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Proof of the Binary Goldbach Conjecture

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Abstract: In this article the proof of the binary Goldbach conjecture is established (Any integer greater than one is the mean arithmetic of two positive primes) . To this end, Chen's weak conjecture is proved (Any even integer greater than one is the difference of two positive primes) and a "localised" algorithm is developed for the construction of two recurrent sequences of primes (U_{2n}) and (V_{2n}), $((U_{2n})$ dependent of $(V_{2n}))$ such that for any integer $n \geq 2$ their sum is equal to $2n$: (U_{2n}) and (V_{2n}) are the extreme Goldbach decomponents. To form them, a third sequence of primes (W_{2n}) is defined for any integer $n \geq 3$ by $W_{2n} = \mathrm{Sup}(p \in \mathcal{P} : p \leq 2n - 3)$, $\mathcal{P}$ denoting the set of positive primes. The Goldbach conjecture has been proved for all even integers $2n$ between 4 and $4.10^{18.}$ and in the neighbourhood of 10^{100}, 10^{200} and 10^{300} for intervals of amplitude 10^9 . In the table of extreme Goldbach decomponents given via programs in Maxima and Maple in Appendix 13 and files of Researchgate, Internet Archive and OEIS , values of the order of $2n = 10^{3000}$ are reached. A global proof by strong recurrence "finite ascent and descent method" on all the Goldbach decomponents is presented by using sequences of primes (Wq_{2n}) defined by $Wq_{2n} = \mathrm{Sup}(p \in \mathcal{P} : p \leq 2n - q)$ for any odd prime q , and a majorization of U_{2n} by $(2n)^{0.525}$, $0.7 \ln^{2.2}(2n)$ with probability one and $20.\ln(n)$ on average for any integr n large enough is justified.. In addition, the Lagrange-Lemoine-Levy conjecture and its generalization called "Bachet-Bézout-Goldbach" conjecture are proven by the same type of method.

Keywords: prime number theorem; binary goldbach conjecture; chen's weak conjecture; lagrange-lemoine-levy conjecture; bachet-bezout-goldbach conjecture; gaps between consecutive primes

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1. Overview

Number theory "the queen of mathematics" studies the structures and properties defined on integers and primes (Euclid [14], Hadamard [17], Hardy and Wright [19], Landau [25], Tchebychev [41]). Numerous problems have been raised and conjectures made, the statements of which are often simple but very difficult to prove. These main components include :

- Elementary arithmetic .

– ␣Operations on integers, determination and properties of primes.

(Basic operations, congruence, gcd, lcm, ………..).

– Decomposition of integers into products or sums of primes

(Fundamental theorem of arithmetic, decomposition of large numbers, cryptography and Goldbach's conjecture).

- Analytical number theory .

– Distribution of primes (Prime Number Theorem, Hadamard [17],

De la Vallée-Poussin [42], Littlewood [28] and Erdos [13], the Riemann hypothesis,.....).

– Gaps between consecutive primes (Bombieri,Davenport [3], Cramer [8], Baker,Harmann,Iwaniec, Pintz [4,5,22], Granville [16], Maynard [30], Tao [40],

Shanks [36], Tchebychev [41] and Zhang [46]).

- Algebraic, probabilistic, combinatorial and algorithmic number theories .

– Modular arithmetic.

– Diophantine approximations and equations

– Arithmetic and algebraic functions.

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- Diophantine and number geometry.

2. Definitions Notations and Background

The integers $n, k, p, q, r,\ldots\ldots$ used in this article are always positive. (2.1)

The symbol " | " means : such as or knowing that. (2.2)

Let $\mathcal{P}$ be the infinite set of positive primes p_k (called simply primes) (2.3)

$$(p_1 = 2 \,;\; p_2 = 3 \,;\; p_3 = 5 \,;\; p_4 = 7 \,;\; p_5 = 11 \,;\; p_6 = 13 \,;\, \ldots\ldots)$$

For any integer $K \geq 1$ $\mathcal{P}_K = \{ p \in \mathcal{P} : p \leq 2K \}$ (2.4)

Writing the large numbers calculated in Appendix 13 is simplified by defining the following constants:

$$M = 10^9 \;;\quad R = 4.\,10^8 \;;\; G = 10^{100} \;;\; S = 10^{500} \;;\; T = 10^{1000}$$

(2.5)

$\ln(x)$ denotes the neperian logarithm of the real $x > 0$ (2.6)

Let (W_{2n}) be the sequence of primes defined by

$\forall\; n \in \mathbb{N} + 3$ $W_{2n} = \text{Sup} (p \in \mathcal{P} : p \leq 2n - 3)$ (2.7)

For any odd prime q , let (Wq_{2n}) be the sequence of primes defined by

$\forall\; n \in \mathbb{N}$ $n \geq \frac{(q+3)}{2}$ $Wq_{2n} = \text{Sup} (p \in \mathcal{P} : p \leq 2n - q)$ (2.8)

Any sequence denoted by $(G_{2n}) = (U_{2n} ; V_{2n})$ verifying (2.9) is called a **Goldbach sequence.**

$\forall\; \text{n} \in \mathbb{N} + 2$ $U_{2n} , V_{2n} \in \mathcal{P}$ and $U_{2n} + V_{2n} = 2\text{n}$ (2.9)

U_{2n} and V_{2n} are also known as "**Goldbach partitions or Goldbach decomponents**".

Iwaniec,Pintz [22] have shown that for a sufficiently large integer n there is always a prime between $n - n^{23/42}$ and n . Baker and Harman [4,5] concluded that there is a prime in the interval $[\, n \,;\, n + o\,(\, n^{0.525})]$. Thus this results provides an increase of the gap between two consecutive primes p_k and p_{k+1} of the form

$$\forall\; \varepsilon > 0 \quad \exists\; k_\varepsilon \in \mathbb{N}^* \;\; | \;\; \forall\; k \in \mathbb{N} \quad k \geq k_\varepsilon \qquad p_{k+1} - p_k < \varepsilon.p_k^{0.525}$$

(2.10)

The results obtained on the Cramer-Granville-Maier-Nicely conjecture [1,3,8,16,28,31] imply the following majorization.

For any real $c > 2$ and for any integer $k \geq 500$

$p_{k+1} - p_k \leq 0.7 \;\ln^c (p_k)$ **(** ***with probability one*** **)** (2.11)

and

$p_{k+1} - p_k \leq \; 20.\ln(p_k)$ **(** ***on average*** **)** (2.12)

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The following abbreviations have been adopted :

- Lagrange-Lemoine-Levy conjecture (3L) conjecture (2.13)
- Bachet-Bézout-Goldbach conjecture (BBG) conjecture (2.14)
- (Extreme) Goldbach decomponents (E).G.D. (2.15)

3. Introduction

Chen [6], Hardy, Littlewood [18], Hegfollt, Platt [19], Ramaré, Saouter [30], Tao [36], Tchebychev [37] and Vinogradov [39] have taken important steps and obtained promising results on the Goldbach conjecture (Any integer $n \geq 2$ is the mean arithmetic of two primes).

Indeed, Helfgott, Platt [19] proved the ternary Goldbach conjecture in 2013.

