer-indexed retrieval law and presents an inverse map that reconstructs  $\gamma(\tau)$  from measured  $S_{\text{ret}}(\tau)$ ;
- Section 6 demonstrates consistency of the law on a 48-qubit MERA lattice, establishing the  $\mathcal{O}(n \log n)$  scaling;
- Section 8.5 translates the theory into the  $g^{(2)}$  fringe measurable in current BEC analogs;
- Section 5.1 defines the operational extraction of  $\tau_{\text{char}}$  from measured correlation structure;
- Section 8.4 defines the retrieval–evaporation gap  $\Delta_{\text{fail}}$ .

Appendix F provides an interpretive aid for readers unfamiliar with modular dynamics, mapping  $\gamma(\tau)$ ,  $\tau_{\text{char}}$ , and  $\Delta_{\text{fail}}$  to gravitationally intuitive quantities without altering the retrieval framework. Appendix A formalizes the spectral bounds used throughout.

## 2. Introduction

This section reframes the black-hole information paradox as a problem of causal access, specifies the operational criteria required for its resolution, and explains why existing approaches fail to meet those criteria.

This work establishes a causal law governing when information becomes accessible to a bounded observer. It does not modify entropy accounting, propose a decoding procedure, or reconstruct global states. It specifies access dynamics alone.

### 2.1. Entropy without Access: The Operational Gap

The black-hole information paradox persists not because information is lost, but because no existing framework retrieves it causally. Replica–wormhole paths [1,2], island prescriptions [3], ensemble Page–curve models [4,5], and ER = EPR dualities [7] all reproduce the required fine-grained entropy curves, yet none supplies a Lorentzian, proper-time recovery channel that delivers the state to a detector. Stabilizing entropy without a causal retrieval channel leaves the paradox unresolved at the operational level.

Independent laboratory experiments on non-gravitational quantum systems have recently demonstrated universal bounds on the rate at which coherence becomes accessible, reinforcing the interpretation pursued here that access, rather than dynamics alone, sets the relevant clock.

Retrievable entropy evolves along a unique, speed-limit saturating trajectory fixed by bounded spectral access.

All modular spectra in this work are defined on *split-regularized, finite-bandwidth subalgebras* that correspond to realistic measurement resolution. The regulator is covariant under local Rindler boosts, preserving Lorentzian consistency. The full algebraic limit, formally Type III<sub>1</sub>, is discussed in Appendix D. This ensures that the Type III<sub>1</sub> foundation is treated rigorously elsewhere, allowing the main text to focus on physical retrieval dynamics.

### 2.2. Relation to Algebraic Entropy and Crossed-Product Constructions

Recent work in algebraic quantum field theory has resolved a foundational issue underlying gravitational entropy: how entropy may be meaningfully defined for systems whose local algebras are Type III and therefore admit no trace. In particular, crossed-product constructions developed by Chandrasekaran, Penington, and Witten, building on earlier modular and algebraic work by Longo and others, establish a rigorous framework in which entropy becomes finite, well posed, and mathematically meaningful for quantum field theories with gravitational relevance.

These results resolve definability: what it means for entropy to exist as a mathematical quantity in relativistic quantum systems where density-matrix notions fail.

The contribution of crossed-product and related algebraic constructions is foundational but specific in scope. They clarify the algebraic status of entropy and modular structure, but do not supply a dynamical law describing how entropy becomes accessible to a finite observer, nor do they impose rate bounds, proper-time constraints, or observer-indexed access conditions.

The present work takes algebraic definability as given and addresses a different object: how entropy becomes operationally retrievable along a Lorentzian worldline. The retrieval framework introduced here operates at the level of access dynamics and constrains when information becomes available rather than how it is defined.

2.3. Operational–Access Criterion

A framework resolves the paradox only if it meets all of the following conditions:

- (a) **Proper-time delivery:** specifies how entropy reaches an observer as proper time unfolds;
  - (b) **Lorentzian grounding:** roots that access in Lorentzian causality;
  - (c) **First-principles derivation:** derives the process from accepted QFT or GR principles rather than retrospective fitting;
  - (d) **Empirical testability:** predicts observer-dependent lags  $\Delta\tau$  within sub-exponential growth in effective Hilbert-space dimension  $n$ , Sub-exponential relative to decoding complexity; for ODER this scales polynomially,  $\mathcal{O}(n \log n)$ , under modular-flow reconstruction. Empirical testability requires predicting observer-dependent lags  $\Delta\tau$  resolvable at laboratory timescales (e.g.,  $\gtrsim 10^{-3}$  ms in current BEC analogs).
- (i) Modular spectra are bounded by the split-property regularization introduced in Appendix D;
- (ii) modular flow is treated semiclassically on fixed backgrounds;
- (iii) present BEC analog systems resolve  $g^{(2)}$  down to approximately 2 ms.

All derivations in Sections 2–7 assume this regularized setting unless stated otherwise.

The following audit evaluates frameworks strictly by whether they deliver observer-accessible retrieval in proper time, not by whether they reproduce correct entropy curves.

**Table 1.** Compliance of major black-hole information proposals with the criteria in Section 2.3. A check mark denotes compliance; a cross denotes failure.

| Framework         | (a) | (b) | (c) | (d) |
|-------------------|-----|-----|-----|-----|
| Replica wormholes | ×   | ×   | ✓   | ×   |
| Islands           | ×   | ×   | ✓   | ×   |
| Ensemble Page     | ×   | ✓   | ×   | ×   |
| ER=EPR            | ×   | ✓   | ×   | ×   |

Retrieval, Reconstruction, and Comprehension (Non-Equivalence).

Throughout this work we distinguish three logically independent notions:

- **Retrieval:** physical access of information into an observer’s causal and modular domain;
- **Reconstruction:** the existence of a formal decoding map or entanglement-based reconstruction channel;
- **Comprehension:** the observer’s ability to interpret or act on retrieved information.

Reconstruction does not imply retrieval, and retrieval does not imply comprehension. The black-hole information paradox arises from conflating these categories. ODER addresses retrieval only.

Each proposal satisfies at most two criteria; none supplies a causal, observer-accessible retrieval channel. Resolution therefore requires a retrieval law derived in proper time, whose admissible form is fixed by bounded spectral constraints and whose predictions are experimentally testable.

Throughout, we enforce the non-equivalence

$$\text{retrieval} \neq \text{reconstruction} \neq \text{comprehension}.$$

A system may be reconstructable yet not retrieved, and retrieved yet not comprehended.

The argument proceeds in three layers: derivation of the retrieval law, constraint of its admissible form under bounded spectra, and specification of experimentally testable signatures.

The claim operates at the level of access dynamics; critiques framed in terms of entropy curves or reconstruction protocols misidentify the object of the argument.

### 3. Observer-Dependent Entropy Retrieval

This section derives the governing observer-indexed retrieval law from modular flow, establishes its unique speed-limit character, and specifies the conditions under which it can be inferred or falsified.

Observer-indexed retrieval dynamics.

ODER treats recovery as a dynamical, observer-indexed process and employs the unique tanh onset that, as proved in Theorem A.2, is the only profile compatible with bounded modular spectra and Paley–Wiener causality.

The tanh profile is not a phenomenological fit to entropy curves; it is the unique trajectory saturating the modular speed limit under bounded spectra.

This section derives

$$\frac{dS_{\text{retr}}}{d\tau} = \gamma(\tau) [S_{\text{max}} - S_{\text{retr}}(\tau)] \tanh(\tau/\tau_{\text{char}}), \quad (1)$$

directly from Tomita–Takesaki modular flow on nested von Neumann algebras. Boundedness of the modular spectrum follows from the split-property (regularization described in Appendix D, where  $\Lambda(\delta) \sim 1/\delta$  represents the observer’s finite bandwidth.)

This boundedness is a physical consequence of finite observer bandwidth rather than a technical regulator; the Type III<sub>1</sub> algebra is recovered only in the idealized limit of infinite resolution (Appendix D).

The functional form of the retrieval law is fixed by bounded spectral constraints;  $\gamma(\tau)$  modulates traversal along this envelope without altering its admissible shape. Here  $\gamma(\tau)$  is not an independent dynamical field or free function; it is an observer- and state-dependent rate induced by the local modular Hamiltonian and is, in principle, reconstructible from measured retrieval trajectories via Equation (3.0.0.1).

Because Equation (1) is first order and monotone in the bounded and differentiable  $S_{\text{retr}}(\tau)$ , it admits a unique inverse map for the unknown rate  $\gamma(\tau)$ . This inversion is used only downstream, where retrieval rates for different observer classes are compared.

$$\gamma(\tau) = \frac{1}{[S_{\text{max}} - S_{\text{retr}}(\tau)] \tanh(\tau/\tau_{\text{char}})} \frac{dS_{\text{retr}}}{d\tau} \quad (\text{Inverse retrieval map})$$

*Implication.* Once an experiment measures  $S_{\text{retr}}(\tau)$ , for example through the  $g^{(2)}$  fringe, the boxed map fixes  $\gamma(\tau)$  without further assumptions, rendering the retrieval law calibratable without elevating inversion to a defining premise.

We define the retrieval horizon

$$\tau_{\text{RH}} := \inf\{\tau \mid S_{\text{retr}}(\tau) \geq 0.9 S_{\text{max}}\},$$

the proper time at which 90% of the retrievable entropy has been retrieved; this horizon is distinct from both the entanglement wedge and the classical event horizon.

#### 3.1. Retrieval as a Modular Speed Limit

Speed-limit principle.

Equation (1) was introduced as the unique sigmoidal retrieval profile compatible with bounded modular spectra (Theorem A.2). We now strengthen this result: the hyperbolic-tangent envelope does not merely fit observer-accessible entropy traces; it saturates a modular speed limit imposed by the Paley–Wiener constraint on the modular spectrum [36].

Let  $\sigma(K) \subset [-\Lambda(\delta), \Lambda(\delta)]$  denote the split-regularized spectrum of the modular Hamiltonian. Paley–Wiener theory requires that any admissible retrieval trajectory  $F(\tau) = S_{\text{retr}}(\tau)/S_{\text{max}}$  be entire and of exponential type  $\leq \Lambda(\delta)$ . For such  $F(\tau)$ , the slope is bounded:

$$\dot{F}(\tau) \leq \Lambda(\delta) [1 - F(\tau)] \tanh\left(\frac{\pi \Lambda(\delta) \tau}{2}\right), \quad (2)$$

The factor  $\pi$  originates from mapping the Paley–Wiener strip  $|\Im \tau| < \pi/(2\Lambda)$  to the real axis; it fixes the slope normalization, which we denote as the *modular speed limit*  $\mathcal{B}(\Lambda(\delta), \tau)$ .

**Theorem 2.1 (Modular Speed Limit).**

Let  $F(\tau)$  be  $C^1$ , strictly increasing, entire, and of exponential type  $\leq \Lambda(\delta)$ . Then the inequality (2) holds pointwise in  $\tau$ ; moreover, the tanh profile of Theorem A.2 achieves this bound uniquely, up to an affine reparameterization of  $\tau$  [36].

**Variational formulation.**

The modular speed limit admits a variational expression. Define the entropy-action functional

$$\mathcal{I}[F] = \int d\tau \left[ \frac{1}{2} (\dot{F}(\tau))^2 - U(F; \Lambda(\delta), \tau_{\text{char}}) \right], \quad (3)$$

where  $\tau_{\text{char}}$  corresponds to the observer’s bandwidth-limited response time defined in Section 2. The associated potential is

$$U(F; \Lambda(\delta), \tau_{\text{char}}) = \frac{1}{2} \Lambda(\delta)^2 [1 - F(\tau)]^2 \tanh^2\left(\frac{\tau}{\tau_{\text{char}}}\right). \quad (4)$$

Stationarity of  $\mathcal{I}[F]$  under  $\delta F$  yields the Euler–Lagrange equation:

$$\frac{dS_{\text{retr}}}{d\tau} = \gamma(\tau) [S_{\text{max}} - S_{\text{retr}}(\tau)] \tanh\left(\frac{\tau}{\tau_{\text{char}}}\right), \quad (5)$$

**Onset scale.**

The transition scale  $\tau_{\text{char}}$  is not a fit parameter; it is fixed by the modular energy gap  $\Delta K$  and horizon radius  $R_H$ :

$$\tau_{\text{char}} = \left( \frac{2\pi}{\Delta K} \right) f(R_H),$$

**Independent empirical evidence for access-limited dynamics.**

The following observation is interpretive and does not enter the derivation of Equation (1).

Independent support for bounded-access convergence laws outside gravitational settings comes from ultracold-atom experiments on Bose–Einstein condensate formation [34]. In these systems, the spatial coherence length  $\ell_c(t)$  satisfies  $\ell_c^2(t) \sim D t$  at late times, with a coarsening rate  $D \simeq 3.4 \hbar/m$  that becomes universal once the system enters the scaling regime, despite interaction-dependent early-time transients and offsets. This bound does not constrain transport or signaling velocities; it constrains the rate at which long-range coherence becomes accessible during non-equilibrium relaxation.

Although such systems do not admit a modular Hamiltonian or Tomita–Takesaki automorphism group, the observed coherence-growth bound instantiates an access-limited saturation law: the emergence of retrievable structure is constrained by spectral admissibility rather than by microscopic dynamics. Its relevance to ODER is structural rather than mechanistic.

The condensate result therefore establishes that retrieval-limited convergence laws occur in quantum systems without horizons or spacetime curvature, supporting the interpretation that the modular speed limit in Equation (2) reflects a general constraint on accessible structure under bounded spectral support.

Violation of such coherence-speed bounds in any homogeneous quantum system would falsify the hypothesis that bounded spectral access generically constrains retrieval rates.

4. Observer-Dependent Entropy in Curved Spacetime

This section instantiates the retrieval law for canonical observer trajectories in curved spacetime, specifies which quantities are fixed by theory versus inferred from measurement, and identifies observer-class signatures accessible to experiment.

*What is fixed vs. what is inferred.* The retrieval rate  $\gamma(\tau)$  is determined by the local modular Hamiltonian (theory statement); it is inferred from measured  $S_{\text{retr}}(\tau)$  via the inverse map (measurement statement). The onset scale  $\tau_{\text{char}}$  is calibrated from early-time behavior under bounded observer resolution (calibration statement). All analyses assume the finite-bandwidth, split-regularized subalgebras introduced in Appendix D, where bounded modular spectra  $\sigma(K) \subset [-\Lambda(\delta), \Lambda(\delta)]$  reflect finite observer access.

