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Sixfold Discrete Symmetry of Fermion Fields as Explanation for Dark Matter

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Abstract: We address the question, what is dark matter? The method used is the Pauli algebra form of the Dirac equation, equivalent to the standard one but allowing to use the multiplicative structure of the algebra. In this form discrete symmetries of fermion fields are the same as the automorphisms of the Pauli group. We construct the prototype Dirac equation in the Clifford algebra and then use its six representations by complex two by two matrices to construct the six symmetric versions, indexed by permutations of three letters. The solutions of symmetric equations form the six sectors of fermion fields. It is shown that the sectors are genuinely distinct, by proving that any fermion field belonging to two different sectors must have mass zero. Also shown is the lack of electromagnetic interaction from one sector to another, since each sector has its own matrix coupling the fermion field to the electromagnetic field. The key tool used is the mass inversion symmetry, introduced in [1]. The sixfold symmetry predicts the ratio of dark to ordinary matter of 5:1 which is close to the observed ratio of 5.2:1. However this symmetry is constructed only for interactions between fermion fields and the electromagnetic field, not yet taking into account the weak and strong interactions. So this article is an indication that maybe the complete answer can be found if the sixfold symmetry extends to these interactions.

Keywords: fermion fields; discrete symmetry; mass inversion; triality

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1. Introduction

The motivation for this article is the sixfold symmetry of fermion fields which is apparent in the Pauli algebra form of the Dirac equation but hidden in its standard four-component form. The mathematical origin of this symmetry is that the sigma matrices and their multiples with the imaginary unit form the single qubit Pauli group [2], and its group of outer automorphisms contains the group of permutations of three letters, of order six [3]. The solutions to the six symmetric versions of the Dirac equation form what is referred to as the six sectors of fermion fields. The six matrix versions of the Dirac equation have the same four scalar equations each, which raises a question, will their solutions be distinct? This question is answered positively by showing that if a field belongs to two different sectors then its mass equals zero.

The discrete symmetries of fermion fields correspond to the automorphisms of the first Pauli group G_1, with inner automorphisms containing the charge conjugation and the mass inversion symmetries, and their composition [1].

The outer automorphisms correspond to the group of order two containing the parity symmetry involution, taken direct product with the permutation group on three letters, obtained by permuting the three spatial derivatives with three sigma matrices.

We construct an operator that calculates the values of the electric and magnetic fields coupled to the fermion field out of the fermion field spinor, with all quantities taking values in the Pauli algebra of two by two complex matrices. The resulting second order wave equation has a coupling matrix which is unique for each of the six versions, obtained by applying the outer automorphisms of the Pauli algebra. This precludes electromagnetic interaction between different sectors.

It is essential to show that the Pauli algebra version of the Dirac equation taken together with its bar-star image is fully equivalent to the standard Dirac equation taken together with its Hermitian conjugate. The proof of equivalence appears in its own section.

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1.1. The Main Results

1. The second order wave equation coupling the fermion field to EM field:

$$\left(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \psi\,(i\sigma_3) = \mathrm{q}\,(\underline{F} + 2\,A_\alpha \partial_\alpha)\,\psi$$

   where ψ is the Pauli algebra spinor, and $i\sigma_3$ is the coupling matrix. The Riemann-Silberstein vector $\underline{F} = \underline{E} + i\underline{B}$ is an operator acting on the Pauli algebra spinor ψ.
2. The Pauli algebra product rule:

$$\partial(fg) = (\partial f)g + f(\partial g) - 2(\underline{f} \wedge \underline{\partial})\underline{g}$$

3. The six symmetric versions of the wave equation: they are obtained from the six different representations of the Clifford algebra $\mathrm{Cl}_{1,2}$ into the algebra of matrices $M_2(\mathbb{C})$. Each of these versions has its own coupling matrix that connects between the fermion field and the EM field.
4. The group of discrete symmetries of fermion fields: they correspond to the group of automorphisms of the first Pauli group G_1, with inner automorphisms being the charge conjugation, the mass inversion, and their composition, and the outer automorphisms being the parity symmetry involution taken direct product with the group of permutations on three letters.