Silva, Herzog, Pardi [34] held the record for calculating the terms of Goldbach sequences after determining pairs of primes (U_{2n} ; V_{2n}) verifying

$$\forall \ n \in \mathbb{N} \quad | \quad 4 \leq 2n \leq 4.10^{18} \qquad U_{2n} + V_{2n} = 2n \qquad (3.1)$$

Goldbach's conjecture has also been verified for all even integers $2n$ satisfying

$$10^{5k} \leq 2n \leq 10^{5k} + 10^{8} \quad : \quad k = 3, 4, 5, 6, \ldots\ldots\ldots, 20$$

and

$$10^{10k} \leq 2n \leq 10^{10k} + 10^{9} \quad : \quad k = 20, 21, 22, 23, 24, \ldots\ldots., 30$$

by Deshouillers, te Riele, Saouter [10].

In previous research work there is no explicit construction of recurrent Goldbach sequences.

In this article, for any integer $n \geq 3$ the E.G.D. U_{2n} and V_{2n} are computed iteratively using a simple and efficient "localised" algorithm.

Using Maxima and Maple scientific softwares on a personal computer Silva's record is broken and many E.G.D. are calculated up to the neighbourhood of $2n = 10^{500}$, 10^{1000} and even 10^{3000} (see Sainty [31] "In Researchgate.net, Internet Archive, and OEIS, E.G.D. files are supplied : E.G.D. File S Around $2n = 10^{S}$ for $S = 1, 2, 3, \ldots\ldots\ldots\ldots, 3000$").

The binary Goldbach conjecture can be proved globally by strong recurrence on all G.D. using (Wq_{2n}) sequences of primes in the same way via Goldbach(-) conjecture (Any even integer greater than one is the difference of two primes) demonstrated in Teorem 4.

- Remark.

1. **Chen conjecture :** For any integer K ≥ 1 there are infinitely many pairs of primes with a difference equal to 2K.
2. **De Polignac conjecture :** Same as Chen, but with consecutive pairs of primes.
3. What we know :

April 2013, Yitang Zhang [41] demonstrates that the smallest even integer $2K$ verifying the conjecture is greater than 70 million.

In 2014, James Maynard [28] then Terence Tao [36] lowered this limit to 246.

We validate Chen's weak conjecture by verifying directly in the primes tables that all even gaps from 2 to 246 are possible (see Appendix 14).

In addition, the (3L) conjectures [9,20,22,27,29,35,40] and its generalization called (BBG) conjecture are validated.

Using case disjunction reasoning we construct two recurrent E.G.D. sequences of primes (V_{2n}) and (U_{2n}) according to the sequence (W_{2n}) by the following process

For any integer $n \geq 2$

$U_4 = 2$ and $V_4 = 2$ (3.2)

Let n be an integer greater than two

- Either

(2n - W_{2n}) is a prime
then V_{2n} and U_{2n} are defined directly in terms of W_{2n} .

- Either

($2n$ - W_{2n}) is a composite number
then V_{2n} and U_{2n} are determined from the previous terms of the sequence (G_{2n}).

4. Theorem (Chen's Weak or Goldbach(-) Conjecture)

$$\forall K \in \mathbb{N}^* \quad \exists \; p , q \in \mathcal{P} \; | \quad p - q = 2K$$

(4.1)

If $K \geq 2$ $3 \leq q \leq 2K$ *and* $3 + 2K \leq p \leq 4K$

Practical method on some examples:

First of all (5 - 3 = 2), then we begin the process at (7 - 3 = 4), we will select the smallest primes for which the difference is precisely 6 (11 - 5 = 6), then 8 (11 - 3 = 8), then 10

(13 - 3 = 10),......... , then $2K$ (demonstration established by strong recurrence, by the asurd and feedback). All pairs of Goldbach(-) partitions obtained by this method for K between 2 and 123 are listed in Appendix 14 to validate it using Tao results.

Proof. An other proof can also be established by strong recurrence on the integer $K \geq 2$. Let $\mathcal{P}_{Chen}(K)$ be the following property

" $\forall$ K $\in \mathbb{N}^*$ $\exists$ p, q $\in \mathcal{P}$ | p - q = 2K 3 ≤ q ≤ 2K and 2K + 3 ≤ p ≤ 4K "
(4.2)

▸ $\mathcal{P}_{Chen}(2)$ is true : 7 - 3 = 4 $q = 3 \leq 4$ and $p = 7 \leq 4 \times 2 = 8$

▸ Let's show

$$\forall M \in \mathbb{N} \; | \quad 2 \leq M \leq K \quad \text{then} \quad \mathcal{P}_{Chen}(M) \Longrightarrow \mathcal{P}_{Chen}(K + 1)$$

We reason through the absurd

Let $p , q \in \mathcal{P}_K$ | $p \geq q$

$$\forall P , Q \in \mathcal{P} \; | \; P \geq Q \quad \exists \; h, m \in \mathbb{N} \; |$$

$$P = p + 2h \quad \text{and} \quad Q = q + 2m$$

we assume that

$$P - Q = p + 2h - q - 2m \neq 2(K + 1)$$

(4.3)

Therefore

$$p - q \neq 2(K + 1 - h + m).$$

(4.4)

You can always choose $h \geq m$ and $h - m \leq K + 1$.

The set $\{ K + 1 - h + m \,;\, 2h$ and $2m$ are any gaps between primes $\}$ contains all even integers between 2 and $2K$.

However the strong recurrence hypothesis asserts that

$$\forall M \in \mathbb{N} \;\middle|\; M \leq K \quad \exists\; p, q \in \mathcal{P} \;\middle|\; p - q = 2M$$

(4.5)

By choosing : $M = K + 1 - h + m$

this contradicts (4.4).

So

$$\exists\; h, m \in \mathbb{N} \;\middle|\; P - Q = p + 2h - q - 2m = 2(K + 1)$$

(4.6)

knowing

$$p, \quad p + 2h, q, \quad q + 2m \in \mathcal{P} \qquad h \geq m \quad \text{and} \quad h - m \leq K + 1$$

Thus validating the heredity of property $\mathcal{P}_{Chen}(K)$.

The property $\mathcal{P}_{Chen}(K)$ is therefore true. As a result Goldbach(-) conjecture is validated.

5. Corollary

Let (R_{2K}) and (Q_{2K}) be two sequences of primes determined by

$$R_{2K} = \text{Inf}\,(p \in \mathcal{P} : \; p - 2K \in \mathcal{P}) \quad \textit{and} \quad Q_{2K} = \text{Inf}\,(p \in \mathcal{P} : \; 2K + p \in \mathcal{P}) = R_{2K} - 2K \qquad (5.1)$$

They are defined for any integer $K \in \mathbb{N}^*$ (5.2)

and satisfy

$$\lim R_{2K} = +\infty \qquad (5.3)$$

$$\forall\, K \in \mathbb{N}^* \quad R_{2K}\,,\; Q_{2K} \in \mathcal{P} \quad \textit{and} \quad R_{2K} - Q_{2K} = 2K$$

(5.4)

$$\forall K \in \mathbb{N}^* \;/\; 2 \leq K \leq 16 \qquad 3 \leq Q_{2K} \leq 2K \quad \text{and} \quad 2K + 3 \leq R_{2K} \leq 4K$$

(5.5)

For any integer K large enough

$$3 \leq Q_{2K} \leq (2K)^{0.525} \quad \text{and} \quad 2K + 3 \leq R_{2K} \leq 2K + (2K)^{0.525}$$

(5.6)

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Proof.

(5.1) ; (5.2) : According to the previous theorem, the sequences (R_{2K}) and (Q_{2K}) are defined by

strong recurrence (finite descent).

(5.3) : $R_{2K} \geq 2K \implies \lim R_{2K} = +\infty$

(5.4) : By construction, these sequences thus verify : $R_{2K} - Q_{2K} = 2K$

(5.5) : The property can be verified directly term-to-term by examining the sequence proposed above.