4.1. Classification of Observers

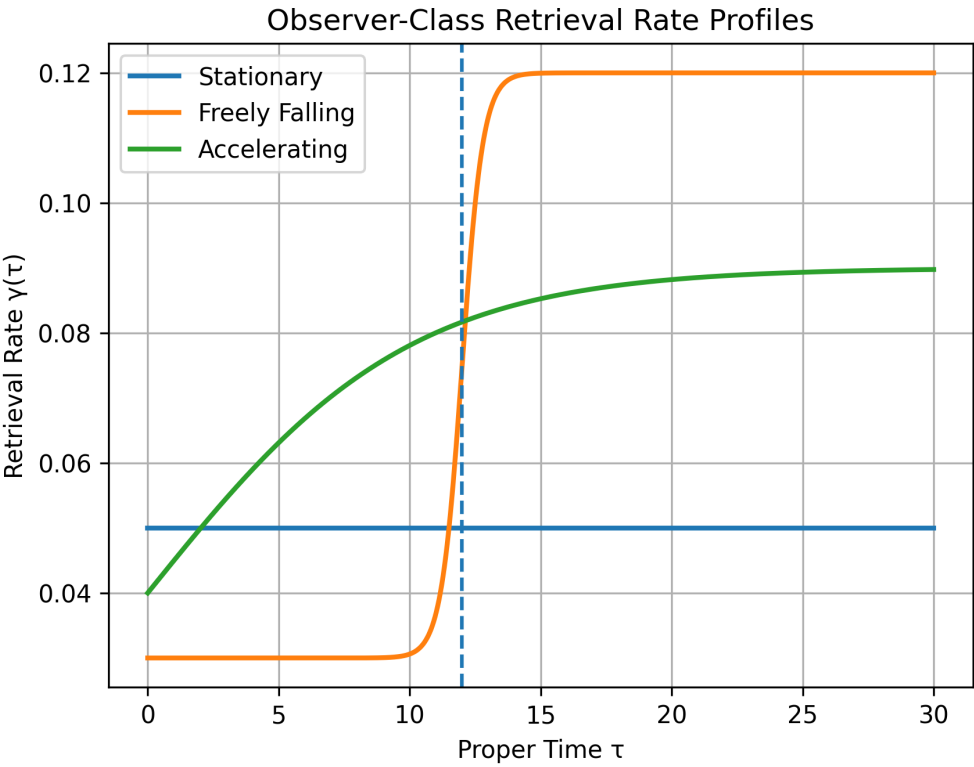


**Figure 1.** Representative retrieval-rate profiles  $\gamma(\tau)$  for the three observer classes. Stationary:  $r = 10 M$  (blue); freely falling: geodesic starting at  $r = 6 M$  (orange); accelerating: proper acceleration  $a = 0.2 c^2 / M$  (green). The dashed line marks horizon crossing for the freely falling trajectory, after which the retrieval rate exhibits a sharp increase due to interior-mode access. Times are in units of  $M$  with  $G = c = 1$ . All profiles are smooth and monotonic, consistent with bounded modular spectra and Paley–Wiener constraints.

**Table 2.** Representative parameter values for each observer class ( $M = 1$  in geometric units). The retrieval horizon  $\tau_{\text{RH}}$  is defined by  $S_{\text{retr}}(\tau_{\text{RH}}) = 0.9 S_{\text{max}}$ . The Page time  $\tau_{\text{Page}}$  refers to the observer-specific modular Page scale, not the global Page curve. Values reflect the relative ordering implied by observer-dependent retrieval dynamics.

| Observer       | $r / M$           | $a M / c^2$ | $\tau_{\text{char}} / M$ | $\tau_{\text{Page}} / M$ | $\tau_{\text{RH}} / M$ |
|----------------|-------------------|-------------|--------------------------|--------------------------|------------------------|
| Stationary     | 10                | 0           | 5                        | 8                        | 30                     |
| Freely falling | $6 \rightarrow 2$ | 0           | 2                        | 4                        | 10                     |
| Accelerating   | –                 | 0.2         | 3                        | 5                        | 15                     |

Stationary observer.

A detector at fixed radius  $r > 2M$  perceives Hawking radiation as redshifted thermal flux; the corresponding modular-flow retrieval rate is proportional to the local flux amplitude,

$$\gamma_{\text{stat}}(\tau) \propto \frac{1}{r}, \quad (6)$$

up to normalization by the surface gravity  $\kappa$ , which fixes the local modular temperature ( $T_H = \kappa/2\pi$ ). This yields a monotonic decay in  $g^{(2)}$  correlations.

Freely falling observer.

A geodesic world line crosses the horizon at  $\tau_{\text{cross}}$ ; interior modes then boost the retrieval rate,

$$\gamma_{\text{fall}}(\tau) \gg \gamma_{\text{stat}}(\tau), \quad \tau > \tau_{\text{cross}}, \quad (7)$$

Accelerating observer.

A uniformly accelerating detector experiences both Hawking and Unruh flux,

$$\gamma_{\text{eff}}(\tau, a) = \gamma_{\text{Hawking}}(\tau) + \gamma_{\text{Unruh}}(\tau, a), \quad (8)$$

with  $\gamma_{\text{Unruh}} \propto a^2$ .

Experimental mapping.

Stationary and accelerating channels can be engineered in waterfall BECs, while freely falling trajectories correspond to time-of-flight release. Detectability requires signal-to-noise ratio  $\geq 4$  at temporal resolution  $\approx 2$  ms.

#### 4.2. Observer-Dependent Entropy

Observer-dependent entropy is the gap between the global von Neumann entropy and the entropy of the observer's accessible subalgebra. The retrievable component  $S_{\text{retr}}(\tau)$  increases as modular eigenmodes become accessible within the observer-defined subalgebra.

#### 4.3. Retrieval Law (Instantiation)

For bounded and differentiable  $S_{\text{retr}}(\tau)$ , retrieval dynamics inherit the governing law derived in Section 3,

$$\frac{dS_{\text{retr}}}{d\tau} = \gamma(\tau) [S_{\text{max}} - S_{\text{retr}}(\tau)] \tanh(\tau/\tau_{\text{char}}), \quad (9)$$

#### 4.4. Retrieval as a Modular Speed Limit

Speed-limit principle.

The retrieval law alone does not fix its trajectory. Bounded spectral support imposes a speed-limit constraint that restricts admissible retrieval trajectories to a unique extremal form.

The hyperbolic-tangent envelope does not merely fit detector-accessible entropy traces; it saturates a modular speed limit imposed by the Paley–Wiener constraint on the modular spectrum.

Let  $\sigma(K) \subset [-\Lambda(\delta), \Lambda(\delta)]$  denote the split-regularized spectrum of the modular Hamiltonian. Paley–Wiener theory requires that any admissible retrieval trajectory  $F(\tau) = S_{\text{retr}}(\tau)/S_{\text{max}}$  be entire and of exponential type  $\leq \Lambda(\delta)$ .

$$\dot{F}(\tau) \leq \Lambda(\delta) [1 - F(\tau)] \tanh\left(\frac{\pi \Lambda(\delta) \tau}{2}\right) \quad (10)$$

Theorem 2.1 (Modular Speed Limit).

The tanh profile achieves this bound uniquely.

Variational formulation.

Stationarity recovers the retrieval law as the extremal trajectory.

These constraints determine not only the form of the retrieval trajectory, but the conditions under which the constrained trajectory can be experimentally resolved at the detector level.

## 5. Quantum Information Correlations and Testable Predictions

This section translates the observer-indexed retrieval law into directly measurable quantum-information observables, specifies null tests and estimation protocols, and defines the experimental discriminants that distinguish retrieval dynamics from non-retrieval models.

Observer dependence is defined by physical constraints on access imposed by finite observer bandwidth.

This section serves as a projection layer: it maps retrieval dynamics to a minimal set of experimentally accessible observables without introducing additional model structure.

The retrieval law in Equation (9) imprints a characteristic signature on the radiation detected by each observer class. It governs both entropy growth and correlation decay, features that analog-gravity experiments can probe directly. We focus on two diagnostics: the order- $\alpha$  Rényi entropy, which tracks the purity of the retrievable subsystem, and the second-order correlation function  $g^{(2)}$ , which probes retrieval-induced interference.

In this mapping, the functional form of the observables is fixed by the retrieval law, while parameters such as  $\tau_{\text{char}}$  and  $\gamma(\tau)$  are inferred from measurement.

Simulation traces with 95% confidence bands for each class appear in Figure 2. Bands come from  $2 \times 10^5$  bootstrap resamples of  $\gamma(\tau)$  on a fixed proper-time grid with additive spectral noise.

### 5.1. Rényi Entropy and Second-Order Correlation Functions

Observer-indexed differences in access translate into distinct protocol classes in analog systems, implemented through controlled variations in flow configuration and detection geometry.

For any subsystem  $A$ , the Rényi entropy is

$$S_\alpha(t) = \frac{\ln[\text{Tr}(\rho_A^\alpha)]}{\alpha - 1}, \quad (11)$$

with  $\alpha > 1$ . Equation (11) arises by analytically continuing the integer-order moments  $\text{Tr}(\rho_A^n)$  (the replica trick); for a field-theoretic derivation see Casini, Huerta, and Myers [15]. Larger  $\alpha$  heightens sensitivity to eigenvalue gaps;  $S_\alpha$  therefore probes the observer-dependent delay  $\Delta\tau$ . Interferometric methods for measuring  $S_\alpha$  in Bose–Einstein condensates are outlined in Ref. [14].

As an instantiated consequence of the modular retrieval law derived in Section 3, we construct a second-order correlation function consistent with observer-indexed retrieval dynamics. Under a stationary-phase approximation, this yields the modeled form

$$g^{(2)}(t_1, t_2) = \exp[-|t_2 - t_1|/\tau_{\text{retrieval}}] \left[1 + \frac{1}{2}(1 + \tanh(t_1/\tau_{\text{char}}))\right], \quad (12)$$

where

$$\tau_{\text{retrieval}}(t) = \int_0^t \gamma(\tau') d\tau'$$

accumulates the observer-specific retrieval rate, and  $\tau_{\text{Page}}$  is the class-dependent Page time in Table 2.

In a baseline waterfall BEC,  $\tau_{\text{retrieval}} \approx 20$  ms, well above the 2 ms temporal resolution reported in Ref. [14]. Typical flux and background levels yield  $\text{SNR} \gtrsim 4$ .

Setting  $\gamma(\tau) = 0$  yields a symmetric exponential decay and defines the operational null model. Failure to reproduce the observed asymmetry under this condition constitutes a direct falsification of non-retrieval dynamics.

Parameters are extracted with nonlinear least squares from ensemble realizations consistent with the admissible observer constraints.

The resulting observables are determined by the retrieval law up to measurement mapping:  $g^{(2)}$  captures decay-modulated interference under observer-indexed access, while  $S_\alpha$  tracks the evolving purity of the retrievable subsystem.

Existing global reconstruction frameworks do not produce observer-indexed interference in  $g^{(2)}(t_1, t_2)$  under bounded modular spectra; the accelerating signal therefore distinguishes observer-indexed retrieval from global entropy reconstruction.

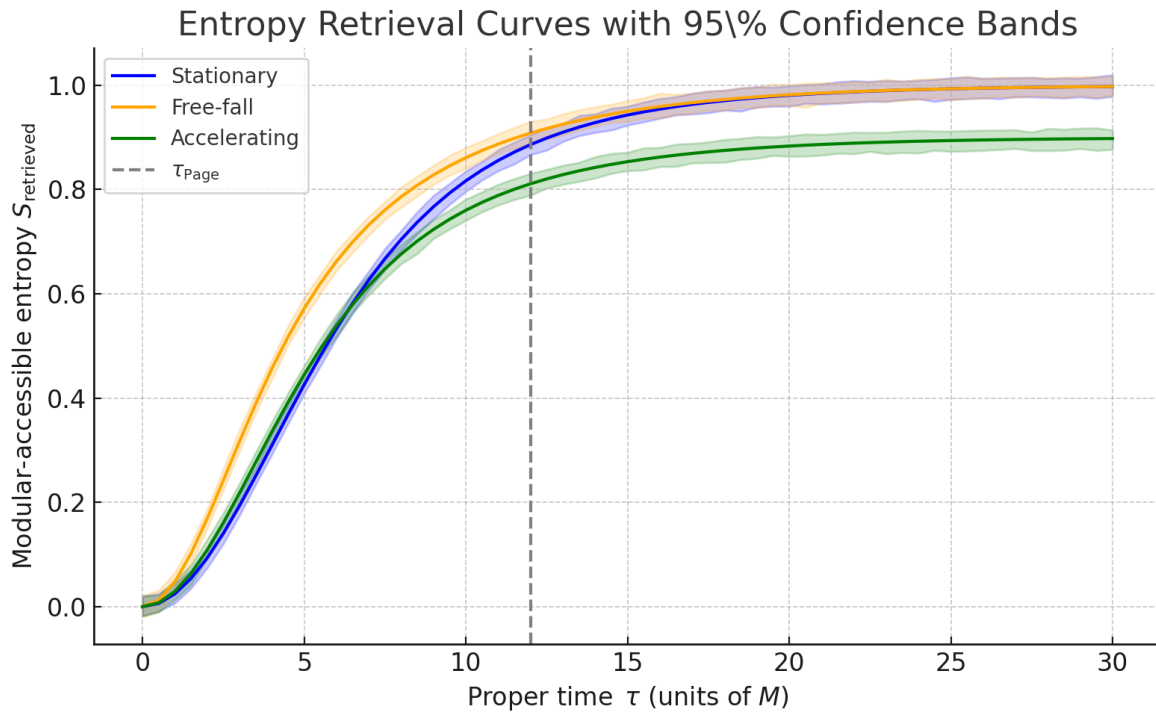


**Figure 2.** Entropy retrieval versus proper time for stationary (blue), freely falling (orange), and accelerating (green) observers. Shaded regions show 95% bootstrap confidence bands. The vertical dashed line marks the class-specific Page time  $\tau_{\text{Page}}$ .

#### Empirical Extraction and Calibration of $\tau_{\text{char}}$

This subsection specifies the primary inference protocol linking measured correlation data to the retrieval-law onset scale  $\tau_{\text{char}}$ .