1.2. Notation

All quantities considered here are elements of the Pauli algebra of complex 2×2 matrices, written in the basis of Pauli matrices. Only lower indices are used and instead of upper and lower indices the change of sign of the vector component is indicated by the bar.

Any element can be written as sum of its scalar component and its vector component. The "bar" operation switches the sign of the vector component leaving the scalar component unchanged:

$$a = a_0 + \underline{a} = \sigma_0 a_0 + \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3$$
$$\bar{a} = a_0 - \underline{a} = \sigma_0 a_0 - \sigma_1 a_1 - \sigma_2 a_2 - \sigma_3 a_3$$

The differential, where ∂_α stands for $\frac{\partial}{\partial x_\alpha}$:

$$\partial = \partial_0 + \underline{\partial} = \sigma_0 \partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3$$
$$\bar{\partial} = \partial_0 - \underline{\partial} = \sigma_0 \partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3$$

The EM four-potential in the interaction term:

$$A = A_0 + \underline{A} = \sigma_0 A_0 + \sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3$$
$$\bar{A} = A_0 - \underline{A} = \sigma_0 A_0 - \sigma_1 A_1 - \sigma_2 A_2 - \sigma_3 A_3$$

and similarly for the conserved current J and its bar $\bar{J}$.

The product of any element with its bar is the determinant:

$$a\bar{a} \;=\; \bar{a}a \;=\; \det(a) I_2$$

The composition of differential with its bar is the "box" operator D'Alembertian:

$$\bar{\partial}\partial = \partial\bar{\partial} \;=\; \Box \;=\; \left(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2\right) I_2$$

Inner and outer products:

$$\underline{a}\cdot\underline{b} = a_1b_1 + a_2b_2 + a_3b_3 \qquad \underline{a}\wedge\underline{b} = \begin{vmatrix} i\sigma_1 & i\sigma_2 & i\sigma_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

An important outer product is the magnetic field $\underline{B}$ written in the basis of sigma matrices:

$$i\underline{B} = \underline{\partial}\wedge\underline{A} = \begin{vmatrix} i\sigma_1 & i\sigma_2 & i\sigma_3 \\ \partial_1 & \partial_2 & \partial_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

A product of vectors is a sum of inner and outer products:

$$\underline{ab} = \underline{a}\cdot\underline{b} + \underline{a}\wedge\underline{b}$$

A product also splits as sum of scalar and vector components:

$$\begin{aligned} ab &= a_0b_0 + \underline{a}\cdot\underline{b} + a_0\underline{b} + b_0\underline{a} + \underline{a}\wedge\underline{b} \\ ba &= b_0a_0 + \underline{b}\cdot\underline{a} + b_0\underline{a} + a_0\underline{b} + \underline{b}\wedge\underline{a} \end{aligned}$$

where the first two summands are the scalar component and the last three summands are the vector component. From this we obtain the relation between the commutator and the outer product: $[a,b] = [\underline{a},\underline{b}] = 2\underline{a}\wedge\underline{b}$.

The bar-star automorphism of Pauli algebra is the composition of the bar operation with Hermitian conjugation, in either order (the "star" is also used for complex conjugation). It is an outer automorphism since it changes the determinant. The bar-star automorphism fixes the subalgebra of real quaternions and reverses the sign of the imaginary quaternions, so it acts like complex conjugation on the Pauli algebra. The real quaternions have real scalar term and imaginary vector terms, vice versa for imaginary quaternions:

$$x = x_0 + \underline{x} \qquad \overline{x}^* = x_0^* - \underline{x}^*$$

The four-component spinor of the Dirac equation is written using letters rather than indices and is completed with the second column as follows:

$$\Psi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b^* \\ b & -a^* \\ c & -d^* \\ d & c^* \end{bmatrix} = \begin{bmatrix} iv \\ \\ u \end{bmatrix}$$