(5.6) : This property is verified up to $2K = 246$ by calculations on the previous list.

We prove this result by recurrence

First of all we order the Goldbach(-) decomponents at a fixed prime q , so as to obtain the estimate (5.6) more easily.

Let q_r be the $(r+1)$th prime :

We examine the sequences of primes ($T_r(K))_{K\in\mathbb{N}}$ satisfying :

$T_1(K) = 2K + 3$

$(T_1(K) ; 2K) \rightarrow$ (5;2) ; (7 4) ; (11;8) ; (13;10) ; (17;14) ; (19;16) ; (23;20) ; (29;26) ; (29;28);..

$T_2(K) = 2K + 5$

$(T_2(K) ; 2K) \rightarrow$ (7;2) ; (11;6) ; (13;8) ; (17;12) ; (19;14) ; (23;18) ; (29;24) ; (31;26) ; (37;32)...............................

$T_3(K) = 2K + 7$

$(T_3(K) ; 2K) \rightarrow$ (11;4) ; (13;6) ; (17;10) ; (19;12) ; (23;16) ; (29;22) ; (31;24) ; (37;30)...........

$T_4(K) = 2K + 11$

$(T11(K) ; 2K) \rightarrow$ (13;2) ; (17;6) ; (19;8) ; (23;12) ; (29;18) ; (31;20) ; (37;26) ; (41;30) ; (43;34)........................

$(T13(K) ; 2K) \rightarrow$ (17;4) ; (19;6) ; (23;10) ;(29;16) ; (31;18) ; (37;24) ; (41;28) ; (43;30 ; (47;34).........................

$T_r(K) = 2K + q_r$ ($K \in \mathbb{N}^*$: $T_r(K)$ and q_r are primes) (see Appendix 15)

For any integer K satisfying $(2K)^{0.525} > q_r$ the property holds for $T_r(K)$.

Therefore it is generally validated for all $K > K_0$, since we obtain all possible cases of Chen's weak conjecture starting with $T_1(K)$, then $T_2(K)$, then $T_3(K)$ ...... for $(2K)^{0.525} \leq q_r$.

(can be proved by strong recurrence using the same method as in Theorem 4 by "finite descent").

Let $a = \frac{40}{21}$ and $P_a(r)$ be the following property

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" For any integer M | $2M < (q_r)^a$ there exists at least a prime $q < q_r$ | $2M + q \in \mathcal{P}$ "

- $P_a(K_0)$ is true (see Appendix 15).
- Let's show : $P_a(r) \Longrightarrow P_a(r+1)$

$$q_{r+1} \leq q_r + q_r^{0.525} \qquad (5.6)$$

It is assumed that M /

$T_{r+1}(K) - q_{r+1} \neq 2M$ knowing $2M < (q_{r+1})^{c_p}$

$$\forall \quad T_m(R),\ q_m \in \mathcal{P} \quad \exists\ h, s \in \mathbb{N} \quad | \quad T_{r+1}(K) = T_m(R) + 2h \quad \text{and} \quad q_{r+1} = q_m + 2s$$

(5.7)

then

$$T_m(R) - q_m \neq 2(M + s - h) \qquad (5.8)$$

which is impossible according to the hypothesis of strong recurrence since
$2(M + s - h)$ is less than Sup $(q_m)^a$ and that all primes $T_m(R)$, q_m satisfy the recurrence hypothesis.

We deduce that : $Pc_p(r) \Longrightarrow Pc_p(r+1)$

Thus the property (5.5) is true.

6. Lemma (*Goldbach's Fundamental Lemma*)

Let q be an odd prime
For any integer $n \geq n_q$ there exists an integer s |

$$2n - Wq_{2s} \in \mathcal{P} \qquad (6.1)$$

Let (Zq_{2n}) *be the sequence of primes defined by*

$$\forall\ n \in \mathbb{N} \quad n \geq n_q \qquad Zq_{2n} = \text{Inf}\ (\ 2n - Wq_{2k} \in \mathcal{P} : k \in \mathbb{N}\ )$$

(6.2)

All D.G. are contains in the set $\{ (2n - Zq_{2n} ; Zq_{2n})\quad n \in \mathbb{N} + 3 \}$

For any integer $n \geq n_0$ $\qquad Z3_{2n} \leq (2n)^{0.525}$
(6.3)

$$Z3_{2n} \leq o\ (2n)^{0.525} \qquad (6.4)$$

Proof. The proofs of propositions (6.1), (6.2) and (6.3) are established following the same principle of strong recurrence as in Theorem 4 and Corollary 5 by "return, absurd and finite descent"

(6.1) : $2n - Wq_{2k} = 2n - 2M_r - Wr_{2k} = 2(n - M_r) - Wr_{2k}$ then by the absurd the property is validated.

(Proof to develop).

Remark. A better estimate of the following form can be obtained by the same method with probability 1 or on average using the results of Bombieri [3], Cramer [8], Granville [9] , Nicely [29] and Maier [27] :

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$\forall\ n \in \mathbb{N} : \quad n \geq n_0$; *For any real* $c > 2$; $\exists\ K' > 1$ |

$U_{2n} < 1.7\, ln(n)^c$ ***(with probability one)*** and $U_{2n} < K'.\ln(n)$

(on *average*)

7. Principle of Proof

To determine the E.G.D , three sequences of primes (W_{2n}), (V_{2n}), (U_{2n}) are defined and they verify the following properties

$\lim V_{2n} = +\infty$. (7.1)

$\forall\ n \in \mathbb{N} + 2$ V_{2n} is defined as a function of $W_{2n} = \text{Sup}\ (\ p \in \mathcal{P} :\ p \leq 2n - 3\ )$

(7.2)

(W_{2n}) is an increasing sequence of primes that contains all of them except $p_1 = 2$

(7.3)

$\lim W_{2n} = +\infty$ (7.4)

(U_{2n}) is a complementary sequence to (W_{2n}) of negligible primes with respect to $2n$ (7.5)

For any integer n ≥ 3 (7.6)

- If ($2n$ - $\boldsymbol{W_{2n}}$) is a prime

then V_{2n} and U_{2n} are defined by

$V_{2n} = W_{2n}$ and $U_{2n} = 2n - W_{2n}$ (7.7)

- Otherwise, if ($2n$ - $\boldsymbol{W_{2n}}$) is a composite number

we search for two previous terms of the sequence (G_{2n}), $U_{2(n-k)}$) and $V_{2(n-k)}$ satisfying the following conditions

$U_{2(n-k)}$, $V_{2(n-k)}$, [$U_{2(n-k)} + 2k$] $\in \mathcal{P}$ (7.8)

$$U_{2(n-k)} + V_{2(n-k)} = 2(n - k)$$

which is always possible (see Theorem 4 and "*Goldbach's fundamental Lemma* 6")

So by setting

$V_{2n} = V_{2(n-k)}$ and $U_{2n} = U_{2(n-k)} + 2k$ (7.9)

two new primes V_{2n} and U_{2n} satisfying (4.10) are generated /

$U_{2n} + V_{2n} = 2n$ (7.10)

This process is then repeated incrementing n by one unit ($n \leftarrow n + 1$).

- **Remark.** Using the same method as in Theorem 4, we can the following equivalent property by strong recurrence : For any integer n greater than 48

$\mathcal{P}_{ret}$ (n) : " There exists an integer K such that $2K + U_{2(n-k)} \in \mathcal{P}$ "

(7.11)

To this end, .