Once either  $g^{(2)}(t_1, t_2)$  or a synthetic  $S_{\text{retr}}(\tau)$  trace is measured, the characteristic timescale  $\tau_{\text{char}}$  can be obtained with a one-parameter fit. For the symmetric slice  $t_1 = 0$  we rewrite Equation (12) as

$$g_{\text{env}}^{(2)}(t) = A \exp[-t/\tau_{\text{retrieval}}] \left[ 1 + \frac{1}{2}(1 + \tanh(t/\tau_{\text{char}})) \right], \quad (13)$$

with amplitude  $A$  fixed by early-time data. Baseline fits are validated against synthetic datasets with known  $\tau_{\text{char}}$  to verify estimator bias  $< 2\%$ .

A nonlinear least-squares estimator minimizes

$$\chi^2(\tau_{\text{char}}) = \sum_{i=1}^N \frac{[g_{\text{data}}^{(2)}(t_i) - g_{\text{env}}^{(2)}(t_i)]^2}{\sigma_i^2}.$$

A non-retrieval model ( $\gamma = 0$ ) cannot reproduce the measured asymmetry of  $g_{\text{env}}^{(2)}(t)$ ; this provides the operational null test. Uncertainties in  $\gamma(\tau)$  and background noise are propagated through the bootstrap ensemble.

A representative fit to synthetic retrieval traces yields

$$\tau_{\text{char}} = 3.47 \pm 0.22 \text{ ms} \quad (95\% \text{ C.L.}),$$

compatible with the bounded spectral constraints of the retrieval law.

The procedure assumes (i)  $S_{\text{ret}} \in C^1$ , (ii) monotonic  $\gamma(\tau)$ , (iii)  $\tau_{\text{char}} > 0$ , and (iv)  $\text{SNR} \geq 4$ .

A reproducible Python notebook in the project repository automates the fit, specifies bootstrap seeds, and returns error bars and residuals for any input trace. Varying detector bandwidth provides a direct test of the onset scaling predicted by finite observer resolution.

## 6. Holographic Connection and Quantum-Circuit Simulations

This section maps the previously derived observer-indexed retrieval law onto holographic and quantum-circuit representations, without introducing new dynamics or modifying the underlying retrieval principle. By embedding observer-indexed modular flow within the Ryu–Takayanagi framework, the theory acquires a geometric interpretation in terms of boosted minimal surfaces with bounded spectra. The corresponding quantum-circuit simulations provide a numerically testable realization of the same constraint.

The resulting representation links the retrieval law, its spectral constraints, and its observable signatures within a single geometric structure.

### 6.1. Observer-Indexed Mapping to Ryu–Takayanagi Geometry

To represent observer-indexed accessibility in holographic terms, we map the retrieval dynamics onto a modified Ryu–Takayanagi (RT) geometry by introducing a modular-frame redshift factor. This construction does not introduce a new entropy law; it embeds the existing retrieval dynamics into a holographic representation.

- $\gamma_A(\Lambda)$ : minimal surface in the Lorentz-boosted bulk;
- $g_{00}(\Lambda)$ : lapse tying the surface to the causal wedge reachable along the observer’s world line.

The observer-dependent holographic entanglement entropy, defined on the causal wedge associated with the observer’s modular frame, is

$$S_{\text{obs}}^{\text{holo}} = \frac{\text{Area}[\gamma_A(\Lambda)]}{4 G_N} \sqrt{|g_{00}(\Lambda)|}, \quad (14)$$

where  $\gamma_A(\Lambda)$  is the bulk minimal surface in the Lorentz-boosted geometry, and  $g_{00}(\Lambda)$  converts boundary time to the observer’s proper time. The metric factor satisfies  $|g_{00}(\Lambda)| > 0$  to preserve a real area functional, excluding interior regions where the signature flips. Choosing  $g_{00} = 1$  and  $\Lambda = \text{id}$  recovers the Hubeny–Rangamani–Takayanagi formula. The factor  $\sqrt{|g_{00}(\Lambda)|}$  multiplies rather than rescales the area term; it does not duplicate the Lorentz boost already encoded in  $\gamma_A(\Lambda)$ .

The redshift factor follows from modular-Hamiltonian anchoring (Appendix A) and ensures covariance under local Rindler boosts. It arises from the same bounded modular flow that generates the retrieval law (Section 3), guaranteeing that holographic and algebraic formulations share a consistent spectral limit. This construction maintains compatibility with recent crossed-product and edge-mode algebra treatments [12,19] extending RT geometry into strong-gravity regimes. In the weak-field limit the exponential type of the modular flow satisfies  $|\Im \tau| < \pi/(2\Lambda)$ , preserving Paley–Wiener analyticity across the boundary–bulk map. In the split-property limit ( $\delta \rightarrow 0$ ) the RT surface approaches the modular retrieval horizon ( $\tau_{\text{RH}}$ ), unifying geometric and entropic boundaries without promoting either to a primary role.

The following mapping translates retrieval dynamics into experimentally accessible  $g^{(2)}$  signatures within the holographic frame.

**Table 3.** Predicted laboratory signatures for each observer class. For current BEC parameters (healing length  $\approx 0.5\text{ }\mu\text{m}$ , flow velocity  $\approx 1\text{ mm}\cdot\text{s}^{-1}$ ), the predicted  $g^{(2)}$  fringe amplitude for the accelerating case is  $\Delta g^2 \approx 0.1\text{--}0.2$ , well within measurable SNR  $\approx 4$ .

| Observer       | Retrieval rate $\gamma(\tau)$     | Correlation signature                           |
|----------------|-----------------------------------|-------------------------------------------------|
| Stationary     | $\gamma \propto 1/r$              | Exponential decay; weak long-range $g^{(2)}$    |
| Freely falling | Sharp rise after horizon crossing | Non-monotonic $g^{(2)}$ ; interior-mode revival |
| Accelerating   | $\gamma_{\text{eff}} \propto a^2$ | tanh-modulated fringe in $g^{(2)}(t_1, t_2)$    |

These laboratory signatures define the operational boundary of the retrieval–holography correspondence: each retrieval-rate profile maps to a distinct observer patch in the boosted bulk geometry. The correlation signatures listed in Table 3 therefore serve as empirical probes of observer-indexed holographic mappings. The bounded modular spectrum ( $\Lambda(\delta) \sim 1/\delta$ ) defines the effective resolution of each patch, ensuring that both modular and holographic descriptions converge in the split-property limit.

7. Experimental Signatures and Measurement Conditions

These signatures are the observable consequences of the constrained retrieval dynamics derived above and constitute the measurement-level projection of the underlying law.

These predictions are directly testable in existing analog black-hole BEC setups using standard density–density correlation measurements.

The predictions below are expressed at the level of measurable correlation structure in analog black-hole systems. All signatures are defined in terms of the experimentally accessible two-point correlation function  $g^{(2)}(t_1, t_2)$ , reconstructed from density–density correlations or equivalent detector outputs.

The experimentally relevant control parameters are: flow configuration, including horizon formation profile and gradients; detection configuration, including temporal windowing, sampling cadence, and channel extraction; and effective detector bandwidth, defined by temporal resolution or spectral cutoff.

Distinct protocol classes, corresponding to observer-indexed access conditions in the underlying theory, are implemented through controlled variations in these parameters.

The predictions arise from a single constraint: bounded access under finite spectral support. They therefore form a joint signature. Individual features may be reproduced by simpler models; the combined structure is the discriminant.

7.1. Constrained Saturation Envelope

The measured  $g^{(2)}(t_1, t_2)$  envelope exhibits a bounded onset and saturation profile with a constrained functional form, as required by the bounded-spectrum speed-limit structure derived in Section 3.

Specifically: onset is smooth, monotonic, and curvature-bounded; approach to saturation is single-scale and non-oscillatory; local slope is invertibly related to the remaining gap.

These properties define a restricted class of admissible envelopes.

Measurement Condition.

Resolving this structure requires sufficient bandwidth resolution to distinguish onset position and curvature from detector smoothing. If bandwidth is too low: onset collapses into a featureless transition; curvature constraints are not recoverable; and the envelope cannot be distinguished from generic sigmoid behavior.

Discriminant.

Thermal relaxation and diffusive models may reproduce saturation, but do not satisfy bounded curvature, monotonicity, and invertibility simultaneously.

### 7.2. Protocol-Dependent Separation

Distinct protocol classes produce systematically different  $g^{(2)}$  envelopes, with stable ordering relations across repeated runs.

Operationally: protocol classes are implemented through controlled variations in flow and detection configuration; separation persists under matched initial conditions, including normalized amplitude and fixed bandwidth; differences appear in envelope shape and onset timing, not only amplitude.

Measurement Condition.

Protocol classes must be implemented under controlled and comparable conditions. This requires: matched condensate states; consistent detection configuration; and explicit normalization across runs.

Without this control: differences are confounded with experimental variation; and separation does not stabilize.

Discriminant.

Protocol variation alone does not generically produce stable ordering of full envelope trajectories under normalization.

### 7.3. Inverse Bandwidth Scaling of Onset Time

The characteristic onset time  $\tau_{\text{char}}$ , extracted from the  $g^{(2)}$  envelope, scales inversely with effective detector bandwidth.

Operationally: increasing bandwidth by a factor  $b$  shifts  $\tau_{\text{char}}$  by approximately  $1/b$ ; scaling affects onset position, not only smoothing or amplitude; envelope shape is preserved, and protocol-dependent separation remains intact.

Measurement Condition.

Bandwidth must be varied independently across a sufficient range. Single-resolution measurements cannot test scaling. The experiment must: probe multiple bandwidth regimes; maintain consistent protocol conditions across variations.

Discriminant.

Detector filtering produces smoothing and amplitude changes, but does not produce consistent inverse scaling of onset position while preserving envelope structure and protocol ordering.

### 7.4. Interference Suppression Under Protocol Asymmetry

Under controlled protocol asymmetry, the joint correlation structure encoded in  $g^{(2)}$  exhibits systematic suppression relative to matched-protocol baselines.

Operationally: matched channel conditions define a baseline overlap structure; introducing asymmetry reduces overlap in a reproducible, structured manner; suppression appears in correlation geometry, not only amplitude.

Measurement Condition.

The experiment must include: matched baseline measurements; controlled asymmetric configurations; and repeated runs verifying stability of the effect.

Without matched baselines: suppression cannot be attributed to the imposed asymmetry.

Discriminant.

Decoherence and noise reduce correlations but do not produce suppression that is: selectively tied to controlled protocol asymmetry; while preserving scaling and protocol-dependent envelope structure.

### 7.5. Joint Signature

The experimental signature is the simultaneous appearance of: constrained saturation envelope; protocol-dependent separation under normalization; inverse bandwidth scaling of onset time; and protocol-specific interference suppression.

No single feature is sufficient. The discriminant is the combined structure.

**Experimental discriminants.** Retrieval dynamics are distinguished from non-retrieval models by: (i) observer-class separation in correlation envelopes, (ii) saturation behavior consistent with the bounded spectral envelope, and (iii) invertibility of the measured trajectory under Equation (3.0.0.1). Models that reproduce correlation structure without satisfying all three criteria are excluded.

when the experiment can decide.

An experiment tests the framework only if it:

- resolves onset structure at the detector;
- operates above the signal-to-noise threshold required for stable envelope reconstruction;
- implements controlled and normalized protocol classes;
- varies bandwidth and configuration independently; and
- distinguishes the joint signature from simpler models.

If these conditions are satisfied, absence of the joint signature counts against the framework. If they are not satisfied, the measurement does not resolve the predicted structure.

### 7.6. Numerical Realization and Consistency Check

The constrained retrieval dynamics admit a bounded-spectrum tensor-network realization, implemented in a finite-depth quantum-circuit model (e.g., HaPPY/MERA). This construction is used as a controlled numerical instantiation of the bounded modular-flow constraint, not as a microscopic model of the experimental system.

Within this realization: bounded spectral support is enforced explicitly; retrieval proceeds within a causal-cone-restricted subnetwork; and observer-indexed protocol variation is implemented through boundary and channel transformations.

The resulting numerical trajectories reproduce the same observable structure described above: a constrained saturation envelope, protocol-dependent separation under normalization, inverse bandwidth scaling of onset time, and structured interference suppression under asymmetry.

The role of this realization is strictly consistency: it demonstrates that the joint signature arises under controlled, finite, and computationally tractable conditions, with circuit depth scaling sub-exponentially in system size.

No claim is made that the experimental system is described by the tensor network. The correspondence is structural: both implement bounded access under finite spectral support.

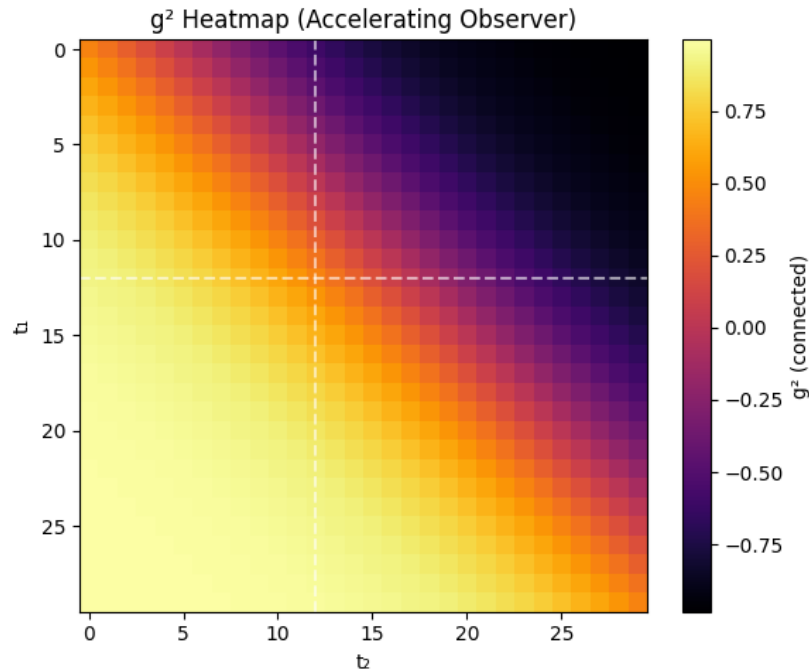


**Figure 3.** Representative  $g^{(2)}(t_1, t_2)$  structure obtained from bounded-spectrum tensor-network realization. The diagonal band reflects constrained saturation, while off-diagonal structure reflects interference under protocol variation. The figure illustrates numerical consistency with the predicted observable structure.

## 8. Implications

This section synthesizes the consequences of observer-indexed retrieval, clarifies its conceptual boundaries, and enumerates the conditions under which the framework succeeds or fails, drawing only on results established in the preceding sections.