The lower square of the two-column matrix is an element u of the subalgebra of real quaternions $\mathbb{H}$ and is unchanged by the bar-star automorphism. The upper square iv is a product of a real quaternion with imaginary unit i and its sign is changed by the bar-star automorphism. Their sum is called the Pauli algebra spinor:

$$u = \begin{bmatrix} c & -d^* \\ \\ d & c^* \end{bmatrix} \quad iv = \begin{bmatrix} a & b^* \\ \\ b & -a^* \end{bmatrix} \quad \psi = u + iv \qquad \overline{\psi}^* = u - iv$$

2. Proof of Equivalence of The Pauli Algebra Dirac Equation to the Standard Dirac Equation

Proposition 1. *The standard Dirac equation taken together with its Hermitian conjugate is equivalent to the Pauli algebra Dirac equation taken together with its bar-star image.*

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Proof. The original Dirac equation: $(i\gamma^{\mu} D_{\mu} - \mathrm{m})\Psi = 0$ is multiplied by $-i$ and is rewritten in two by two blocks with separate matrices for the differentials and for the EM interaction term. The left column of the equation below are the four scalar equations of the standard Dirac equation and the right column are the equations of the Hermitian conjugate Dirac equation, though they appear in different order. The matrices σ_3 appear because without them the right column would have the opposite sign.

$$\begin{bmatrix} \partial_0 & \underline{\partial} \\ -\underline{\partial} & -\partial_0 \end{bmatrix} \begin{bmatrix} iv \\ u \end{bmatrix} + i\,\mathrm{q} \begin{bmatrix} -A_0 & \underline{A} \\ -\underline{A} & A_0 \end{bmatrix} \begin{bmatrix} iv \\ u \end{bmatrix} \begin{bmatrix} \sigma_3 \end{bmatrix} + i\,\mathrm{m} \begin{bmatrix} iv \\ u \end{bmatrix} \begin{bmatrix} \sigma_3 \end{bmatrix} = 0$$

(It is useful at this point to rewrite the equation without blocks and check that the four equations in the left column are those of the Dirac equation and the four equations in the right column are of its Hermitian conjugate).

We now multiply the equation on the right by $i\sigma_3$ and rewrite it as two equations in Pauli algebra:

$$\begin{aligned} \partial_0 iv(i\sigma_3) + \underline{\partial}\, u(i\sigma_3) + \mathrm{q}\, A_0 iv - \mathrm{q}\, \underline{A} u - \mathrm{m}\, iv = 0 \\ -\partial_0 u(i\sigma_3) - \underline{\partial}\, iv(i\sigma_3) - \mathrm{q}\, A_0 u + \mathrm{q}\, \underline{A} iv - \mathrm{m}\, u = 0 \end{aligned}$$

Next add and subtract the two equations recalling that $\psi = u + iv \quad \overline{\psi}^{*} = u - iv$:

$$\begin{aligned} \partial_0 \psi\,(i\sigma_3) + \underline{\partial}\,\psi\,(i\sigma_3) + \mathrm{q}\, A_0 \psi - \mathrm{q}\, \underline{A} \psi + \mathrm{m}\, \overline{\psi}^{*} = 0 \\ \partial_0 \overline{\psi}^{*}(i\sigma_3) - \underline{\partial}\overline{\psi}^{*}(i\sigma_3) + \mathrm{q}\, A_0 \overline{\psi}^{*} + \mathrm{q}\, \underline{A} \overline{\psi}^{*} + \mathrm{m}\, \psi = 0 \end{aligned}$$

Now we can combine the scalar and vector components and obtain the two Pauli algebra Dirac equations:

$$\begin{aligned} \partial\,\psi\,(i\sigma_3) + \mathrm{q}\,\bar{A}\,\psi + \mathrm{m}\,\overline{\psi}^{*} = 0 \\ \bar{\partial}\,\overline{\psi}^{*}(i\sigma_3) + \mathrm{q}\, A\,\overline{\psi}^{*} + \mathrm{m}\,\psi = 0 \end{aligned} \tag{1}$$

These two equations are transformed one into the other by the bar-star automorphism, because ∂ and A are real. □