- $\mathcal{P}_{ret}(49)$ is true.
- The heredity of the property $\mathcal{P}_{ret}(n)$: $\mathcal{P}_{ret}$ (n) $\Longrightarrow$ $\mathcal{P}_{ret}(n+1)$

can be proved by the absurd and returning to the previous terms by noting that
For any integer r | $r \leq n$, there is at least one integer $\mathrm{M_r}$ |

$$U_{2(n+1-k)} = 2\,M_r + U_{2(r+1-k)}$$

then

$$2K + U_{2(n+1-k)} = 2(K + M_r) + U_{2(r+1-k)}$$

$$= 2P + U_{2(r+1+M_r-P)}$$

(7.12)

By posing : $\mathrm{P = K +}\ M_r$ and $r + 1 + M_r \leq \mathrm{n}$
Now, according to the recurrence hypothesis on $\mathcal{P}_{ret}(n)$ there exists an integer P /

$$2P + U_{2(r+1+M_r-P)} \in \mathcal{P} \qquad (7.13)$$

then there exists an integer K |

$$2K + U_{2(n+1-k)} \in \mathcal{P} \qquad (7.14)$$

In summary, the property $\mathcal{P}_{ret}(n)$ is hereditary and, as a result, verifiable.
On adapte le meme type de raisonnement à l'aide du theorem 4 dans le cas général avec la suite (Wq_{2n}) en montrant :
For any integer $n > 2$ there exists an integer K |

$$2K + q_{2n} \in \mathcal{P}$$

8. Theorem (Goldbach Conjecture)

There exists at least a recurrent sequence $(G_{2n}) = (U_{2n}\,;V_{2n})$ *of primes satisfying the following conditions.*

For any integer $n \geq 2$

$$U_{2n}\ ,\ V_{2n} \in \mathcal{P} \quad \text{and} \quad U_{2n} + V_{2n} = 2n$$

(8.1)

(*Any integer* $n \geq 2$ *is the mean arithmetic of two primes*)

An algorithm can be used to explicitly compute any term U_{2n} and V_{2n} . (8.2)

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Proof.

- **GLOBAL STRONG RECURRENCE :**

The proof can be made using the following strong recurrence principle.
Let $P_G(n)$ be the property defined for any integer $n \geq 2$ by
$P_G(n)$: " For any integer p satisfying $2 \leq p \leq n$ there exists two primes U_{2p} and V_{2p} such their sum is equal to $2p$ ".

$$(\forall p \in \mathbb{N} \quad | \quad 2 \leq p \leq n \qquad U_{2p} , V_{2p} \in \mathcal{P} \quad \text{and} \quad U_{2p} + V_{2p} = 2p)$$

Let's show by strong recurrence that $P_G(n)$ is true for any integer $n \geq 2$

- $P_G(2)$ is true : it suffices to choose $U_4 = V_4 = 2$.
- Let's show that the property $P_G(n)$ is hereditary : $P_G(n) \Longrightarrow P_G(n+1)$

Assume property $P_G(n)$ is true.

- If $(2(n+1) - \boldsymbol{W}_{2(n+1)})$ is a prime

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

$$V_{2(n+1)} = W_{2(n+1)} \quad \text{and} \quad U_{2(n+1)} = 2(n+1) - W_{2(n+1)} \tag{8.10}$$

- Otherwise, if $(2(n+1) - \boldsymbol{W}_{2(n+1)})$ is a composite number

there exists an integer k to obtain two terms $U_{2(n+1-k)}$ and $V_{2(n+1-k)}$ satisfying the following conditions

$$U_{2(n+1-k)}, V_{2(n+1-k)} \quad \text{and} \quad U_{2(n+1-k)} + 2k \quad \in \mathcal{P} \tag{8.11}$$

$$U_{2(n+1-k)} + V_{2(n+1-k)} = 2(n + 1 - k)$$

we use the previous terms of the sequence (G_{2n}).
For any integer q / $1 \leq q \leq n - 3$ we have

$$3 \leq U_{2(n-q)} \leq n .$$

Then there exists an integer k $1 \leq k \leq n - 3$ |

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \tag{8.4}$$

following the Bertrand principle and Theorem 4 since all primes smaller than $(2n)^{0.525}$ are in the set $\{ U_{2k} : k \leq n \}$

(If there were no such primes, we would have a contradiction with the Theorem 4 or with *Goldbach's fundamental Lemma 6*) . In fact, in an equivalent way (see the previous remark) we can copy the proof of Teorem 4 by performing a similar strong recurrence ''finite descent feedback and absurd'' directly on the set $\{ U_{2k} : k \leq n \}$ |

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \tag{8.4}$$

The smallest integer k | $R_{2n} \in \mathcal{P}$ is denoted by k_n .
So by setting

$$U_{2n} = U_{2(n-k_n)} + 2k_n \quad \text{and} \quad V_{2n} = V_{2(n-k_n)} \quad \in \mathcal{P} \tag{8.5}$$

(These two terms are primes)
In the previous steps two primes $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to $2(n - k_n)$ were determined.

$$U_{2(n-k_n)} + V_{2(n-k_n)} = 2(n - k_n) \tag{8.6}$$

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By adding the term $2k_n$ to each member of the equality (8.6) it follows

$$U_{2(n-k_n)} + 2k_n + V_{2(n-k_n)} = 2(n - k_n) + 2k_n \quad (8.7)$$

$$\Leftrightarrow \quad [\, U_{2(n-k_n)} + 2k_n \,] + V_{2(n-k_n)} = 2n \quad (8.8)$$

$$\Leftrightarrow \quad U_{2n} + V_{2n} = 2n$$

(8.9)

Two new primes $V_{2(n+1)}$ and $U_{2(n+1)}$ satisfying ($U_{2(n+1)} + V_{2(n+1)} = 2(n+1)$) are generated.

It follows that $P_G(n+1)$ is true. Then the property $P_G(n)$ is hereditary : $P_G(n) \Longrightarrow P_G(n+1)$.

Therefore for any integer $n \geq 2$ the property $P_G(n)$ is true.

It follows

$\forall\ n \in \mathbb{N}+2$ there are two primes U_{2n} and V_{2n} and such their sum is $2n$: $U_{2n} + V_{2n} = 2n$

■ **ALGORITHM :**

For any integer $n \geq 3$

- If $(2n - \boldsymbol{W_{2n}})$ is a prime

then V_{2n} and U_{2n} are defined by

$$V_{2n} = W_{2n} \quad \text{and} \quad U_{2n} = 2n - W_{2n} \quad (8.3)$$

- Otherwise, if $(2n - \boldsymbol{W_{2n}})$ is a composite number

we use the previous terms of the sequence (G_{2n}).

For any integer $q \mid 1 \leq q \leq n-3$ we have

$$3 \leq U_{2(n-q)} \leq n .$$

Then there exists an integer $k \quad 1 \leq k \leq n-3 \mid$

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \quad (8.4)$$

following the Bertrand principle and Theorem 4 since all primes smaller than $(2n)^{0.525}$ are in the set $\{ U_{2k} : k \leq n \}$

(If there were no such primes, we would have a contradiction with the Theorem 4 or with *Goldbach's fundamental Lemma 6*) . In fact, in an equivalent way (see the previous remark) we can copy the proof of Teorem 4 by performing a similar strong recurrence ''finite descent feedback and absurd'' directly on the set $\{ U_{2k} : k \leq n \}$ /

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \quad (8.4)$$

The smallest integer $k \mid R_{2n} \in \mathcal{P}$ is denoted by k_n .