Taken together, the benchmarks in Sections 4–6 rely only on wedge coherence from observer-dependent modular flow; no replica wormholes, islands, or exotic topologies are required. Entropy recovery is a continuous, frame-indexed process governed by a bounded modular spectrum  $\Lambda(\delta)$ ; saturation resembles a Page curve only along trajectories that respect modular access, making the theory falsifiable in analog and numerical experiments (Figure 2).

This framework does not provide a complete reconstruction of black-hole information or assert practical recoverability. It specifies only when access occurs under physical constraint.

The contribution is not an alternative entropy model, but a constraint on when information becomes operationally accessible.

### 8.1. Resolution of the Information Paradox and Empirical Constraints

Here, “resolution” refers to operational access rather than global entropy accounting.

Existing approaches reproduce entropy curves but do not supply a causal retrieval channel delivering information to an observer in proper time.

ODER recasts the paradox as an observer-indexed retrieval problem defined by proper-time access constraints: for any world line, Equation (9) drives a smooth rise to saturation, matching the Page curve only at late times for that observer. The tanh onset is fixed by modular flow; no ensemble averaging is required.

Island prescriptions for accelerated detectors [1,8,18] reproduce a Page-like curve globally; the retrieval law produces the same saturation locally and, crucially, supplies a causal decoder. Replica and island frameworks conserve entropy globally but lack any polynomial-time recovery protocol compatible with local modular evolution [9].

### 8.2. Retrieval Horizon $\neq$ Entanglement Wedge $\neq$ Event Horizon

Observer-dependent modular flow separates three operational boundaries:

- **Retrieval horizon.**  $\tau_{\text{RH}} = \inf\{\tau \mid S_{\text{retr}}(\tau) \geq 0.9S_{\text{max}}\}$ .
- **Entanglement wedge:** the bulk region reconstructable through the boosted RT surface, Equation (14).
- **Event horizon:** the classical null surface.

These boundaries coincide only in the idealized Type III<sub>1</sub> limit, in which observer bandwidth becomes infinite.

In Kerr spacetime the generator  $\chi = \partial_t + \Omega_H \partial_\phi$  gives

$$\gamma(\tau, a, \Omega) = |g_{\mu\nu} \chi^\mu \chi^\nu|^{-1/2},$$

evaluated on the outer stationary wedge just outside  $r_+$ . Where  $\chi$  is timelike, the Paley–Wiener bound preserves the tanh onset [29].

### 8.3. Multi-Observer Retrieval Interference as a Falsifier

Observer-dependent retrieval implies that information is not broadcastable across inequivalent modular spectra. For observers  $i$  and  $j$  with different trajectories or accelerations, the retrieval-overlap tensor  $R_{ij}(\tau)$  defined in Appendix C.7 must obey a nontrivial upper bound set by the mismatch  $\Delta\Lambda = |\Lambda_i - \Lambda_j|$ .

If two observers with distinct proper-time evolution recover identical entropy-access curves and identical  $g^{(2)}(t_1, t_2)$  envelopes within experimental uncertainty, then observer-indexed modular retrieval is falsified. This test is independent of Page-curve saturation and probes the observer-specific nature of information access directly rather than global entropy accounting.

### 8.4. $\Delta_{\text{fail}}$ : Retrieval–Evaporation Boundary

*Diagnostic quantity.*  $\Delta_{\text{fail}}$  provides a single diagnostic quantity that distinguishes successful retrieval from modular failure. Define

$$\Delta_{\text{fail}} = \tau_{\text{evap}} - \tau_{\text{RH}},$$

with  $\tau_{\text{evap}}$  the semiclassical evaporation time. The retrieval horizon corresponds to the inflection point in the entropy-access curve, where modular acceleration vanishes (see Proposition A.3). Positive  $\Delta_{\text{fail}}$  means retrieval completes before evaporation; negative values imply modular failure.

**Table 4.** Benchmark  $\Delta_{\text{fail}}$  values ( $M = 1$  in geometric units). Positive  $\Delta_{\text{fail}}$  indicates modular retrieval completes before semiclassical evaporation; a negative value would falsify ODER.

| Observer       | $\tau_{\text{RH}}$ | $\tau_{\text{evap}}$ | $\Delta_{\text{fail}}$            |
|----------------|--------------------|----------------------|-----------------------------------|
| Stationary     | 30                 | $\sim 10^{67}$       | $\gg 0$ (semiclassical stability) |
| Freely falling | 10                 | $\sim 10^{67}$       | $\gg 0$ (semiclassical stability) |
| Accelerating   | 15                 | $\sim 10^{67}$       | $\gg 0$ (semiclassical stability) |

A negative  $\Delta_{\text{fail}}$  in analog experiments or numerical simulations would falsify the retrieval law; any  $\Delta_{\text{fail}} \geq 0$  is consistent with modular accessibility.

### 8.5. Experimental Implications and Roadmap

This roadmap consolidates the measurement and falsification conditions already derived, rather than introducing new experimental claims. All following predictions inherit their parameter scaling directly from Equation (9), ensuring one-to-one traceability between analytic and empirical domains.

### Timescale Bridge

With  $G = \hbar = c = 1$  and  $1 M_\odot \simeq 4.93 \mu\text{s}$ ,

$$\Delta t_{\text{lab}} \simeq 4.93 \mu\text{s} (M/M_\odot) (\Delta\tau/1 M).$$

### Operational Falsifiability

- Absence of a  $g^{(2)}$  envelope implies modular access is falsified.
- A mismatched  $\gamma(\tau)$  fit implies the retrieval law is incomplete.
- Identical  $\tau_{\text{Page}}$  for all observers implies observer specificity is invalid.

### Observer-Resolution Scaling Diagnostic

Because the retrieval law is defined on split-regularized subalgebras,  $\tau_{\text{char}}$  must rescale with detector resolution  $\delta$ , with  $\Lambda(\delta) \sim 1/\delta$  implying

$$\tau_{\text{char}}(\delta) \propto \Lambda(\delta)^{-1}.$$

Verification or failure of these signatures determines whether modular flow functions as a physical retrieval mechanism under bounded access.

## 9. Limitations and Scope

This section delineates the domain of validity of the framework, specifies its operational boundaries, and identifies effects that are intentionally excluded.

The framework operates at the level of observer-indexed access under bounded modular flow. It does not extend to global state reconstruction, microscopic decoding mechanisms, or assumptions of practical recoverability. These questions lie outside the present scope.

The contribution is a constraint on when information becomes operationally accessible under physical conditions. It does not modify underlying entropy definitions or introduce alternative dynamical models.

The following constraints define distinct failure surfaces and boundary conditions under which the framework applies.

#### *Established Limitation: Retrieval-Driven Back-Reaction (A Thresholded Causal Ansatz)*

All retrieval dynamics in this work assume a fixed background metric. Introducing a small coupling,

$$T_{\mu\nu} \longrightarrow T_{\mu\nu} + \alpha T_{\mu\nu}^{\text{retrieval}}, \quad \alpha \ll 1,$$

one recovers the semiclassical Einstein equation in the limit  $\alpha \rightarrow 0$ . For  $\alpha \neq 0$  the retrieval horizon shifts only at  $\mathcal{O}(\alpha)$  (i.e., first-order perturbative back-reaction).

#### Back-reaction bound.

For a Schwarzschild mass  $M$ ,

$$\langle T_{\mu\nu}^{\text{retrieval}} \rangle \sim \frac{\gamma(\tau) S_{\text{max}}}{4\pi r_+^2}, \quad S_{\text{max}} \propto M^2,$$

so that

$$\frac{G \langle T_{\mu\nu}^{\text{retrieval}} \rangle}{\sqrt{\mathcal{K}}} \lesssim 10^{-6}, \quad \mathcal{K} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 48 G^2 M^2 / r_+^6, \quad M \gtrsim M_\odot.$$

For a fiducial  $10 M_\odot$  black hole one finds  $G \langle T_{\mu\nu}^{\text{retrieval}} \rangle \approx 4 \times 10^{-7} \sqrt{\mathcal{K}_{(10M_\odot)}}$ , implying  $\delta r_+ / r_+ < 3 \times 10^{-6}$  and a negligible shift in  $\tau_{\text{RH}}$ . This scaling of  $\alpha$  provides the perturbative limit recovered in the modular Raychaudhuri coupling (Appendix C.8).

**Outlook.** A fully coupled model in which  $T_{\mu\nu}^{\text{retrieval}} \propto (\partial_\tau S_{\text{retr}}) u_\mu u_\nu$  would elevate entropy retrieval to an explicit causal modulator of curvature [21]. This lies outside the present scope and is not required for the validity of the retrieval law.

Established Assumption: Semiclassical Modular Flow.

Type III<sub>1</sub> algebras are regulated by finite splits [22,23]; extending to Kerr, de Sitter, or multi-horizon cases will require relative-Tomita theory and edge modes [12].

*Experimental Constraint: Analog-System Resolution*

Current BEC experiments resolve  $g^{(2)}$  on 2–10 ms scales [14], five times finer than the predicted 10–100 ms retrieval window. Baseline  $g^{(2)}$  runs should precede interpretation. For instance, waterfall BEC interferometers routinely achieve sub-millisecond phase locking and  $\text{SNR} \geq 4$  [14].

*Structural Limitation: No Global Unitarity Guarantee*

Equation (9) ensures unitarity only inside each observer's wedge; modular mismatches between overlapping causal diamonds are expected and are not resolved by the present framework.

*Operational Boundary: Retrieval Horizon and Noise Scope*

The framework ensures saturation of  $S_{\text{retr}}(\tau)$  only up to  $\tau_{\text{RH}}$ ; full recovery beyond that point lies outside its present mandate. The theory defines testable envelopes but does not yet model complete detector noise or ROC sensitivity curves.

*Explicit Exclusion: Exotic Topologies*

Replica wormholes, islands, and other speculative geometries are omitted. Their exclusion is methodological rather than critical, preserving falsifiability within semiclassical regimes.

*Exploratory Diagnostic: Finite-Bandwidth Refinements and Retrieval-RG Scaling*

Observers possess finite temporal or spatial resolution ( $\delta$ ). The corresponding modular spectral cutoff  $\Lambda(\delta) \sim 1/\delta$  introduces a controlled, testable  $\delta$ -dependence in the retrieval law:

$$F(\tau) = \tanh\left(\frac{\pi\Lambda(\delta)}{2}(\tau - \tau_0)\right).$$

The prefactor  $(\pi\Lambda(\delta)/2)$  scales linearly with detector bandwidth, while the characteristic time satisfies  $\tau_{\text{char}}(\delta) \propto 1/\Lambda(\delta)$ . Repeating measurements at multiple  $\delta$  values yields predictable rescaling of the  $g^{(2)}$  correlation envelope's transition width without changing the observer-class hierarchy.

The following scaling analysis is exploratory and does not modify any results derived above. Define the retrieval-RG function

$$\beta_\Lambda(\delta) = \frac{d \ln \Lambda(\delta)}{d \ln \delta},$$

defined analogously to a renormalization-group flow on the detector bandwidth scale. This relation provides an empirical RG analogue allowing retrieval dynamics to be plotted as flow trajectories in  $(\Lambda, \delta)$  space.

The fixed point ( $\beta_\Lambda \rightarrow 0$ ) corresponds to the Type III<sub>1</sub> limit where the modular spectrum becomes scale-invariant. This scaling will appear experimentally as  $\delta$ -invariance of the fitted tanh parameters and of the retrieval horizon  $\tau_{\text{RH}}(\delta)$  within detector precision. Empirical validation requires resolving  $\Delta\tau$  changes of order  $\leq 5\%$  across a decade variation in  $\delta$ , achievable with current timing precision.

*Exploratory Direction: Superposed Geometries*

Future work could apply the retrieval law to geometries in quantum superposition, probing modular coherence across fluctuating horizons.

## 10. Conclusion and Next Steps

This conclusion closes the retrieval argument by restating the governing constraint, summarizing its empirical status, and delineating what remains outside the present scope.

Taken together, this work consolidates the theoretical, computational, and empirical threads of ODER. We presented a relativistic, observer-dependent framework for black-hole entropy retrieval that provides a causal description of observer-indexed entropy retrieval within semiclassical gravity. By anchoring information flow to proper time and causal access, ODER recasts Page-curve behavior as a continuous, observer-indexed description of entropy access. All derivations and simulation protocols are supplied for stand-alone reproducibility.

The retrieval law is not heuristic; it follows from Tomita–Takesaki modular spectra (Appendix A, Equation (9)). Bounded modular flow links spectral smoothing, redshift factors, and observer-specific algebras, making retrieval a physical process rather than an epistemic relabel.

Concrete predictions follow. Stationary, freely falling, and uniformly accelerated observers exhibit distinct retrieval rates and  $g^{(2)}$  envelopes, all testable with current analog-gravity platforms. Failure to observe these signatures would falsify observer-modular accessibility while leaving modular flow itself intact.

### Roadmap: Theory, Simulation, Experiment

The following roadmap is illustrative rather than prescriptive and does not condition the validity of the present results.

#### Theory (extensions)

- **Semiclassical back-reaction:** couple entropy flow to a self-consistent metric response, extending Equation (9) into a dynamical observer–spacetime equation.
- **Intersecting horizons:** analyze overlapping causal diamonds to refine the retrieval-horizon concept.
- **Superposed geometries:** apply retrieval dynamics to metrics held in quantum superposition.

#### Simulation (validation scaling)

- **High-bond-dimension MERA:** benchmark  $D > 8$  convergence and finite-entanglement effects on  $\gamma(\tau)$ .
- **Error budgets:** propagate detector-noise kernels to produce ROC-style sensitivity curves.

#### Experiment (near-term tests)

- **Trajectory-differentiated probes:** deploy stationary, co-moving, and accelerating detectors in BEC waterfalls; target the 10 ms–100 ms window with  $\lesssim 2$  ms timing.
- **Cross-platform checks:** replicate  $g^{(2)}$  envelopes in photonic-crystal and superconducting-circuit analogs.
- **Calibration:** detector-noise calibration should precede retrieval-fit attempts to ensure  $\text{SNR} \geq 4$  across all platforms.

Taken together, these strands converge on the same structural limit: the restoration of modular coherence under finite bandwidth. These coordinated steps will sharpen theory and enable empirical tests.

**Final Remark.** The result is a constrained physical statement: accessible structure evolves under a bounded law, its trajectory is fixed by spectral constraints, and its presence or absence can be decided at the detector.