3. Second Order Wave Equation Coupling the Fermion and the EM Fields

3.1. Reciprocal Expressions for the Spinor and Its Bar-Star Image

The equations (1) allow to express $\overline{\psi}^{*}$ in terms of ψ and vice versa express ψ in terms of $\overline{\psi}^{*}$. Then combine them to obtain second order equations for both:

$$\overline{\psi}^{*} = -\frac{1}{\mathrm{m}}(\partial\,\psi\,(i\sigma_3) + \mathrm{q}\,\bar{A}\,\psi) \qquad \psi = -\frac{1}{\mathrm{m}}\left(\bar{\partial}\,\overline{\psi}^{*}(i\sigma_3) + \mathrm{q}\,A\,\overline{\psi}^{*}\right)$$

Now plug in and simplify:

$$\begin{aligned} \mathrm{m}^2\psi &= -\Box\psi + \mathrm{q}\,\bar{\partial}\bar{A}\psi\,(i\sigma_3) + \mathrm{q}\,A\partial\,\psi\,(i\sigma_3) + \mathrm{q}^2\det(A)\,\psi \\ \mathrm{m}^2\overline{\psi}^{*} &= -\Box\overline{\psi}^{*} + \mathrm{q}\,\partial A\overline{\psi}^{*}(i\sigma_3) + \mathrm{q}\,\bar{A}\bar{\partial}\,\overline{\psi}^{*}(i\sigma_3) + \mathrm{q}^2\det(A)\,\overline{\psi}^{*} \end{aligned}$$

Rewriting we have:

$$\begin{aligned} \left(\Box + \mathrm{m}^2 - \mathrm{q}^2\det A\right)\psi &= \mathrm{q}\,(\bar{\partial}\bar{A} + A\partial)\,\psi\,(i\sigma_3) \\ \left(\Box + \mathrm{m}^2 - \mathrm{q}^2\det A\right)\overline{\psi}^{*} &= \mathrm{q}\,(\partial A + \bar{A}\bar{\partial})\,\overline{\psi}^{*}(i\sigma_3) \end{aligned}$$

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Multiplying on the right by $i\sigma_3$ we have:

$$\left(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \psi \,(i\sigma_3) = -\,\mathrm{q}\,(\bar\partial \bar A + A\partial)\ \psi$$
$$\left(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \overline{\psi}^* (i\sigma_3) = -\,\mathrm{q}\,(\partial A + \bar A\bar\partial)\ \overline{\psi}^*$$

Again, these two equations are transformed one into the other by the bar-star automorphism. Taking into account the placement of brackets, the right sides should be understood as

$$\begin{aligned}\left(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \psi \,(i\sigma_3) &= -\,\mathrm{q}\,\bar\partial(\bar A\,\psi) - \mathrm{q}\,A(\partial\psi) \\ \left(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \overline{\psi}^* (i\sigma_3) &= -\,\mathrm{q}\,\partial(A\overline{\psi}^*) - \mathrm{q}\,\bar A(\bar\partial\overline{\psi}^*)\end{aligned}$$

To calculate $\bar\partial(\bar A\psi)$ and $\partial(A\overline{\psi}^*)$ we need the Pauli algebra product rule, next.

3.2. The Pauli Algebra Product Rule

(it differs from the ordinary product rule by the twist term due to non-commutativity)

Proposition 2. *Let the differential ∂ and the functions f, g have the vector components $\underline{\partial}$, $\underline{f}$ and $\underline{g}$. Then:*

$$\partial(fg) = (\partial f)g + f(\partial g) - 2(\underline{f} \wedge \underline{\partial})\underline{g}$$