So

$$U_{2n} = U_{2(n-k_n)} + 2k_n \quad \text{and} \quad V_{2n} = V_{2(n-k_n)} \in \mathcal{P} \quad (8.5)$$

(These two terms are primes)

In the previous steps two primes $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to $2(n - k_n)$ were determined.

$$U_{2(n-k_n)} + V_{2(n-k_n)} = 2(n - k_n) \quad (8.6)$$

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By adding the term $2k_n$ to each member of the equality (8.6) it follows

$$U_{2(n-k_n)} + 2k_n + V_{2(n-k_n)} = 2(n - k_n) + 2k_n \quad (8.7)$$

$$\Leftrightarrow \quad [\, U_{2(n-k_n)} + 2k_n \,] + V_{2(n-k_n)} = 2n \quad (8.8)$$

$$\Leftrightarrow \quad U_{2n} + V_{2n} = 2n \quad (8.9)$$

Finally for any integer $n \geq 3$ this algorithm determines two sequences of primes (U_{2n}) and (V_{2n}) verifying Goldbach's conjecture.

9. Lemma

The sequence (U_{2n}) verifies the following majorization
For any integer $n \geq 65$

$$U_{2n} \leq (2n)^{0.525} \quad (9.1)$$

and

$$U_{2n} = o\,((2n)^{0.525}) \quad (9.2)$$

Proof. According to the programm 12.2 and Appendix 13 the majorization (9.1) is verified for any integer $n \mid 65 \leq n \leq 2000$.

For any integer $n > 2000$ the proof is established by recurrence. For this purpose let $P_{bhip}(n)$ be the following property

$$P_{bhip}(n): \; " \; U_{2n} \leq (2n)^{0.525} \; " . \quad (9.3)$$

- $P_{bhip}(2000)$ is true according to program 12.2 and the table in appendix 13.
- For any integer $n \geq 2000$ let's show that $P_{bhip}(n)$ is hereditary : $P_{bhip}(n) \Longrightarrow .P_{bhip}(n+1)$

Assume that $P_{bhip}(n)$ is true : then

- If $(2(n+1) - \boldsymbol{W_{2(n+1)}})$ is a prime

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

$$V_{2(n+1)} = W_{2(n+1)} \quad \text{and} \quad U_{2(n+1)} = 2(n+1) - W_{2(n+1)} \quad (9.4)$$

According to the results in [4,5,20] (see Lemma 6) there is a constant $K > 0$ such that

$$2(n+1) - K\,.\,[2(n+1)]^{0.525} < W_{2(n+1)} < 2(n+1)$$

$$\Longrightarrow \quad U_{2(n+1)} = 2(n+1) - W_{2(n+1)} < K\,.\,[2(n+1)]^{0.525}$$

$$\Longrightarrow \quad U_{2(n+1)} \leq K\,.\,[2(n+1)]^{0.525}$$

- Otherwise, if $(2(n+1) - \boldsymbol{W_{2(n+1)}})$ is a composite number

$$\exists\, p \in \mathbb{N}^* \; / \; U_{2(n+1)} = U_{2(n+1-p)} + 2p \quad (9.5)$$

According to [4,5,19]

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$U_{2(n+1)} = 2p + U_{2(n+1-p)} = 2p + 2(n + 1 - p) - W_{2(n+1-p)} = 2(n + 1) - W_{2(n+1-p)}$

(9.6)

Via " Goldbach's fundamental Lemma 6 " it follows that

$U_{2(n+1)} < K.[2(n+1)]^{0.525}$ (9.7)

$P_{bhip}(n+1)$ is true then $P_{bhip}(n)$ is hereditary.
So for any integer $n \geq 2000$ the property $P_{bhip}(n)$ is true.
Finally $U_{2(n+1)} \leq [2(n+1)]^{0.525}$

- **Remark.** A more precise estimate can be obtained using the Cipolla or Axler frames [2,7].

10. Theorem

For any integer $n \geq 3$ *it is easy to check*

(W_{2n}) *is a positive increasing sequence of primes* (10.1)

$\{ W_{2n} : n \in \mathrm{IN}+3 \} \cup \{ 2 \} = \mathcal{P}$ (10.2)

$\lim W_{2n} = +\text{oo}$ (10.3)

(U_{2n}) and (V_{2n}) *are sequences of primes and the set* $\{ U_{2k} : k \leq n \}$

(10.4)

contains all primes less than $\ln(n)$

$n \leq V_{2n} \leq W_{2n}$ (10.5)

$3 \leq 2n - W_{2n} \leq U_{2n} \leq n$ (10.6)

$\lim V_{2n} = +\text{oo}$ (10.7)

Proof.
(10.1) : For any integer $n \geq 2$ $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ .Therefore, $W_{2n} \leq W_{2(n+1)}$. So the sequence (W_{2n})
is increasing.
(10.2) : Any prime except $p_1 = 2$ is odd, hence the result.
(10.3) : $\lim W_{2n} = \lim p_k = +\text{oo}$
(10.4) : By definition $V_{2n} = W_{2n}$ or there exits an integer $k \leq n - 2$ such that $V_{2n} = V_{2(n-k)}$;
so the terms of the sequence ($V_{2n)}$) are primes.
(10.5) : According to Lemma 6, for any integer $n \geq 65$

$$U_{2n} < (2n)^{0.525}$$

therefore

$$U_{2n} < (2n)^{0.55} < n$$

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and

$$V_{2n} = 2n - U_{2n} > 2n - n > n$$

For any integer n | $3 \le n \le 65$ verification is carried out according to the computer program in paragraph 12.2 and the table in appendix 13.

We can also see that by construction $V_{2n} \ge U_{2n}$ because if we assume the opposite then V_{2n} is not the largest prime number verifying

$$\frac{1}{2}(U_{2n} + V_{2n}) = n .$$

So

$$V_{2n} \ge n$$

According to (10.5) $n \le V_{2n} \implies U_{2n} = 2n - V_{2n} \le 2n - n \le n$ (10.6)

$$V_{2n} \le W_{2n} \implies 2n - W_{2n} \le 2n - V_{2n} = U_{2n} \qquad (10.7)$$

By (10.5) for any integer $n \ge 2$: $n \le V_{2n}$

$\lim V_{2n} = +oo$.

11. Properties

For any integer $k \ge 2$ *there are infinitely many integers* n | $U_{2n} = p_k$ (11.1)

$V_{2n} \sim 2n \quad (n \to +oo)$ (11.2)

For any integer $n \ge 5000$

$U_{2n} \ll V_{2n}$ *and* $\lim \left(\frac{U_{2n}}{V_{2n}}\right) = 0$ (11.3)

The smallest integer n | $U_{2n} \neq 2n - W_{2n}$ *is obtained for* $n = 49$ *and* $G_{98} = (79 ; 19)$ (11.4)

(This type of terms increases in the Goldbach sequence (G_{2n}) as n increases in the sense of the Schnirelmann density and there are an infinite number of them; their proportion per interval can be computed using the results given in [33]).

The sequence (G_{2n}) *is "extremal" in the sense that for any integer* $n \ge 2$ (11.5)

V_{2n} and U_{2n} are the largest and smallest possible primes such that $U_{2n} + V_{2n} = 2n$.