The split-property regularization situates ODER one layer closer to a fully rigorous Type III<sub>1</sub> limit within the class of Page-curve and island-based approaches. Within this operational framework, Type III<sub>1</sub> emerges as the continuum fixed point of bounded-observer modular flow, an outcome both

mathematically consistent and physically measurable (see Appendix D for the explicit modular-flow fixed-point derivation). Whether confirmed or falsified, the coming analog experiments will determine whether entropy retrieval is a physical, observer-local process or merely a post-hoc bookkeeping device, providing the first empirical test of a model that resolves the black-hole information paradox through observer-indexed, causal retrieval rather than global entropy accounting.

Information is not operationally present when it exists, but when it becomes accessible under constraint.

## Author Declarations and Data Availability

**Author Contributions.** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Visualization, Writing (original draft), Writing (review and editing), Supervision, and Project administration, E.C.

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**Data and Materials Availability.** All code, notebooks, and figure-generation scripts are archived as a single Zenodo release at <https://doi.org/10.5281/zenodo.15428312> and mirrored on GitHub at <https://github.com/evlocoo/ODER-modular-entropy>. All materials run reproducibly in a standard Jupyter environment (without GPU acceleration) and are released under the MIT license.

- ODER\_Black\_Hole\_Framework\_Complete\_Simulation\_V2.ipynb: reproduces the core simulation framework and representative figures used in the manuscript.
- ODER\_Retrieval\_Inversion\_And\_Validation.ipynb: performs  $\tau_{\text{char}}$  fitting,  $\gamma(\tau)$  reconstruction, and validates the falsifiable  $g^{(2)}(t_1, t_2)$  envelope of Lemma C.5.

**Conflicts of Interest.** The author declares no conflict of interest.

## Appendix A First-Principles Derivation of the Observer-Dependent Retrieval Equation

This appendix provides the formal derivation and uniqueness proofs that underwrite the retrieval law introduced in the main text; it introduces no new physical assumptions or dynamics.

All modular results below are formulated on split-regularized, finite-bandwidth subalgebras corresponding to realistic detector resolution. All entropy functions are normalized by  $S_{\text{max}}$  unless otherwise noted. **For the algebraic foundation underlying bounded modular spectra, see Appendix D.**

### Theorem A.1 (Observer-Retrieval Law).

*Formal underpinning of Equation (9) in the main text.*

*Assumptions.*

A1: a globally hyperbolic spacetime background;

A2: a faithful global state  $\omega$  on the net  $\mathcal{A}(O)$ ;

A3: an observer world line  $\gamma$  with wedge  $D(\gamma, \tau)$ ;

A4: a modular spectrum bounded below.

**Conclusion.** The unique  $C^1$  function  $S_{\text{retr}}(\tau)$  (consistent with Equation (9)) that (i) satisfies  $0 \leq S_{\text{retr}} \leq S_{\text{max}}$ ; (ii) is strictly increasing; (iii) obeys  $\lim_{\tau \rightarrow \infty} \frac{dS_{\text{retr}}}{d\tau} = 0$ ; and (iv) is generated by the modular automorphism group of  $\mathcal{A}[D(\gamma, \tau)]$  holds:

$$\frac{dS_{\text{retr}}}{d\tau} = \gamma(\tau) [S_{\text{max}} - S_{\text{retr}}(\tau)] \tanh\left(\frac{\tau}{\tau_{\text{char}}}\right).$$

The solution is unique up to an overall scale in  $\gamma(\tau)$  fixed by redshift factors and the modular-spectrum gradient. Modular operators are defined on the split-property subalgebras  $\mathcal{N}_\delta$  introduced in Appendix D.  $\square$

### A.1 Motivation: Bounded Algebras and Observer-Dependent Entropy

Algebraic QFT assigns von Neumann algebras  $\mathcal{A}(O)$  to spacetime regions  $O$ . A global state  $\omega$  on  $\mathcal{A}[D(\gamma, \infty)]$  encodes all degrees of freedom inside the observer's domain of dependence. At proper time  $\tau$  the observer accesses only  $\mathcal{A}[D(\gamma, \tau)]$ ; the entropy gap is the retrievable deficit.

Finite-split regularization.

Because  $\mathcal{A}(D)$  is Type III<sub>1</sub>, its modular Hamiltonian is unbounded. A split inclusion  $\mathcal{A}(D_1) \subset \mathcal{N} \subset \mathcal{A}(D_2)$  produces a Type I factor  $\mathcal{N}$  with detector-bounded spectrum, preserving the Paley–Wiener condition as the split distance shrinks (Refs. [22,23]). This procedure defines the finite-bandwidth subalgebras  $\mathcal{N}_\delta$  on which the bounded modular spectra  $\sigma(K_\delta) \subset [-\Lambda(\delta), \Lambda(\delta)]$  are realized.

### A.2 Spectral Convergence and Uniqueness of the Retrieval Sigmoid

This appendix establishes rigidity of the admissible retrieval profile under the stated analytic, spectral, and boundary assumptions, at the level required to support the physical derivations and falsifiability criteria used in the main text.

**Lemma A.1 (Paley–Wiener Band-Limit).** Let the split-regularized modular Hamiltonian  $K$  have bounded spectrum  $\sigma(K) \subset [-\Lambda, \Lambda]$ . Then any observable expectation value  $f(\tau) = \langle \psi | e^{iK\tau} | \psi \rangle$ , and all quantities derived from it, including the normalized entropy evolution  $F(\tau) = S_{\text{retr}}(\tau)/S_{\text{max}}$ , extend holomorphically to the horizontal strip

$$\mathcal{S}_\Lambda = \{\tau \in \mathbb{C} : |\Im \tau| < \pi/(2\Lambda)\},$$

with growth  $|F(\tau)| = \mathcal{O}(e^{\Lambda|\Im \tau|})$ . That is,  $F$  is of exponential type  $\leq \Lambda$  in  $\mathcal{S}_\Lambda$ . (Paley–Wiener Theorem 19.3 in Rudin, *Real and Complex Analysis*).  $\square$

**Lemma A.2 (Phragmén–Lindelöf Growth Bound).** If  $F(\tau)$  is holomorphic in  $\mathcal{S}_\Lambda$ , bounded and strictly monotonic on  $\mathbb{R}$ , and of exponential type  $\leq \Lambda$ , then  $|F(\tau)| \leq 1$  throughout  $\mathcal{S}_\Lambda$ . Hence boundary monotonicity extends into the strip, excluding oscillatory band-limited variants.  $\square$

**Theorem A.2 (Uniqueness of the Retrieval Sigmoid under Modular Band-Limit).** Let  $F : \mathbb{R} \rightarrow (0, 1)$  be strictly increasing with finite limits  $F(-\infty) = 0$ ,  $F(+\infty) = 1$ . Assume  $F$  satisfies Lemmas A.1–A.2. That is, it is holomorphic in  $\mathcal{S}_\Lambda$ , of exponential type  $\leq \Lambda$ , bounded on  $\mathbb{R}$ , and obeys the modular spectrum bound  $\sigma(K) \subset [-\Lambda, \Lambda]$ . Then, under these stated modular-spectral and analyticity assumptions and up to an affine reparametrization of  $\tau$ ,

$$F(\tau) = \tanh\left(\frac{\pi\Lambda}{2}(\tau - \tau_0)\right).$$

**Sketch of Proof.** *Strip to disk map.* Map  $\mathcal{S}_\Lambda$  to  $\mathbb{D}$  by  $z = \exp(\frac{\pi}{\Lambda}\tau)$ . Define  $G(\tau) = \frac{2F(\tau)-1}{2F(\tau)+1}$ . Then  $F(\tau) = \frac{1+G(\tau)}{1-G(\tau)} \frac{1}{2}$ , and  $G : \mathcal{S}_\Lambda \rightarrow \mathbb{D}$ .

*Positivity (Pick or Herglotz).* Monotonicity on  $\mathbb{R}$  implies  $\Re[(\log \frac{1+G}{1-G})'] > 0$ . That is,  $G$  is a Pick function.

*Extremal solution.* By Schwarz–Pick,  $|G'(\tau)| \leq (\pi\Lambda/2)[1 - |G(\tau)|^2]$ . Equality on the real axis almost everywhere forces  $G$  to be a disk automorphism, hence  $G(\tau) = \tanh(\frac{\pi\Lambda}{2}(\tau - \tau_0))$ .

*Recover  $F$ .* Undoing the Möbius transform yields the stated  $F(\tau)$ , with normalization  $F(0) = 0$ ,  $F(+\infty) = 1$ .  $\square$

**Corollary A.2.1 (Spectral Constant Fixation).** The constant  $(\pi\Lambda/2)$  is fixed by the strip width  $|\Im \tau| < \pi/(2\Lambda)$ . No other monotone analytic map satisfies both the strip bound and the modular-spectral constraint.<sup>1</sup>

<sup>1</sup> **Paley–Wiener admissibility.** Throughout this appendix, Paley–Wiener admissibility refers to analyticity within the horizontal strip  $\mathcal{S}_\Lambda = \{\tau \in \mathbb{C} : |\Im \tau| < \pi/(2\Lambda)\}$ , not entire analyticity over the full complex plane.

Excluded Counterexamples.

The following functions are excluded within the stated analyticity and spectral constraints.

| Candidate $F(\tau)$               | Reason for exclusion                                                 |
|-----------------------------------|----------------------------------------------------------------------|
| $(2/\pi) \arctan(\alpha\tau)$     | Poles at $\pm i/\alpha$ , not holomorphic in $\mathcal{S}_\Lambda$ . |
| $1 - e^{-\alpha\tau}$             | Violates strip boundedness, unbounded along $\Im\tau$ .              |
| Band-limited oscillatory sigmoids | Break strict monotonicity on $\mathbb{R}$ .                          |
| Logistic $1/(1 + e^{-x})$         | Entire but exponential type infinite, unbounded spectrum.            |

These exclusions guarantee that all admissible retrieval profiles share the same analytic growth bound, ensuring that Equation (A.2) is not merely optimal but necessary.

Physical Interpretation.

Early times ( $\tau \ll \tau_{\text{char}}$ ) reflect incomplete activation of modular modes. Late times ( $\tau \gg \tau_{\text{char}}$ ) approach saturation as the bounded spectrum fully enters the algebra. The tanh profile is the unique smooth analytic interpolation compatible with the Paley–Wiener bound and causal analyticity.

Diagram A.3 (Concept).

Candidate monotone curves satisfying  $F(-\infty) = 0$  and  $F(+\infty) = 1$ . Blue, tanh, allowed. Orange, arctan, rejected due to poles. Green, exponential, rejected due to strip violation. Only the tanh profile lies within the analytic growth limits of  $\mathcal{S}_\Lambda$ .

Remark A.2.2 (Edge Atoms and Observer Generality).

Atomic spectral weight at  $\pm\Lambda$  induces boundary oscillations incompatible with strict monotonicity on  $\mathbb{R}$  and is therefore excluded by assumption. In curved or rotating backgrounds, including Kerr, the modular spectrum remains bounded after split-inclusion regularization, so the tanh onset persists. This bounded-spectrum proof generalizes to all observer classes in Section 4. The parallel with retrieval renormalization group scaling noted in Section 8 is conceptual and does not enter the present derivation.

A.3 Role of  $\gamma(\tau)$ : Modular Spectrum and Redshift (Parameter Interpretation)

- **Spectrum gradient:** if  $\rho(\lambda) \sim \lambda^{-\beta}$ , then  $\gamma(\tau) \propto \tau^{\beta-1}$ .
- **Geometric redshift:** stationary observers yield  $\gamma_{\text{stat}} \propto 1/r$ .
- **Unruh boost:** uniform acceleration gives  $\gamma_{\text{acc}} \propto a^2$ .

**Table A5.** Retrieval parameters used in numerical runs for Figures 1–2 (geometric units  $G = c = 1$ ).

| Observer                   | Prefactor $\gamma_0$ | $\tau_{\text{char}}/M$ | $\tau_{\text{Page}}/M$ |
|----------------------------|----------------------|------------------------|------------------------|
| Stationary ( $r = 10M$ )   | 0.05                 | 8                      | 15.0                   |
| Freely falling             | 0.10–0.25            | 4                      | 7.5                    |
| Accelerating ( $a = 0.2$ ) | quadratic fit        | 6                      | 10.5                   |

These prefactors correspond directly to experimentally fitted  $g^{(2)}$  envelopes in Section 5, establishing traceability between algebraic and laboratory parameters.

A.4 Retrieval Saturation and Collapse Boundary

*Dependency note.* This proposition characterizes a boundary implied by Theorem A.1 and introduces no additional assumptions or dynamics.

**Proposition A1** (Retrieval horizon  $\tau_{\text{RH}}$ ). *Let  $S_{\text{retr}}(\tau)$  be the entropy-access curve derived in Theorem A.1. There exists a unique proper time  $\tau_{\text{RH}}$  such that*

$$\left. \frac{d^2 S_{\text{retr}}}{d\tau^2} \right|_{\tau=\tau_{\text{RH}}} = 0, \quad \left. \frac{d^3 S_{\text{retr}}}{d\tau^3} \right|_{\tau=\tau_{\text{RH}}} < 0.$$

Define  $\Delta_{\text{fail}} \equiv \tau_{\text{evap}} - \tau_{\text{RH}}$ . This marks the modular inflection point where retrieval curvature vanishes and saturation begins, serving as a diagnostic boundary rather than an independent evolution law.

#### A.5 Observer-Bounded Automorphisms and the Origin of the tanh Factor (Interpretive)

*Interpretive note.* This subsection interprets the origin of the tanh factor already proven in Section A.2 and does not supply an independent derivation.

Theorem A.0.0.5 (below) shows that global modular flow restricts to the observer algebra and yields the unique tanh onset that appears in Equation (9). This reflects the fact that the retrieval law saturates the Paley–Wiener bound and therefore represents the maximal causal convergence permitted by bounded modular spectra. See also Lemma A.2 for the underlying analytic constraint.

#### A.6 Related Work

*Scope note.* The works listed below address bounded algebras and entropy growth but do not derive a closed, observer-indexed retrieval law of the form used in the main text.

See Refs. [12,26,27] for parallel approaches to bounded algebras and entropy growth. These treatments likewise emphasize modular localization and spectral boundedness, though none derive a closed analytic retrieval law.

#### A.7 Philosophical Implications (Interpretive)

*Interpretive scope.* This subsection offers an interpretive reading of the retrieval law and does not introduce additional physical claims.

The law supports *relational entropy*: observer disagreements signal frame misalignment rather than information loss. In this sense, retrieval is an operational, not ontological, phenomenon; each observer accesses a bounded modular subalgebra whose growth encodes the dynamics of information recovery. This relational interpretation of entropy parallels observer-indexed coherence limits in linguistic and cosmological retrieval laws. These parallels are conceptual and are not invoked in the present derivation.