Proof. Calculate the difference $\partial(fg) - (\partial f)g - f(\partial g)$. By additivity it is enough to consider the case: $\underline{f} = \sigma_j f \quad \underline{g} = \sigma_k g$. Since for the time derivative ∂_0 the ordinary product rule applies we only need to consider $\underline{\partial} = \sigma_i \partial_i$. Recall that all terms apart from sigma matrices are scalar and can be moved around. Calculating the three summands of the twist term in the product rule:

$$\partial(fg) = \sigma_i \partial_i(\sigma_j f \sigma_k g) = \sigma_i\sigma_j\sigma_k \partial_i(fg) = \sigma_i\sigma_j\sigma_k\left(\frac{\partial f}{\partial x_i}g + f\frac{\partial g}{\partial x_i}\right)$$
$$(\partial f)g = \sigma_i\sigma_j\sigma_k\frac{\partial f}{\partial x_i}g$$
$$f(\partial g) = \sigma_j f\left(\sigma_i\sigma_k\frac{\partial g}{\partial x_i}\right) = \sigma_j\sigma_i\sigma_k f\frac{\partial g}{\partial x_i}$$

Subtracting we have the commutator:

$$\partial(fg) - (\partial f)g - f(\partial g) = [\sigma_i, \sigma_j]f\left(\sigma_k\frac{\partial g}{\partial x_i}\right) = -2(\sigma_j f) \wedge (\sigma_i\partial_i)(\sigma_k g) = -2(\underline{f} \wedge \underline{\partial})\underline{g}$$

□

3.3. The Second Order Fermion-EM Wave Equation

Calculate the right sides of the wave equation using the Pauli algebra product rule:

3.3.1. The Right Side of the First Equation

We wish to calculate $\mathrm{q}\,\bar\partial(\bar A\psi)\ +\ \mathrm{q}\,A(\partial\psi)$.

$$\begin{aligned}&\bar\partial(\bar A\psi) = (\partial_0 - \underline{\partial})(A_0\psi - \underline{A}\psi) = \\ &= (\partial_0 A_0)\psi + A_0(\partial_0\psi) + (\underline{\partial}\,\underline{A})\psi + \underline{A}(\underline{\partial}\psi) - 2(\underline{A} \wedge \underline{\partial})\psi - \\ &-(\partial_0\underline{A})\psi - \underline{A}(\partial_0\psi) - (\underline{\partial}A_0)\psi - A_0(\underline{\partial}\psi)\end{aligned}$$

Add to this:

$$A(\partial\psi) = (A_0 + \underline{A})(\partial_0\psi + \underline{\partial}\psi) =$$
$$A_0(\partial_0\psi) + A_0(\underline{\partial}\psi) + \underline{A}(\partial_0\psi) + \underline{A}(\underline{\partial}\psi)$$

Note that $(\partial_0 A_0)\psi - (\partial_0\underline{A})\psi = (\partial_0\bar{A})\psi$ and also $-(\underline{\partial}A_0)\psi + (\underline{\partial}\,\underline{A})\psi = -(\underline{\partial}\bar{A})\psi$. Together these four terms become $(\bar{\partial}\bar{A})\psi$. Note also that $2\underline{A}(\underline{\partial}\psi) = 2\underline{A}\cdot(\underline{\partial}\psi) + 2\underline{A}\wedge(\underline{\partial}\psi)$.

After canceling and combining we have

$$\bar{\partial}(\bar{A}\psi) + A(\partial\psi) = (\bar{\partial}\bar{A})\psi + 2(A_\mu\partial_\mu)\psi$$
$$R.S. = -\,\mathrm{q}\,(\bar{\partial}\bar{A})\psi - 2\,\mathrm{q}\,(A_\mu\partial_\mu)\psi$$

3.3.2. The Right Side of the Second Equation

We wish to calculate $\mathrm{q}\,\partial(A\overline{\psi}^*) + \mathrm{q}\,\bar{A}(\bar{\partial}\overline{\psi}^*)$.