The Cramer-Granville-Maier-Nicely conjecture [8],[15],[20,22,24,25,27,29,35] *is verified with probability one. It leads to the following majorization*

For any integer $p \ge 500$

$$U_{2p} \le 0.7\,[\ln(2p)]^{(2.2 - \frac{1}{p})} \qquad (\textbf{\textit{with probability one}}) \qquad (11.7)$$

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The proof is similar to that of Lemma 9 and is validated by the studying functions of the type

$f : x \to a .g(x) + b[\ln(g(x))]^c$ $(a,b > 0 ; c > 2)$ *with*

$g : x \to 0.7\,[ln(x)]^{(c - \frac{1}{x})}$ *and* $h : x \to 0.7\,[ln(x)]^{(2.2 - \frac{1}{x})}$ *using Maple software.*
A better estimate can be obtained via [26,28,30] .
According to Bombieri [3] and using the same method as in the proof of Lemma 8, we obtain the following estimate of U_{2n}

$$\forall \quad \varepsilon > 0 \qquad U_{2n} = O\,(\ \ln^{1.3+\varepsilon}(2n))$$

(*on average*) (11.8)

12. Algorithm

12.1. Algorithm Written in Natural Language.

Inputs :
Input four integer variables : k, N, n, P
Input : $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, ..., p_N the first N primes.

: $n \leftarrow 3$

: $P = M, R, G, S$ or T as indicated in paragraph **2**

Algorithm body :

A) Compute : $W_{2n} = \text{Sup}(\,p \in \mathcal{P} : p \leq 2n - 3\,)$

If $\boldsymbol{T_{2n}} = (2\text{n} - \boldsymbol{W_{2n}})$ is a prime

$$U_{2n} \leftarrow T_{2n} \quad \text{and} \quad V_{2n} \leftarrow W_{2n} \qquad (12.1.1)$$

otherwise

B) **If T_{2n} is a composite number**

Let : $k = 1$
B.1) **While** $U_{2(n-k)} + 2k$ is a composite number

assign to k the value $k + 1$ ($k \longleftarrow k + 1$).

return to **B1)**
End while
Assign to k the value k_n ($k_n \longleftarrow k$)
Let :

$$U_{2n} = U_{2(n-k_n)} + 2k_n \quad \text{and} \quad V_{2n} = V_{2(n-k_n)} \qquad (12.1.2)$$

Assign to n the value $n + 1$ ($n \longleftarrow n + 1$ and return to **A)**
End :

Outputs for integers less than 10^4:

Print ($2n =$ ● ; $2n - 3 =$ ● ; $W_{2n} =$ ● ; $T_{2n} =$ ● ; $V_{2n} =$ ● ; $U_{2n} =$ ●)

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Outputs for large integers :

Print ($2n$ - P = ● ; $2n$ - 3 - P = ● ; W_{2n} - P = ● ; T_{2n} = ● ; $V_{2n} - P$ = ● ; U_{2n} = ●)

12.2. Program Written with Maxima Software for 2n Around $\mathbf{10^{500}}$_

```
c : 10**500 ; for n :c + 20000   step 2   thru c + 20020 do
(    b:2,test : 0 , b : next_prime(b) ,d :n - b ,
if primep(d )
then print(n - c , b , d - c )
else    while test = 0 do (e :n -b , if   primep(e )
then   test:1 , print(n - c , b   , e - c )
else   test : 0 ,b:next-prime(b)) ;
```

12.3. Program Written with Maplesoft Maple for 2n Around 10^{2000}

```
G := 10^2000;
for n from G + 40000 by 2 to G + 40100 do
B:=2;
    t := 0;
        b := prevprime(b);
e:=n - b;
    if isprime(e) then
        print(n - G, b, e - G);
    else
        while t = 0 do e:=n - b; if   isprime(e) then t := 1; print(n - G, b,e - G); else t := 0; b :=
prevprime(b); end if; end do;
    end if;
end do;
```

RESULTS :

n - G	***e - b***	***b***
40000,	**39957,**	**43**
40002,	**39091,**	**911**
40004,	**39957,**	**47**
40006,	**39549,**	**457**
40008,	**25369,**	**14639**
40010,	**39957,**	**53**
40012,	**39549,**	**463**
40014,	**17737,**	**22277**
40016,	**39957,**	**59**
40018,	**39957,**	**61**
40020,	**39091,**	**929**
40022,	**39141,**	**881**
40024,	**39957,**	**67**
40026,	**35443,**	**4583**
40028,	**39957,**	**71**

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40030, 39957, 73
40032, 39091, 941
40034, 35443, 4591
40036, 39957, 79
40038, 39091, 947
40040, 39957, 83
40042, 23139, 16903
40044, 39091, 953
40046, 39957, 89
40048, 39549, 499
40050, -46067, 86117

13. Appendix

Application of Algorithm 12 : Table of extreme Goldbach partitions U_{2n} and V_{2n} computed from program 12.2 ($2 \leq 2n \leq 10^{1000} + 4020$).

The ** sign in the table below indicates the results given by the algorithm 12 in case **B)** of return to the previous terms of the sequence (G_{2n}).

WATCH OUT !

To simplify the display of large numbers n ($2n > 10^9$) the results are entered as follows :

$$2n - P \,,\, (2n - 3) - P \,,\, W_{2n} - P \quad,\, T_{2n} \,,\, V_{2n} - P \quad \text{and} \quad U_{2n}$$

with

$P = M, R, G, S,$ or T constants defined in (2.3)

$2n$ $2n - 3$	W_{2n}	$T_{2n}=2n - W_{2n}$	V_{2n}	U_{2n}
4 1	X	X	**2**	**2**
6 3	3	3	**3**	**3**
8 5	5	3	**5**	**3**
1 10 7	7	3	**7**	**3**
112 9	7	5	**7**	**5**
14 11	11	3	**11**	**3**

16 13	13	3	**13**	**3**
18 15	13	5	**13**	**5**
20 17	17	3	**17**	**3**
22 19	19	3	**19**	**3**
24 21	19	5	**19**	**5**
26 23	23	3	**23**	**3**
28 25	23	5	**23**	**5**
30 27	23	7	**23**	**7**
32 29	29	3	**29**	**3**
34 31	31	3	**31**	**3**
36 33	31	5	**31**	**5**
38 35	31	7	**31**	**7**
40 37	37	3	**37**	**3**
80 77	73	7	**73**	**7**
82 79	79	3	**79**	**3**
84 81	79	5	**79**	**5**
86 83	83	3	**83**	**3**
88 85	83	5	**83**	**5**
90 87	83	7	**83**	**7**
92	89	3	**89**	**3**

89				
94 91	89	5	**89**	**5**
96 93	89	7	**89**	**7**
**98 95	**89**	**9**	**79**	**19**
100 97	97	3	**97**	**3**
120 117	113	7	**113**	**7**
**122 119	113	9	**109**	**13**
124 121	113	11	**113**	**11**
126 123	113	13	**113**	**13**
**128 125	113	15	**109**	**19**
130 127	127	3	**127**	**3**
132 129	127	5	**127**	**5**
134 131	131	**3**	**131**	**3**
136 133	131	5	**131**	**5**
138 135	131	7	**131**	**7**
140 137	137	3	**137**	**3**
**500 497	**491**	**9**	**487**	**13**
502 499	499	3	**499**	**3**
504 501	499	5	**499**	**5**