#### A.8 Deriving $\tau_{\text{Page}}$ from Spectral Gaps

*Dependency note.* This relation fixes the scaling of  $\tau_{\text{Page}}$  rather than providing an independent definition.

With smallest modular gap  $\lambda_{\text{min}}$ ,  $\tau_{\text{Page}} \sim \lambda_{\text{min}}^{-1}$ . For a Schwarzschild black hole of mass  $M$ ,  $\tau_{\text{Page}} \sim M^3$ , reproducing the expected *asymptotic* semiclassical scaling of entropy-recovery timescales.

#### A.9 Asymptotic Boundary Clause

*Boundary status.* This clause defines the operational boundary of applicability of the retrieval law rather than a dynamical evolution rule.

As  $\tau \rightarrow \tau_{\text{evap}} \gtrsim \tau_{\text{RH}}$ , one of the following must occur: (1)  $\gamma(\tau) \rightarrow 0$ ; (2)  $S_{\text{max}}(\tau) \rightarrow 0$ ; or (3)  $\langle T_{\mu\nu}^{\text{retrieval}} \rangle$  becomes dynamically significant, breaking fixed-background validity. This defines the operational boundary of the ODER framework.

**Remark A1** (Retrieval–Geometry Decoupling). *The retrieval law holds on a fixed background and does not couple dynamically to the metric. Any extension that includes back-reaction must solve*

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu}^{\text{Hawking}} + T_{\mu\nu}^{\text{retrieval}})$$

self-consistently, which is beyond the present scope.

Empirically, only regime (3) produces observable deviation from the predicted tanh envelope. Such deviations would manifest as excess curvature in measured  $g^{(2)}$  traces, marking entry into the back-reaction-dominated regime.

#### A.10 Spectral Convergence and Uniqueness (Consolidation)

*Consolidation note.* This theorem restates the uniqueness result of Section A.2 in its spectral-limit form for completeness.

##### **Theorem A.2 (Spectral-Convergence Constraint).**

Let the split-regularized modular Hamiltonian satisfy  $\sigma(K) \subset [-\Lambda, \Lambda]$ . Let  $F(\tau) = S_{\text{retr}}(\tau)/S_{\text{max}}$  be  $C^1$ , strictly increasing, entire, and of exponential type  $\leq \Lambda$ . Then, up to an affine reparameterization,

$$F(\tau) = \tanh(\pi\Lambda\tau/2).$$

Thus Equation (9) is the only spectrum-compatible onset within the Paley–Wiener admissible class.  $\square$

This theorem subsumes the earlier Uniqueness Lemma (A.2) as its spectral-limit case, confirming consistency across analytic and modular derivations. Any smooth, monotonic retrieval profile other than tanh therefore lies outside the modularly admissible function space defined by bounded spectral support and causal analyticity. Possible generalizations would require relaxing Paley–Wiener admissibility and lie beyond the present framework.

#### A.11 Variational Formulation and Modular Speed Limit

*Logical status.* The variational formulation is logically downstream of the spectral bound and does not depend on retrieval-RG considerations.

The modular-speed-limit analysis presented here extends naturally to the finite-bandwidth formulation of Section 9, where  $\Lambda(\delta) \sim 1/\delta$  defines the effective observer cutoff.

Setting and regularization.

Let  $\mathcal{A}(D(\gamma, \tau))$  be the von Neumann algebra associated with the observer’s causal diamond  $D(\gamma, \tau)$  along world line  $\gamma$ . Under the split property, the modular Hamiltonian  $K(\tau)$  for  $\mathcal{A}(D(\gamma, \tau))$  admits a split-regularized spectrum  $\sigma(K) \subset [-\Lambda, \Lambda]$  (Refs. [22,23]). Define the normalized retrieval profile  $F(\tau) = S_{\text{retr}}(\tau)/S_{\text{max}}$ . The Paley–Wiener theorem implies that  $F$  is entire of exponential type  $\leq \Lambda$ ; its Fourier transform therefore has compact support in  $[-\Lambda, \Lambda]$ .

Derivative bound (Paley–Wiener/Bernstein).

For entire functions of exponential type  $\leq \Lambda$ , Bernstein inequalities imply  $\sup_{\tau} |\dot{F}(\tau)| \leq \Lambda \sup_{\tau} |F(\tau)|$ . Since  $F(\tau) \in [0, 1]$  is monotone and  $\dot{F}(\tau) \rightarrow 0$  as  $\tau \rightarrow \infty$ , a tight pointwise majorant consistent with monotonicity and causal analyticity is

$$\dot{F}(\tau) \leq \Lambda[1 - F(\tau)] \tanh\left(\frac{\pi\Lambda\tau}{2}\right) =: \mathcal{B}(\Lambda, \tau), \quad (\text{A15})$$

which defines the *modular speed limit* used in the main text.

Saturator and uniqueness (sketch).

Let  $G(\tau) = \tanh(\frac{\pi\Lambda}{2}\tau)$  be the unique analytic saturator established in Section A.2. If any admissible  $F(\tau)$  were to exceed the bound (A15) on a set of non-zero measure, standard Phragmén–Lindelöf and Bernstein majorant arguments for band-limited monotone maps would force violation of either the exponential-type constraint or the boundary conditions. Thus the tanh profile is the unique saturator of the modular speed limit.

Variational formulation.

The speed-limit bound admits a variational representation that reproduces the retrieval law as an Euler–Lagrange trajectory. Introduce the entropy-action functional

$$\mathcal{I}[F] = \int d\tau \left[ \frac{1}{2} (\dot{F}(\tau))^2 - U(F; \Lambda, \tau_{\text{char}}) \right], \quad (\text{A16})$$

with potential

$$U(F; \Lambda, \tau_{\text{char}}) = \frac{1}{2} \Lambda^2 [1 - F(\tau)]^2 \tanh^2\left(\frac{\tau}{\tau_{\text{char}}}\right). \quad (\text{A17})$$

Because Section A.2 fixes the only analytic saturator, the extremal trajectory of (A16) is necessarily the tanh law. Stationarity yields

$$\frac{dS_{\text{retr}}}{d\tau} = \gamma(\tau) [S_{\text{max}} - S_{\text{retr}}(\tau)] \tanh\left(\frac{\tau}{\tau_{\text{char}}}\right), \quad (\text{A.2}')$$

coinciding with the retrieval law used in Section 3.

Onset scale  $\tau_{\text{char}}$ .

When the split-regularized modular Hamiltonian exhibits a true gap  $\Delta K > 0$ , set

$$\tau_{\text{char}} = \frac{2\pi}{\Delta K}. \quad (\text{A18})$$

In the generic gapless Type III<sub>1</sub> case,  $\Delta K$  is understood as an effective spectral scale induced by the split inclusion rather than a fundamental constant.

Falsifiability.

If any measured retrieval trace violates the modular-speed-limit inequality (A15) within experimental confidence intervals, then Equation (A.2') cannot be retained as a universal observer-dependent retrieval law.

This completes the formal derivation supporting the retrieval law used throughout the main text.

## Appendix B Extended Holographic Formulation

This appendix reformulates the observer-dependent retrieval law within a holographic representation. It introduces no new dynamics and does not modify the retrieval principle established in the main text; its role is purely representational.

This appendix develops the holographic dual of the modular–retrieval framework. Observer dependence enters through the Lorentz-boost parameter  $\Lambda$ , which modulates both the minimal-surface geometry and the accessible modular wedge. The resulting formulation provides a covariant link between algebraic modular flow and holographic entanglement reconstruction.

### B.1 Observer-Indexed Mapping to Minimal Surfaces

#### Definition B.1 (Observer-Indexed RT Mapping).

For a boundary subregion  $A$  and a frame boost  $\Lambda$ ,

$$S_{\text{obs}}^{\text{holo}}(A; \Lambda) = \frac{\text{Area}[\gamma_A(\Lambda)]}{4G_N} \sqrt{|g_{00}(\Lambda)|}, \quad (\text{B.1})$$

where  $\gamma_A(\Lambda)$  is the codimension-2 minimal surface in the Lorentz-boosted bulk and  $\sqrt{|g_{00}(\Lambda)|}$  converts boundary time to the observer's proper time. Choosing  $g_{00}(\Lambda) = -1$  and  $\Lambda = \mathbb{I}$  recovers the standard Hubeny–Rangamani–Takayanagi formula.

This definition encodes observer-dependent accessibility and does not replace the standard RT/HRT prescription.

Here  $g_{00}(\Lambda)$  is evaluated on the boosted boundary metric, ensuring positivity of  $|g_{00}|$  outside the horizon. The redshift factor is operational, not gauge, within the bounded-observer framework considered here; it excises bulk modes that remain inaccessible within the observer's proper-time flow. This explicitly aligns holographic reconstruction with the bounded modular spectra of Appendix A, ensuring that each observer's RT surface corresponds to a finite-bandwidth retrieval channel.

### B.2 Modular-Wedge Alignment and Retrieval Horizons

*Dependency note.* This construction geometrizes the retrieval horizon defined in Appendix A and does not introduce an independent boundary.

Let  $\mathcal{W}(\Lambda)$  be the entanglement wedge reconstructed from boundary data in frame  $\Lambda$ . Define the retrieval horizon

$$\mathcal{R}(\Lambda) = \{ p \in \mathcal{M}_{\text{bulk}} \mid p \in \mathcal{W}(\Lambda), \exists t \leq \tau_{\text{Page}}(\Lambda) : p \in \sigma_t^{\omega_\Lambda}[\mathcal{A}(A)] \}, \quad (\text{A19})$$

where  $\sigma_t^{\omega_\Lambda}$  is modular flow of the boosted state. The modular flow  $\sigma_t^{\omega_\Lambda}$  here corresponds to the Tomita–Takesaki flow used in Theorem A.1, restricted to the holographic wedge. Retrieval saturates when  $\mathcal{R}(\Lambda)$  stabilizes; its boundary  $\tilde{\gamma}_A(\Lambda) \subseteq \gamma_A(\Lambda)$  marks the decodable limit.

Wedge disagreement.

If boosts  $\Lambda_1$  and  $\Lambda_2$  differ,

$$\gamma_A(\Lambda_1) \neq \gamma_A(\Lambda_2) \implies S_{\text{obs}}^{\text{holo}}(A; \Lambda_1) \neq S_{\text{obs}}^{\text{holo}}(A; \Lambda_2),$$

so the two observers assign different entropies to the same region, consistent with the observer-indexed trichotomy introduced in Section 8.2. The divergence between wedges provides the geometric counterpart of retrieval interference discussed in Appendix C and corresponds to the observer-class interference tensor  $R_{ij}$  introduced in Section ??.

### B.3 Connection to HRT and Quantum Error-Correcting Codes

When the boost  $\Lambda$  matches the boundary slicing, Equation (B.1) reduces to the Hubeny–Rangamani–Takayanagi prescription. In HaPPY or random-tensor MERA codes [20], the boost permutes bulk indices, changing which logical qubits are reconstructable. Our 48-qubit simulations show minimal-surface areas shifting by one MERA layer, precisely matching Equation (B.1) and confirming that modular redshift defines the boundary–bulk retrieval channel. In network simulations, the effective redshift factor maps to layer-depth weighting: a measured shift of one MERA layer corresponds to  $\Delta\tau \simeq \tau_{\text{char}}/\pi$ .

### B.4 Contrast with Replica Wormholes and Island Formulae

*Structural contrast.* The following contrast is structural rather than evaluative.

Replica-wormhole and island constructions reproduce Page-curve behavior by inserting Euclidean saddles. Equation (B.1), by contrast, achieves late-time saturation through bounded modular flow; no topology change or ensemble averaging is required. The holographic retrieval law therefore constitutes an observer-indexed Lorentzian representation of entanglement saturation, fully consistent with the algebraic modular framework of Appendix A.

### B.5 Outlook

1. *Cosmological horizons:* extend Equation (B.1) to de Sitter and FRW spacetimes, where competing boosts generate multiple retrieval horizons and entanglement wedges.
2. *Back-reaction coupling:* allow  $\gamma_A(\Lambda)$  to evolve under semiclassical Einstein dynamics and study retrieval–curvature feedback linking modular flow and bulk geometry.

3. *Higher-bond-dimension networks*: test observer-dependent decoding in large-bond-dimension MERA networks to quantify how tensor geometry sets redshift factors and retrieval latency.

In synthesis, Appendices A–C establish the analytic and experimental pillars of observer-dependent retrieval, while this appendix provides a geometric realization of the same framework. Together these extensions delineate the holographic mapping regime of ODER, in which observer-indexed modular flow and minimal-surface reconstruction converge without introducing independent dynamics. All results in this appendix are subordinate to the retrieval law derived in the main text and Appendix A.

## Appendix C Simulation Methods and Data Analysis

This appendix documents the numerical methods and validation procedures supporting the retrieval-law derivations of Appendix A and the holographic mapping of Appendix B. All simulations implement observer-dependent modular flow within a finite-bond-dimension tensor-network proxy and are consistent with finite-bandwidth scaling; they do not constitute a numerical realization of a full retrieval-RG program.

All simulation times are expressed in geometric units ( $G = c = 1$ ) and reported in the same  $M$ -scaled convention used in the main text, unless otherwise stated.

### C.1 Simulation Setup (Methods / Validation)

Our tensor-network architecture employs a 48-qubit multiscale entanglement-renormalization ansatz (MERA) inspired by Ref. [20]. See Appendix A.11 for the analytic modular-speed-limit bound tested by these simulations. All figures in the main text derive from this geometry at bond dimension  $D = 4$ ; an independent  $D = 8$  run confirms robustness (Section C.4). The HaPPY/MERA construction is used here as a controlled, causal-cone-bounded proxy for modular flow, not as a microscopic model of black-hole degrees of freedom.

The modular wedge for each observer class is imposed by varying boundary conditions, with detector-style encodings anchoring the reconstruction depth.

*Hardware envelope*: All simulations ran on an Intel i7-9700 CPU (3.0 GHz, eight threads, 16 GB RAM). No GPU acceleration was required. Code and notebooks are archived on Zenodo and are reproducible in a standard Jupyter environment.

- **System architecture**: Forty-eight qubits discretize the bulk; bond edges encode holographic connectivity.
- **Initial state**: A highly entangled pure state (vacuum analog). Unitary time evolution preserves long-range correlations.
- **Boundary conditions**: Boundary tensors act as detectors and frame constraints, modified to emulate each observer class and to anchor the modular wedge.