Instead of calculating, we can simply apply the bar-star automorphism to the previous calculation, keeping in mind that $\partial^* = \partial$ and $A^* = A$.

$$\partial(A\overline{\psi}^*) + \bar{A}(\bar{\partial}\overline{\psi}^*) = (\partial A)\overline{\psi}^* + 2(A_\nu\partial_\nu)\overline{\psi}^*$$
$$R.S. = -\,\mathrm{q}\,(\partial A)\overline{\psi}^* - 2\,\mathrm{q}\,(A_\nu\partial_\nu)\overline{\psi}^*$$

3.3.3. The Two Equations for the Two Quaternionic Functions

Substitute $\psi = u + iv$ and $\overline{\psi}^* = u - iv$ in the two equations:

$$\begin{aligned}
\Big(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\Big)\psi\ (i\sigma_3) &= -\,\mathrm{q}\,(\bar{\partial}\bar{A})\psi\ - 2\,\mathrm{q}\,(A_\mu\partial_\mu)\psi \\
\Big(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\Big)\overline{\psi}^*(i\sigma_3) &= -\,\mathrm{q}\,(\partial A)\overline{\psi}^* - 2\,\mathrm{q}\,(A_\nu\partial_\nu)\overline{\psi}^*
\end{aligned}$$

We get

$$\Big(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\Big)(u+iv)(i\sigma_3) = -\,\mathrm{q}\,(\bar{\partial}\bar{A})(u+iv) - 2\,\mathrm{q}\,(A_\mu\partial_\mu)(u+iv)$$
$$\Big(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\Big)(u-iv)(i\sigma_3) = -\,\mathrm{q}\,(\partial A)(u-iv) - 2\,\mathrm{q}\ (A_\nu\partial_\nu)(u-iv)$$

Now calculate $\frac{1}{2}(Eq.1 + Eq.2)$ and $\frac{1}{2}(Eq.1 - Eq.2)$.

For this we will first need $\frac{1}{2}(\bar{\partial}\bar{A} + \partial A)$ and $\frac{1}{2}(\bar{\partial}\bar{A} - \partial A)$.

$$\begin{aligned}
\tfrac{1}{2}(\bar{\partial}\bar{A} + \partial A) = \tfrac{1}{2}(\partial_0 - \underline{\partial})(A_0 - \underline{A}) + \tfrac{1}{2}(\partial_0 + \underline{\partial})(A_0 + \underline{A}) &= \partial_\alpha A_\alpha + \underline{\partial}\wedge\underline{A} \\
\tfrac{1}{2}(\bar{\partial}\bar{A} - \partial A) = \tfrac{1}{2}(\partial_0 - \underline{\partial})(A_0 - \underline{A}) - \tfrac{1}{2}(\partial_0 + \underline{\partial})(A_0 + \underline{A}) &= -\partial_0\underline{A} - \underline{\partial}A_0
\end{aligned}$$

Note that $\partial_\alpha A_\alpha$ can be made zero by the choice of Lorenz gauge, that $\underline{\partial}\wedge\underline{A}$ is the magnetic field multiplied by the imaginary unit $i\underline{B}$ and $-\partial_0\underline{A} - \underline{\partial}A_0$ is the electric field $\underline{E}$. Combining all this we obtain two equations:

$$\begin{aligned}
\Big(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\Big)u\ (i\sigma_3) &= -\,\mathrm{q}\,i\underline{B}u - \,\mathrm{q}\,\underline{E}iv - 2\,\mathrm{q}\ (A_\alpha\partial_\alpha)u \\
\Big(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\Big)iv(i\sigma_3) &= -\,\mathrm{q}\,\underline{E}u - \,\mathrm{q}\,i\underline{B}iv - 2\,\mathrm{q}\ (A_\alpha\partial_\alpha)iv
\end{aligned}$$

3.3.4. The Wave Equation Coupling the Fermion and the EM Fields

Adding and subtracting the equations we obtain:

$$\left(\square + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \psi \,(i\sigma_3) = -\,\mathrm{q}\,(i\underline{B} + \underline{E} + 2\,A_\alpha \partial_\alpha)\,\psi$$
$$\left(\square + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \overline{\psi}^* (i\sigma_3) = -\,\mathrm{q}\,(i\underline{B} - \underline{E} + 2\,A_\alpha \partial_\alpha) \overline{\psi}^*$$

It is easier to see the bar-star symmetry if we introduce the Riemann-Silberstein field vector $\underline{F} = \underline{E} + i\underline{B}$ and its bar-star image $\overline{\underline{F}}^* = -\underline{E} + i\underline{B}$.