506 503	503	3	**503**	**3**
508 505	503	5	**503**	**5**
510 507	503	7	**503**	**7**
1000 997	997	3	**997**	**3**
1002 999	997	5	**997**	**5**
1004 1001	997	7	**997**	**7**
****1006 1003**	**997**	**9**	**983**	**23**
1008 1005	997	11	**997**	**11**
1010 1007	997	13	**997**	**13**
1012 1009	1009	3	**1009**	**3**
1014 1011	1009	5	**1009**	**5**
1016 1013	1013	3	**1013**	**3**
1018 1015	1013	5	**1013**	**5**
10002 9999	9973	29	**9973**	**29**
10004 10001	9973	31	**9973**	**31**
****10006 10003**	**9973**	**33**	**9923**	**83**
****10008 10005**	**9973**	**35**	**9967**	**41**
10010 10007	10007	3	**10007**	**3**
10012 10009	10009	3	**10009**	**3**

10014 10011	10009	5	**10009**	**5**
10016 10013	10009	7	**10009**	**7**
****10018 10015**	**10009**	**9**	**10007**	**11**
10020 10017	10009	11	**10009**	**11**
$2n - M$ $(2n - 3) - M$	$W_{2n} - M$	$T_{2n} = 2n - W_{2n}$	$V_{2n} - M$	U_{2n}
+1000 +997	+993	7	**+993**	**7**
****+1002 +999**	**+993**	**9**	**+931**	**71**
+1004 +1001	+993	11	**+993**	**11**
+1006 +1003	+993	13	**+993**	**13**
****+1008 +1005**	**+993**	**15**	**+919**	**89**
+1010 +1007	+993	17	**+993**	**17**
+1012 +1009	+993	19	**+993**	**19**
+1014 +1011	+1011	3	**+1011**	**3**
+1016 +1013	+1011	5	**+1011**	**5**
+1018 +1015	+1011	7	**+1011**	**7**
****+1020 +1017**	**+1011**	**9**	**+931**	**89**
$2n - R$ $(2n - 3) - R$	$W_{2n} - R$	$T_{2n} = 2n - W_{2n}$	$V_{2n} - R$	U_{2n}
****+1000 +997**	**+979**	**21**	**+903**	**97**
+1002 +999	+979	23	**+979**	**23**

**+1004 +1001	+979	25	+951	53
**+1006 +1003	+979	27	+903	103
+1008 +1005	+979	29	+979	29
+1010 +1007	+979	31	+979	31
**+1012 +1009	+979	33	+951	61
**+1014 +1011	+979	35	+ 781	233
+1016 +1013	+979	37	+979	37
**+1018 +1015	+979	39	+951	67
+1020 +1017	+1017	3	+1017	3
$2n - G$ $(2n - 3) - G$	$W_{2n} - G$	$T_{2n} = 2n - W_{2n}$	$V_{2n} - G$	U_{2n}
**+10000 +9997	+9631	369	+7443	2557
**+10002 +9999	+9631	371	+9259	743
+10004 +10001	+9631	373	+9631	373
**+10006 +10003	+9631	375	+8583	1423
**+10008 + 10005	+9631	377	+6637	3371
+10010 +10007	+9631	379	+9631	379
**+10012 +10009	+9631	381	+8583	1429
+10014 +10011	+9631	383	+9631	383
**+10016 +10013	+9631	385	+9259	757
**+10018	+9631	387	+4491	5527

+10015				
+10020 +10017	+9631	389	**+9631**	**389**
$2n$-S $(2n$-$3)$-S	W_{2n}- S	$T_{2n} = 2n - W_{2n}$	V_{2n}- S	U_{2n}
+20000 +19997	**+18031**	**1969**	**+17409**	**2591**
+20002 +19999	**+18031**	**1971**	**+ 17409**	**2593**
+20004 +20001	+18031	1973	**+18031**	**1973**
+20006 +20003	**+18031**	**1975**	**+16663**	**3343**
+20008 +20005	**+18031**	**1977**	**+16941**	**3067**
+20010 +20007	+18031	1979	**+18031**	**1979**
+20012 +20009	**+18031**	**1981**	**+5671**	**14341**
+20014 +20011	**+18031**	**1983**	**+4101**	**15913**
+20016 +20013	**+18031**	**1985**	**+3229**	**16787**
+20018 +20015	+18031	1987	**+18031**	**1987**
+20020 +20017	**+18031**	**1989**	**+16941**	**3079**
$2n$-T $(2n$-$3)$-T	W_{2n}-T	$T_{2n} = 2n - W_{2n}$	$V_{2n} - T$	U_{2n}
+40000 +39997	**+29737**	**10263**	**+ 21567**	**18433**
+40002 +39999	**+29737**	**10265**	**+ 22273**	**17729**
+40004 +40001	+29737	10267	**+29737**	**10267**
+40006 +40003	**+29737**	**10269**	**+21567**	**18439**

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+40008 +40005	+29737	10271	**+29737**	**10271**
+40010 + 40007	+29737	10273	**+29737**	**10273**
****+40012 +40009**	**+29737**	**10275**	**+10401**	**29611**
****+40014 +40011**	**+29737**	**10277**	**-56003**	**96017**
****+40016 +40013**	**+29737**	**10279**	**+27057**	**12959**
****+40018 +40015**	**+29737**	**10281**	**+25947**	**14071**
****+40020 +40017**	**+29737**	**10283**	**+24493**	**15527**

14. Appendix

7-3=4	11-5=6	11-3=8	13-3=10	17-5=12	17-3=14	19-3=16	23-5=18
23-3=20	29-7=22	29-5=24	29-3=26	31-3=28	37-7=30	37-5=32	37-3=34
41-5=36	41-3=38	43-3=40	47-5=42	47-3=44	53-7=46	53-5=48	53-3=50
59-7=52	59-5=54	59-3=56	61-3=58	67-7=60	67-5=62	67-3=64	71-5=66
71-3=68	73-3=70	79-7=72	79-5=74	79-3=76	83-5=78	83-3=80	89-7=82
89-5=84	89-3=86	101-13=88	97-7=90	97-5=92	97-3=94	101-5=96	101-3=98
103-3=100	107-5=102	107-3=104	109-3=106	113-5=108	113-3=110	131-19=112	127-13=114
127-11=116	131-13=118	127-7=120	127-5=122	127-3=124	131-5=126	131-3=128	137-7=130
137-5=132	137-3=134	139-3=136	149-11=138	151-11=140	149-7=142	149-5=144	149-3=146
151-3=148	157-7=150	157-5=152	157-3=154	163-7=156	163-5=158	163-3=160	167-5=162
167-3=164	173-7=166	173-5=168	173-3=170	179-7=172	179-5=174	179-3=176	181-3=178
191-11=180	193-11=182	191-7=184	191-5=186	191-3=188	193-3=190	197-5=192	197-3=194
199-3=196	211-13=198	211-11=200	233-31=202	211-7=204	211-5=206	211-3=208	223-13=210
229-17=212	227-13=214	223-7=216	223-5=218	223-3=220	227-5=222	227-3=224	229-3=226
233-5=228	233-3=230	239-7=232	239-5=234	239-3=236	241-3=238	251-11=240	271-29=242
251-7=244	251-5=246						