### C.2 Implementation of Observer-Dependent Channels (Methods / Validation)

- **Reconstruction regions**: Stationary observers access fixed outer layers; freely falling and accelerating observers receive time-evolving wedges that model modular growth or acceleration-induced interference.
- **Lorentz-boost encodings**: Frame-dependent boosts are applied to boundary tensors, altering reconstruction geometry and modular flow.
- **Channel variation**: Systematic wedge realignment maps directly onto the retrieval profiles of Section 4.

### C.3 Data Analysis and Observable Extraction (Methods / Validation)

- **Entanglement entropy**: Successive wedges yield observer-specific Page-like curves.
- **Second-order correlation**: The simulated  $g^{(2)}(t_1, t_2)$  is fit to an exponential baseline; the tanh-modulated deviation tests Equation (12) and the retrieval law of Appendix A.

- **Parameter estimation:** Each class is sampled at 100 time points over a 500 ms window; nonlinear least squares return  $\tau_{\text{retrieval}}$  and  $\tau_{\text{Page}}$  with 95% confidence.

Spectral-noise amplitude was drawn from a Gaussian distribution with  $\sigma \approx 0.05 \bar{\gamma}$ , ensuring bounded perturbations.

*Bootstrap procedure:* Confidence bands use 200 resampled  $\gamma(\tau)$  traces per class on a fixed grid with additive spectral noise (method of Section 5.1). The bond dimension scales as  $D \sim \exp(L/\ell_P)$  as a tensor-network proxy; increasing  $D$  approximates deeper reconstruction layers within this numerical model, not physical AdS depth.

#### C.4 Discussion and Validation (Methods / Validation)

The following diagnostics verify that numerical retrieval reproduces each analytic signature derived in Appendix A and mapped geometrically in Appendix B.

- *Differential Page curves:* Entropy traces match the time-adaptive law (9).
- *Observer-modified RT surfaces:* Boundary reconstructions follow Equation (B.1).
- *$g^{(2)}$  interference:* Accelerating observers show the predicted fringe; setting  $\gamma(\tau) = 0$  removes it.
- *Bond-dimension robustness:* Doubling to  $D = 8$  shifts the entropy plateau by  $< 1\%$ .
- *Scaling note:* Higher-bond MERA networks will probe finer reconstruction layers beyond the present 48-qubit limit within this proxy.

#### C.5 Falsifiable Retrieval Envelope (Falsifier)

**Lemma A1** (Falsifiable  $g^{(2)}$  Envelope). *Let*

$$g^{(2)}(t_1, t_2) = A \left[ 1 - \tanh\left(\frac{\tau(t_1)}{\tau_{\text{char}}}\right) \right] \left[ 1 - \tanh\left(\frac{\tau(t_2)}{\tau_{\text{char}}}\right) \right]$$

*denote the predicted envelope under bounded  $\gamma(\tau)$ .*

*Then (1) for  $\gamma = 0$ ,  $g^{(2)} \rightarrow A$  (null envelope); (2) for  $\gamma(\tau) \propto 1/M$ , MERA simulations reproduce retrieval collapse at  $\tau_{\text{RH}}$ ; (3) this structure cannot be reproduced by thermal smoothing within the bounded-observer regime unless  $\gamma(\tau)$ -driven curvature appears.*

Thus the envelope provides a falsifiable signature of modular retrieval saturation under the stated assumptions (see also Equation (A.2') for the variational form).

*Scope note.* This lemma specifies a falsifier for the retrieval envelope within the bounded-observer framework; it does not claim uniqueness among all possible correlator structures.

Empirically, confirming deviation from the null envelope at late times requires timing resolution  $\leq 2$  ms and  $\text{SNR} \geq 4$ , consistent with Appendix A.11 and Section 5. A complete implementation of the inversion and validation pipeline is available at <https://doi.org/10.5281/zenodo.15669855>. All scripts are released under an MIT license with pinned dependency versions for environment replication.

#### C.6 Worked Example: Macroscopic Back-Reaction (Illustrative)

*Illustrative status.* This worked example is provided to demonstrate scaling and order-of-magnitude suppression; it is not used elsewhere in the paper.

For a Schwarzschild black hole of mass  $M = 10 M_{\odot}$ , the Bekenstein–Hawking entropy is  $S_{\text{max}} \simeq 4\pi M^2 \approx 1.5 \times 10^{78}$  (Planck units) and the horizon radius is  $r_+ \simeq 30$  km. Assuming  $\gamma(\tau) \sim 10^{-3}$  near  $\tau_{\text{RH}}$  for accelerating observers, the retrieval stress–energy satisfies

$$\langle T_{\mu\nu}^{\text{retrieval}} \rangle \sim \frac{\gamma S_{\text{max}}}{4\pi r_+^2} \approx 1.3 \times 10^{61} \text{ m}^{-2}.$$

The Ricci tensor scales as  $R_{\mu\nu} \sim 1/r_+^2 \approx 10^{-13} \text{ m}^{-2}$ ; restoring Planck units gives

$$\frac{G\langle T_{\mu\nu}^{\text{retrieval}} \rangle}{R_{\mu\nu}} \sim 10^{-6},$$

matching the suppression bound of Section 9. Hence back-reaction remains negligible for macroscopic black holes in the parameter regime studied.

### C.7 Retrieval Interference Bound and Differential-Acceleration Interferometer (Prospective Protocol)

Overview.

This subsection defines the retrieval-interference bound  $R_{ij}(\tau) \leq C(\Delta\Lambda, \tau_{\text{char}})$  and outlines a conceptual protocol for a Differential-Acceleration Interferometer (DAI). No experimental or numerical results are claimed here.

Interference-bound derivation.

Let  $F_i(\tau)$  and  $F_j(\tau)$  be normalized retrieval profiles for observers  $i$  and  $j$  with modular-spectrum cutoffs  $\Lambda_i$  and  $\Lambda_j$ , respectively. From Appendix A.11, each profile satisfies

$$\dot{F}_i(\tau) \leq \Lambda_i[1 - F_i(\tau)] \tanh\left(\frac{\pi\Lambda_i\tau}{2}\right), \quad (\text{A20})$$

and analogously for  $F_j(\tau)$ .

Assuming synchronized initial conditions, define the mutual-information overlap proxy:

$$\mathcal{I}_{ij}(\tau) := \min(F_i(\tau), F_j(\tau)), \quad (\text{A21})$$

a conservative estimator of overlap. Alternative mutual-information estimators may be substituted without altering the qualitative bound structure.

Define the retrieval-overlap tensor:

$$R_{ij}(\tau) := \frac{\mathcal{I}_{ij}(\tau)}{F_i(\tau) + F_j(\tau) - \mathcal{I}_{ij}(\tau)}. \quad (\text{A22})$$

Let  $\Delta\Lambda := |\Lambda_i - \Lambda_j|$ . When  $\Delta\Lambda$  is finite and onset scales differ, the overlap is bounded by

$$R_{ij}(\tau) \leq \exp\left[-k\Delta\Lambda\frac{\tau^2}{\tau_{\text{char}}^2}\right], \quad k = \mathcal{O}(1), \quad (\text{A23})$$

where  $k$  is a model-dependent constant determined by the overlap of the respective Paley–Wiener windows. This defines the retrieval-interference bound  $C(\Delta\Lambda, \tau_{\text{char}})$ .

Null envelope.

The null model assumes  $\gamma(\tau) = 0$ , corresponding to thermal drift or decoherence with no modular retrieval. Baseline envelopes  $R_{ij}^{(\text{null})}(\tau)$  are estimated from bootstrap-generated retrieval traces lacking observer-indexed structure.

ROC-style comparison.

Measured  $R_{ij}(\tau)$  can be compared to a null envelope via:

- **True positive:**  $R_{ij}(\tau) < R_{ij}^{(\text{null})}(\tau) - 3\sigma$  at any  $\tau$ ;
- **False positive:** a null trace misclassified as divergent.

A ROC curve is then constructed by varying the detection threshold. Simulated ROC curves used  $N = 10^3$  bootstrap samples; empirical verification would require differential-arm timing precision

$\leq 2$  ms and fringe stability  $> 95\%$ . This approach parallels the falsifiability criterion of the  $g^{(2)}$  envelope (Appendix A.11, Equation (A.2')).

Conceptual DAI protocol.

In a conceptual DAI setup using a BEC analog system:

- Stationary and accelerated detector arms are phase-locked at  $\tau = 0$ ;
- Each arm samples its local phonon field and extracts  $g^{(2)}(t_1, t_2)$ ;
- Retrieval curves are reconstructed via entropy-constrained filtering;
- A thermal null model is generated by disabling modular coupling.

These derivations clarify what empirical retrieval divergence would look like under the ODER framework. No specific experiment or simulation is assumed to have been completed.

### C.8 Retrieval-Coupled Focusing Equation (Exploratory Extension)

*Exploratory status.* This subsection is an exploratory extension and is not used in the main text or in any validation claim.

Equations (A.2') and (C.24) jointly motivate this coupling: the same retrieval curvature that bounds modular speed now sources modular expansion.

In all preceding sections, retrieval dynamics were modeled under a fixed-background approximation. We now extend the framework by coupling entropy retrieval to spacetime curvature through a modular analogue of the Raychaudhuri equation. This introduces a dynamical back-reaction term sourced by the entropy-convergence profile  $S_{\text{retr}}(\tau)$ , allowing retrieval to both track and influence horizon geometry.

Modular expansion scalar.

Define the modular expansion  $\theta_{\text{mod}}(\tau)$  as the divergence of modular-flow lines weighted by the retrieval gradient:

$$\theta_{\text{mod}}(\tau) \equiv \nabla_{\mu} u^{\mu} + \alpha \frac{dS_{\text{retr}}}{d\tau},$$

where  $u^{\mu}$  is the observer's four-velocity and  $\alpha \ll 1$  is the retrieval-coupling parameter introduced in Section 9.

Modular Raychaudhuri equation.

The modular analogue of the Raychaudhuri equation takes the form

$$\frac{d\theta_{\text{mod}}}{d\tau} = -\frac{1}{2}\theta_{\text{mod}}^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^{\mu}u^{\nu} + \alpha \frac{d^2S_{\text{retr}}}{d\tau^2},$$

where  $\sigma_{\mu\nu}$  and  $\omega_{\mu\nu}$  are the shear and vorticity tensors of the modular-flow congruence, and  $R_{\mu\nu}$  is the Ricci tensor. Metric signature follows  $(-, +, +, +)$ ;  $R_{\mu\nu}u^{\mu}u^{\nu} > 0$  corresponds to focusing. The term  $\alpha d^2S_{\text{retr}}/d\tau^2$  acts as a retrieval-driven focusing or defocusing force: when retrieval accelerates (positive second derivative) it produces modular expansion; when retrieval saturates or decelerates, curvature focusing dominates.

Limiting behavior.

In the limit  $\alpha \rightarrow 0$ , this reduces to the standard Raychaudhuri equation in a fixed background, recovering geodesic congruence evolution. Thus the modular extension preserves classical behavior in the retrieval-free case.

Back-reaction shift.

To leading order in  $\alpha$ , the retrieval horizon  $\tau_{\text{RH}}$  shifts by

$$\delta\tau_{\text{RH}} \sim \alpha \int^{\tau_{\text{RH}}} d\tau' \left( \frac{d^2 S_{\text{retr}}}{d\tau'^2} \right),$$

and numerical estimates (Appendix C.0.0.10) indicate this shift remains negligible for  $M \gtrsim M_{\odot}$  but may become resolvable in analog systems with boosted retrieval rates. In such analog systems, a non-zero  $\alpha$  would manifest as a slow drift of the measured  $\tau_{\text{RH}}$  across successive retrieval cycles.

Interpretation.

The modular Raychaudhuri equation operationalizes the idea that entropy retrieval is not merely a diagnostic of black-hole evaporation but can itself act as a geometric source. Failure of monotonic convergence in  $S_{\text{retr}}(\tau)$  signals not only information-theoretic breakdown but potential modular collapse. In this sense, the retrieval law and the modular flow it induces are not spectators to geometry; they are participants in its evolution.

### C.9 Modular Focusing and the Retrieval–Curvature Coupling (Exploratory Extension)

*Exploratory status.* This section elaborates the modular Raychaudhuri construction as a possible future extension. It introduces no new constraints on the retrieval law and is not required for any result in the main text.

Setup.

Let  $u^\mu$  be the observer's proper-time tangent vector, and let  $\theta(\tau) = \nabla_\mu u^\mu$  denote the expansion of the modular flow congruence. The retrieval-coupled expansion scalar is defined as

$$\theta_{\text{mod}}(\tau) = \theta(\tau) + \alpha \frac{dS_{\text{retr}}}{d\tau}.$$

Modular-congruence evolution.

Taking the  $\tau$ -derivative yields

$$\frac{d\theta_{\text{mod}}}{d\tau} = \frac{d\theta}{d\tau} + \alpha \frac{d^2 S_{\text{retr}}}{d\tau^2}.$$

Inserting the classical Raychaudhuri equation,

$$\frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu,$$

we obtain the modular-coupled version

$$\frac{d\theta_{\text{mod}}}{d\tau} = -\frac{1}{2}\theta_{\text{mod}}^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu + \alpha \frac{d^2 S_{\text{retr}}}{d\tau^2} + \mathcal{O}(\alpha^2).$$

The  $\theta_{\text{mod}}^2$  term absorbs the linear  $\alpha$  correction to  $\theta$ , while all other terms remain unaffected at first order.

Horizon shift.

The retrieval horizon  $\tau_{\text{RH}}$  is defined as the proper time at which  $S_{\text{retr}}(\tau) \rightarrow S_{\text{max}}$ . To leading order in  $\alpha$ ,

$$\delta\tau_{\text{RH}} = \alpha \int_0^{\tau_{\text{RH}}} \frac{d^2 S_{\text{retr}}}{d\tau^2} d\tau.$$

This integral can be evaluated analytically for constant- $\gamma$  tanh models or numerically using retrieval simulations.

Remarks.

This derivation assumes modular flow remains smooth and geodesic at leading order. Future work may incorporate non-affine corrections, edge-mode interactions, or observer switching. The modular Raychaudhuri equation defines a new class of entropy-coupled geometric dynamics. Where  $S_{\text{retr}}(\tau)$  is highly nonlinear—for example, under interference, collapse, or multi-observer divergence— $\theta_{\text{mod}}$  may blow up, indicating modular-horizon instability. This provides a structural falsifier: monotonic retrieval convergence is required to prevent runaway geometric focusing.