Then the equations become:

$$\left(\square + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \psi \,(i\sigma_3) + \mathrm{q}\,(\,\underline{F} + 2\,A_\alpha \partial_\alpha)\,\psi \;= 0$$
$$\left(\square + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \overline{\psi}^* (i\sigma_3) + \mathrm{q}\,(\overline{\underline{F}}^* + 2\,A_\alpha \partial_\alpha)\,\overline{\psi}^* = 0$$

Again, these two equations are transformed one into the other by the bar-star automorphism. The two equations are essentially the same, because the choice between the spinors ψ and $\overline{\psi}^*$ is arbitrary. The matrix $i\sigma_3$ appears as the coupling matrix between the second order operator in the left summand and the electromagnetic field vector in the right summand.

Here the Riemann-Silberstein vector $\underline{F} = \underline{E} + i\underline{B}$ [4], [5] is understood as operator acting on the Pauli algebra spinor ψ.

We have proven the:

Proposition 3. *The equation coupling the fermion field to the electromagnetic field:*

$$\left(\square + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \psi \,(i\sigma_3) + \mathrm{q}\,(\underline{F} + 2\,A_\alpha \partial_\alpha)\,\psi = 0 \tag{2}$$

4. The Pauli Algebra Lagrangian

The purpose of this section is to rewrite the Lagrangian of quantum electrodynamics in Pauli algebra form. Then the Lagrangian is rewritten in terms of Clifford algebra $\mathrm{Cl}_{1,2}$. Next we use the six representations of the Clifford algebra $\mathrm{Cl}_{1,2}$ into the algebra of two by two complex matrices $M_2(\mathbb{C})$ so as to construct six symmetric forms of the Dirac equation and the automorphisms that connect between them.

The main difference between the standard QED Lagrangian [6–8] :

$$\mathcal{L}_{QED} = \overline{\Psi}(i\gamma^\mu \partial^\mu - m)\Psi - qA_\mu \overline{\Psi}\gamma^\mu \Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

and the Pauli algebra Lagrangian used here is use of the multiplicative structure of the algebra.

The other difference is that unlike the usual recovering the equations of motion from the Lagrangian using the Euler-Lagrange equations, another procedure is used, namely equating to zero the formal derivatives. This results in Pauli algebra Dirac equation [1], and the inhomogeneous wave equations.

Here is the Pauli algebra version of the QED Lagrangian:

$$\begin{aligned}\mathcal{L} \;&=\; \psi^* \partial\, \psi (i\sigma_3) - q\psi^* \bar{A}\, \psi + m\psi^* \overline{\psi}^* + \left(\overline{\partial A}\right)\partial A \,+ \\ &+\; \overline{\psi}\, \bar{\partial}\, \overline{\psi}^* (i\sigma_3) - q\overline{\psi} A\, \overline{\psi}^* + m\, \overline{\psi}\, \psi + \left(\overline{\bar{\partial}\bar{A}}\right)\bar{\partial}\bar{A}\end{aligned}$$

The Lagrangian is intentionally written in two lines because these lines are transformed one into the other by the bar-star automorphism of the Pauli algebra, [1].

The Lagrangian can also be written in terms of the conserved current J:

$$\begin{aligned}\mathcal{L} &= \psi^* \partial \psi(i\sigma_3) - \tfrac{1}{2}(\bar{A}\, J + \bar{J}\, A) + m\psi^*\overline{\psi}^* + \bar{A} \,\Box\, A \\ &+ \overline{\psi}\,\bar{\partial}\,\overline{\psi}^*(i\sigma_3) - \tfrac{1}{2}(A\,\bar{J} + J\,\bar{A}) + m\,\overline{\psi}\,\psi + A \,\Box\, \bar{A}\end{aligned}$$

The reason for this is that $\bar{\partial}\partial A = \partial\bar{\partial} A = \Box\, A$ and also (see 5)

$$q\psi^*\bar{A}\,\psi + q\overline{\psi}A\,\overline{\psi}^* = \tfrac{1}{2}(\bar{A}\,J + A\,\bar{J} + \bar{J}\,A + J\,\bar{A})$$