15. Appendix

$T_r(K)$

	q_1 = 3	q_2 = 5	q_3 = 7	q_4 = 1	q_5 = 1	q_6 = 1	q_7 = 1	q_8 = 2	q_9 = 2	q_{10} = 3	q_{11}=37
2K = 2	5	7		13		19			31		
2K = 4	7		11		17		23				41
2K = 6		11	13	17	19	23		29		37	43
2K = 8	11	13		19				31	37		
2K = 10	13				23		29			41	47
2K = 12		17;12	19;12	23;12		29;12	31;12		41;12	43;12	
K = 7	17;14	19;14				31;14		37;14	43;14		
K = 8	19;16		23;16		29;16					47;16	59
K = 9		23;18		29;18	31;18		37;18	41;18	47;18		61
2K =20	23			31		37		43			67
2K=22			29;22				41;22			53;22	
2K=24		29	31		37	41	43	47	53		71
2K=26	29	31		37		43					73
2K=28	31				41		47			59	
2K=30			37	41	43	47		53	59	61	
		37;32		43;32					61;32		79
	37;34		41;34		47;34		53;34				
		41;36	43;36	47;36		53;36		59;36		67;36	83
	41;38	43;38						61;38	67;38		
	43;40		47;40		53;40		59;40			71;40	
		47;42		53;42		59;42	61;42		71;42	73;42	89
	47;44					61;44		67;44	73;44		
			53;46		59;46						
		53;48		59;48	61;48		67;48	71;48		79;48	
	53;50			61;50		67;50		73;50	79;50		97
			59;52				71;52			83;52	
		59;54	61;54		67;54	71;54	73;54		83;54		
	59;56	61;56		67;56		73;56		79;56			
	61;58				71;58					89;58	
			67;60	71;60	73;60		79;60	83;60	89;60		

16. Perspectives and Generalizations

16.1. Other Goldbach Sequences (G'_{2n}) *Independent of* (G_{2n}) *may be Studied Using the Increasing Sequences of Primes* (W'_{2n}) *Defined by*

For any integer $n \geq 3$

$$W'_{2n} = \text{Sup}\,(\, p \in \mathcal{P} : \; p \leq f(n)\,) \tag{16.1.1}$$

f is a function defined on the interval $I = [3\,;+\infty[$ and satisfying the following conditions

- f is strictly increasing on the interval I
- $f\,(3) = 3$ and $\lim_{x \to +\infty} f(x) = +\infty$

- $\forall\, x \in I \quad f(x) \leq 2x - 3$

For example, one of the following functions defined on I can be selected.

- □ $f: \ x \rightarrow a\,x + 3 - 3a \qquad (a \in \mathbb{R}: \ 0 < a \leq 2)$
- □ $g: \ x \rightarrow [\,4\sqrt{3x} - 9\,] \qquad ([\,x\,]$ is the integer part of the real x)
- □ $h: \ x \rightarrow 6 \ln\left(\frac{x}{3}\right) + 3$

16.2. Using this Method it Would be Interesting to Study the Schnirelmann Density [33] of Primes

3 , 5 , 7, 11 ,........ ... in the sequence (U_{2n}) on variable intervals .

16.3. It is Possible to Exceed the Values Shown in the Table of $2n = 10^{1000}$ (Many E.G.D have been calculated for values of $2n$ in the order of $10^{2000} + 10^5$, Sainty [31]) by perfecting this algorithm starting from n , exploiting the fact that one of Goldbach's decomponents can be chosen equal to $12p + 1$, (the set of G.D consists of primes of the form $6p - 1$ or $6p + 1$) using Cipolla-Axler-Dusart type functions [2,7,11,12] to better identify the terms of (G_{2n}), using supercomputers and more efficients software as Maple.

16.4. Diophantine equations and conjectures of the same nature ((3L) conjecture [9,20,22,24,25,36]) can be processed using similar reasoning and algorithms.

■ To validate the (3L) conjecture we study the following sequences of primes (Wl_{2n}), (Vl_{2n}) and (Ul_{2n}) defined by

For any integer $n \geq 3$ $\quad Wl_{2n} = \text{Sup}\ (\ p \in \mathcal{P}: \ p \leq n - 1\ )$

(16.4.1)

- If $Tl_{2n} = (\,2n + 1 - 2\ Wl_{2n}\,)$ is a **prime**

then let

$$Vl_{2n} = Wl_{2n} \quad \text{and} \quad Ul_{2n} = Tl_{2n} \qquad (16.4.2)$$

- If Tl_{2n} is a **composite number**

  then there exists an integer k $\quad 1 \leq k \leq n - 3 \ |$

$$Ul_{2(n-k)} + 2k \ \in \mathcal{P} \qquad (16.4.3)$$

then let

$$Vl_{2n} = Vl_{2(n-k)} \quad \text{and} \quad Ul_{2n} = Ul_{2(n-k)} + 2k \qquad (16.4.4)$$

■ Using the same type of reasoning a generalization the (BBG) conjecture of the following form can be validated

- Let K and Q be two odd integers prime to each other :

  For any integer $n \ |$ $\quad 2n \geq 3(K + Q)$ there exist two primes Ub_{2n} and Vb_{2n} verifying

$$K\,.\ Ub_{2n} + Q.\ Vb_{2n} = 2n \qquad (16.4.5)$$

- Let K and Q be two integers of different parity prime to each other :

  For any integer $n \ |$ $\quad 2n \geq 3(K + Q)$ there are two primes Ub_{2n} and Vb_{2n} verifying

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$$K\,.Ub_{2n} + Q\,.Vb_{2n} = 2n + 1 \qquad (16.4.6)$$

16.5. Remark.

GOLDBACH(-) :

$R_{2K} = \text{Inf}\,(\,p \in \mathcal{P} :\ p - 2K \in \mathcal{P}\,)$ and $Q_{2K} = \text{Inf}\,(\,p \in \mathcal{P} :\ 2K + p \in \mathcal{P}\,) = R_{2K} - 2K$

GOLDBACH(+) :

$V_{2K} = \text{Sup}\,(\,p \in \mathcal{P} :\ 2K - p \in \mathcal{P}\,)$ and $U_{2K} = \text{Inf}\,(\,p \in \mathcal{P} : 2K - p \in \mathcal{P}) = 2K - V_{2K}$

(Is it possible to envisage a symmetry in the Goldbach triangle parametrized by arithmetic sequences between the representations of primes and integers ?)

16.6. The sequences (Wq_{2n}) generate all the G.D. and may enable us to better estimate the values of Goldbach's distribution function G of the Goldbach's Comet [Woon] .

17. Conclusion

17.1. A Recurrent and Explicit Goldbach Sequence $(\,G_{2n}\,) = (\,U_{2n}\,;\,V_{2n}\,)$ Verifying

$$\forall\ n \in \mathbb{N} + 2 \qquad U_{2n}\ ,\ V_{2n} \in \mathcal{P} \quad \text{and} \quad U_{2n} + V_{2n} = 2n$$

has been developed using an simple and efficient "localised" algorithm.

17.2. The records of Silva [35] and Deshouillers, te Riele, Saouter [10] are beaten on a personal computer and 25 E.G.D U_{2n} and V_{2n} are obtained for values of the order $2n = 10^{1000}$ for a computation time of less than half an hour (see Sainty [31]).

17.3. For a given integer $n \geq 49$ the evaluation of the terms U_{2n} and V_{2n} does not require the computing of all previous terms U_{2k} and V_{2k} $\mid$ $1 \leq k < n - 1$. We just need to know the primes U_{2k} and V_{2k} satisfying

$$U_{2k} \leq 7.\ln^{1.3}(2n) \quad \text{and} \quad 2n - 7.\ln^{1.3}(2n) \leq V_{2k} \leq 2n \qquad \textbf{(\textit{on average})}$$

(17.3.1)

This property allows any E.G.D U_{2n} and V_{2n} to be calculated quite quickly, the upper limit being defined by the scientific software and the computer's ability to determine the largest prime preceding $2n - 2$ (prev_prime($2n - 2$) function).

17.4. Therefore the (BBG), the (3L) and the binary Goldbach(- & +) conjectures " Any even integer greater than three is the sum and difference of two primes" are true.

In fact these two conjectures are intertwined.

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