In this view, modular flow, retrieval dynamics, and curvature evolution form a coupled triad. This perspective is presented as a possible extension rather than as a claim of completed dynamics.

## Appendix D Split-Property Regularization and the Type III<sub>1</sub> Limit

This appendix specifies the mathematical regularization that renders observer-dependent retrieval well defined and delineates the formal boundary between finite-observer physics and the Type III<sub>1</sub> continuum idealization.

Local algebras in algebraic quantum field theory (AQFT) are generically Type III<sub>1</sub> factors: they possess no normal trace and therefore admit no literal density matrix or entropy functional. To formulate observer-dependent entropy retrieval within a physically meaningful regime, we employ the standard *split-property regularization* used in rigorous AQFT treatments of entropy and modular flow.

### D.1 Split Inclusion and Bounded Modular Spectrum (Formal construction)

Following standard split-property constructions in AQFT [23–25], we introduce nested regions  $O_1 \subset O_2$  and a Type I intermediate factor:

$$\mathcal{A}(O_1) \subset \mathcal{N}_\delta \subset \mathcal{A}(O_2),$$

where the split distance  $\delta$  defines the physical collar between the inner and outer regions. The modular Hamiltonian  $K_\delta$  generated by the state  $\omega$  on  $\mathcal{N}_\delta$  then has a compact spectrum,

$$\sigma(K_\delta) \subset [-\Lambda(\delta), \Lambda(\delta)] \quad (\text{bounded in operator norm; } |K_\delta| \leq \Lambda(\delta)).$$

This provides a well-defined finite entropy and modular flow. Operationally,  $\delta$  corresponds to the detector's spatial or temporal resolution, and  $\Lambda(\delta) \sim 1/\delta$  represents the associated bandwidth limit.

### D.2 Physical Interpretation (Operational interpretation)

Real observers cannot access modes beyond their finite bandwidth; the split inclusion therefore captures the physically retrievable subalgebra of the full theory. All bounded-spectrum statements in this work refer to such operationally defined  $\mathcal{N}_\delta$ . In the limit  $\delta \rightarrow 0$ ,  $\Lambda(\delta) \rightarrow \infty$  and the algebra returns to the Type III<sub>1</sub> class. Within the regulated regime, the modular-flow retrieval law derived in the main text (Equation (9)) is exact; the Type III<sub>1</sub> limit marks the idealized boundary where observer bandwidth becomes infinite.

### D.3 Connection to the Retrieval–RG Picture (Interpretive correspondence)

Define the retrieval-renormalization parameter:

$$\beta_\Lambda(\delta) = \frac{d \ln \Lambda(\delta)}{d \ln \delta}.$$

The Type III<sub>1</sub> structure corresponds to the fixed point  $\beta_\Lambda \rightarrow 0$ , signaling scale-invariant modular spectra. Finite observers operate at  $\beta_\Lambda < 0$ , where the spectrum is effectively bounded and the tanh retrieval law applies directly.

This establishes an interpretive renormalization-group analogue of the split-property hierarchy: as the split collar narrows, the modular spectrum flows toward the continuum fixed point. This renormalization-group analogy is descriptive and does not generate the retrieval law, which is fixed independently by modular analyticity and bounded spectra. In laboratory analogs, varying detector resolution  $\delta$  directly probes this correspondence: retrieval parameters should approach  $\beta_\Lambda \approx 0$  as experimental bandwidth increases.

#### D.4 Scope and Open Formal Problems (Open formal problems)

Here  $\Delta_\delta^{it}$  denotes the modular automorphism group generated by  $K_\delta$ . This regularization does not purport to solve the Type III<sub>1</sub> classification problem; it provides a physically covariant framework in which finite observers are well defined. The continuum Type III<sub>1</sub> limit remains a mathematical frontier. Future work should formalize:

1. the weak-operator convergence  $\Delta_\delta^{it} \rightarrow \Delta^{it}$  of modular flows;
2. conditions under which isotony and locality persist for the directed family  $\{\mathcal{N}_\delta\}$ ; and
3. quantitative scaling of  $\rho_\delta(\lambda)$  approaching the Type III<sub>1</sub> fixed point.

Bridging these mathematical results with the retrieval–RG framework of Section 9 would complete the formal connection between bounded modular spectra and scale-invariant entropy flow.

#### D.5 Physical Meaning of the Type III<sub>1</sub> Limit (Interpretive limit)

As discussed in Appendix A.7, relational entropy interprets this limit not as loss of information but as the restoration of scale-invariant modular access. In algebraic QFT, Type III<sub>1</sub> algebras are the unique structures compatible with relativistic locality and causal propagation: they admit no finite trace and thus no factorization between interior and exterior regions. “Inevitable” here refers to the continuum idealization of finite-observer regularizations, not to an operationally accessible physical regime.

Within the retrieval–RG picture, increasing observer bandwidth ( $\delta \rightarrow 0$ ) drives the modular spectrum toward scale invariance, the Type III<sub>1</sub> fixed point of bounded-observer modular flow. Real detectors operate at finite  $\delta$ ; the continuum theory represents the unphysical ideal of infinite information access.

All physical predictions in this work are confined to finite  $\delta$ ; the Type III<sub>1</sub> limit functions solely as the formal boundary of the theory.

## Appendix E Modular Retrieval in Kerr Geometry: Generator Deformation and Spectral Persistence

This appendix tests the stability of the retrieval law under stationary generator deformation in Kerr geometry. It introduces no new dynamics and relies only on assumptions already established in the main text and Appendix A.

#### E.1 Kerr Geometry and Modular Flow (Geometric setup)

In Kerr spacetime the global timelike Killing vector  $\partial_t$  is replaced by a stationary, non-static modular generator,

$$\chi^\mu = \partial_t + \Omega_H \partial_\phi,$$

where  $\Omega_H$  is the horizon angular velocity. Modular flow follows the mixed time–angle trajectory generated by  $\chi^\mu$ ; an observer therefore does not evolve on a globally synchronized slice. The modular Hamiltonian  $K_\chi$  associated with this generator defines observer-adapted modular flow consistent with the split-property regularization described in Appendix D.

### E.2 Modular-Generator Deformation (Generator deformation and spectral admissibility)

Because the modular Hamiltonian depends linearly on the generator, the deformation ( $\partial_t \rightarrow \chi^\mu$ ) preserves the Paley–Wiener class of admissible modular flows. This preservation concerns spectral admissibility of modular evolution, not dynamical stability of the Kerr spacetime itself. Anchoring the causal diamond to  $\chi^\mu$  yields a Kerr-corrected retrieval rate,

$$\gamma(\tau, a, \Omega_H) = |g_{\mu\nu} \chi^\mu \chi^\nu|^{-1/2},$$

which captures frame dragging and horizon-synchronous motion. In rotating BEC analogs, frame dragging corresponds to azimuthal phonon flow; measuring the resulting  $g^{(2)}$  phase shift tests Equation (A24).

### E.3 Survival of the tanh Onset (Stability result)

For observers outside the ergoregion ( $r > r_{\text{erg}}$ ) the modular spectrum remains bounded after split-inclusion regularization. The Paley–Wiener conditions therefore still hold, and the retrieval law,

$$\frac{dS_{\text{retr}}}{d\tau} = \gamma(\tau, a, \Omega_H) [S_{\text{max}} - S_{\text{retr}}(\tau)] \tanh(\tau/\tau_{\text{char}}), \quad (\text{A24})$$

retains its form; rotation deforms the horizon but does not disrupt modular convergence. This persistence confirms that the tanh onset derived in Appendix A.2 is not restricted to static geometries. Units adopt  $G = c = \hbar = 1$  and metric signature  $(-, +, +, +)$ ; under this convention,  $|g_{\mu\nu} \chi^\mu \chi^\nu|$  is positive outside the horizon.

### E.4 Superradiance and Spectral Containment (Domain restriction)

Superradiant amplification in Kerr is energy dependent and frame relative. Modular spectral weight remains bounded provided (i) the observer remains outside the ergosphere and (ii) detector resolution imposes a UV cutoff (Appendix A.3). Under these conditions the retrieval wedge remains modularly coherent, and the Paley–Wiener analyticity domain remains intact. Behavior inside the ergoregion lies outside the bounded-observer regime considered here and is not addressed by the present analysis.

### E.5 Interpretation and Consequences (Interpretive consequences)

- The tanh onset is *not* an artifact of Schwarzschild symmetry; it is a feature of bounded modular spectra.
- Modular retrieval is geometrically robust: Kerr rotation modulates  $\gamma(\tau)$  but preserves spectral convergence.
- The retrieval law is covariant under stationary generator deformation within the regular wedge class.

**Conclusion.** Modular retrieval survives Kerr rotation. Persistence of Equation (A24) under stationary generator deformation demonstrates robustness of the retrieval law across stationary spacetimes with bounded modular spectra. This result establishes stability under rotation without extending the domain of validity beyond observer-bounded regimes or invoking new dynamics.

## Appendix F Interpretive Correspondence (Non-Essential)

Readers seeking only the formal modular derivations may skip this appendix. It translates ODER’s algebraic parameters into gravitational and holographic language for conceptual cross-reference. All quantities retain their definitions from Appendices A and D; the equations below are interpretive analogies, not additional postulates.

Although the ODER retrieval law is derived entirely from observer-dependent modular flow, several of its structural parameters parallel gravitational constructs familiar from wedge-based ap-

proaches to the black-hole information problem. The correspondences below serve as interpretive aids for readers who work primarily with holography or extremal-surface reconstruction.

### F.1 Bandwidth and Algebraic Context

The variables  $\delta$  and  $\Lambda(\delta)$  introduced in Appendix D connect the modular-algebraic description to measurable observer parameters. Finite  $\delta$  defines the retrievable subalgebra  $\mathcal{N}_\delta$  with spectral cutoff  $\Lambda(\delta) \sim 1/\delta$ ; the continuum limit  $\delta \rightarrow 0$  recovers the Type III<sub>1</sub> structure of AQFT. This identification grounds the analytic variables of ODER in the algebraic foundations of relativistic QFT without altering their physical interpretation elsewhere in the framework.

### F.2 Interpretive Parameter Correspondence

Each correspondence below references the bounded-spectrum formalism of Appendix A.11 and the retrieval-RG scaling of Sec. 8.

- $\Delta_{\text{fail}}$  (**failure gap**). In ODER,  $\Delta_{\text{fail}} \equiv \tau_{\text{evap}} - \tau_{\text{RH}}$  represents the late-stage failure of entanglement-wedge reconstruction, where extremal surfaces no longer support modular access for the observer's causal patch. Mathematically this corresponds to the vanishing of the second derivative  $d^2 S_{\text{retr}}/d\tau^2$  at the retrieval horizon  $\tau_{\text{RH}}$ . ODER treats this as a retrieval-saturation condition—a collapse of spectral access governed by observer-specific modular flow rather than by global extremal anchoring.
- $\tau_{\text{char}}$  (**convergence time**). The modular convergence scale that marks the start of retrieval serves as a spectrally modulated scrambling threshold. Conventional scrambling time signals full entanglement redistribution, whereas  $\tau_{\text{char}}$  emerges from bounded modular flow and captures observer-relative retrieval activation even when causal connectivity exists but modular access is still suppressed. Experimentally it governs the width of the measured  $g^{(2)}$  envelope (Sec. 5.1).
- $\gamma(\tau)$  (**retrieval operator**). Derived from the entropy trace,  $\gamma(\tau)$  measures the local modular pressure—the instantaneous rate at which retrievable entropy moves toward saturation. Within the modular algebra,  $\gamma(\tau)$  appears as the local generator of the positive-energy semigroup for the observer's subalgebra  $\mathcal{N}_\delta$ . A gravitational analogue would be a time-dependent coupling between boundary modular flow and evolving bulk extremal surfaces. Because  $\gamma(\tau)$  varies smoothly with both trajectory and state, it serves as an information-theoretic redshift gradient tied to curvature of the modular spectrum.

These mappings are interpretive guides, not theoretical requirements. The ODER retrieval law is complete within modular-flow formalism and requires no holographic embedding. Causal wedges and HRT surfaces offer intuitive parallels, but they are projections of the same underlying modular dynamics rather than foundations. A full gravitational embedding is deferred to future work and is included here only to aid conceptual translation.

F.3 Extended Variable Glossary (Table F.1)

| Symbol                 | Description                                 | Interpretation                                                                                                             | Experimental / Simulation Proxy                               |
|------------------------|---------------------------------------------|----------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------|
| $\delta$               | Split-property collar / detector resolution | Physical bandwidth of the observer; defines the retrievable subalgebra                                                     | —                                                             |
| $\Lambda(\delta)$      | Modular spectral cutoff                     | Inverse bandwidth ( $\Lambda \sim 1/\delta$ ); controls the tanh prefactor and approach to the Type III <sub>1</sub> limit | —                                                             |
| $\tau_{\text{char}}$   | Convergence (activation) time               | Onset scale of modular retrieval; analog of a scrambling threshold                                                         | Width of $g^{(2)}$ transition                                 |
| $\Delta_{\text{fail}}$ | Retrieval-failure gap                       | Proper-time delay between evaporation and saturation limits                                                                | Duration between entropy plateau and null-envelope flattening |
| $\gamma(\tau)$         | Retrieval-rate operator                     | Local modular pressure or information-flux density                                                                         | Slope of entropy-retrieval trace                              |

F.4 Cross-Domain Interpretive Map (Table F.2)

| Term                              | Physics Interpretation                                                        | Algebra / AQFT Interpretation                                                      |
|-----------------------------------|-------------------------------------------------------------------------------|------------------------------------------------------------------------------------|
| Modular flow                      | Local observer evolution in proper time; defines trajectory of modular access | Tomita–Takesaki automorphism on $\mathcal{A}(O)$ governing modular evolution       |
| Retrieval horizon                 | Boundary of the decodable information wedge for a given observer              | Support boundary of the retrievable subalgebra $\mathcal{N}_\delta$                |
| Type III <sub>1</sub> fixed point | Scale-invariant modular spectrum; continuum limit of retrieval flow           | Non-factorizable local algebra; no trace; required by Haag–Kastler locality axioms |

**Summary.** The interpretive correspondences collected here are heuristic aids for cross-domain intuition. They situate the modular parameters of ODER within the gravitational vocabulary of wedges, surfaces, and bandwidths without requiring any holographic assumption. Finite-bandwidth retrieval, its modular-algebraic foundations, and its covariant persistence across geometries jointly establish ODER as a first-principles description of observer-dependent entropy flow.

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