Recovering the equations of motion is done by equating to zero the formal derivatives of the Lagrangian with respect to ψ^*, $\overline{\psi}$, $\bar{A}$, A:

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = 0 \qquad \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0 \qquad \frac{\partial\mathcal{L}}{\partial\bar{A}} = 0 \qquad \frac{\partial\mathcal{L}}{\partial A} = 0$$

This results in four equations

$$\begin{aligned}\partial\,\psi\,(i\sigma_3) - q\bar{A}\,\psi + m\overline{\psi}^* &= 0 \\ \bar{\partial}\,\overline{\psi}^*(i\sigma_3) - qA\,\overline{\psi}^* + m\,\psi &= 0 \\ -J + \Box\, A &= 0 \\ -\bar{J} + \Box\, \bar{A} &= 0\end{aligned} \tag{3}$$

The first two equations are the Pauli algebra Dirac equation [1], and the last two are the inhomogeneous wave equations [9].

The sign of the d'Alembertian used here is the opposite of the sign in [9].

The differential, the four-potential and the four-current are elements of the Pauli algebra $M_2(\mathbb{C})$ split into scalar and vector parts as follows:

$$\begin{aligned}\partial &= \partial_\mu\sigma_\mu = \partial_0 + \underline{\partial} & \bar{\partial} &= \partial_0 - \underline{\partial} \\ A &= A_\mu\sigma_\mu = A_0 + \underline{A} & \bar{A} &= A^0 - \underline{A} \\ J &= J_\mu\,\sigma_\mu = J_0 + \underline{J} & \bar{J} &= J^0 - \underline{J}\end{aligned}$$

If $u \in \mathbb{H}$ then $u^* = \bar{u}, \overline{u}^* = u$. For iv the opposite holds $(iv)^* = -i\bar{v}$.

For any $\psi \in M_2(\mathbb{C})$ $\bar{\psi}\psi = \det(\psi)I_2$

Both compositions of the differential and its bar are the d'Alembertian: $\partial\bar{\partial} = \bar{\partial}\partial = \Box$. Hence:

$$\begin{aligned}\overline{(\partial A)}\partial A &= \bar{A}\bar{\partial}\partial A = \bar{A}\,\Box\, A \\ \overline{(\bar{\partial}\bar{A})}\bar{\partial}\bar{A} &= A\partial\bar{\partial}\bar{A} = A\,\Box\,\bar{A}\end{aligned}$$

The Lagrangian is rewritten separating the elements into their scalar and vector parts. This will be used when writing the representation independent form of the Lagrangian.

$$\begin{aligned}\mathcal{L} = \;& \psi^*\,(\partial_0 + \underline{\partial})\,\psi(i\sigma_3) + m\psi^*\overline{\psi}^* + \overline{\psi}\,(\partial_0 - \underline{\partial})\,\overline{\psi}^*(i\sigma_3) + m\,\overline{\psi}\,\psi - \\ & -\tfrac{1}{2}[(A_0 - \underline{A})\,(J_0 + \underline{J}) + (J_0 - \underline{J})\,(A_0 + \underline{A})] + (A_0 - \underline{A})\,\Box\,(A_0 + \underline{A}) - \\ & -\tfrac{1}{2}[(A_0 + \underline{A})\,(J_0 - \underline{J}) + (J_0 + \underline{J})\,(A_0 - \underline{A})] + (A_0 + \underline{A})\,\Box\,(A_0 - \underline{A})\end{aligned}$$

Taking into account that

$$\psi^* = \bar{u} - i\bar{v} \qquad \bar{\psi} = \bar{u} + i\bar{v} \qquad \overline{\psi}^* = u - iv \qquad \psi = u + iv$$

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and the expressions for the current (6) which imply:

$$\begin{aligned} J_0 &= q(\bar{u}u + \bar{v}v)\sigma_0 \\ \underline{J} &= J^k\sigma_k = iq(\bar{u}\sigma_k v - \bar{v}\sigma_k u)\sigma_k \end{aligned}$$

the Lagrangian is rewritten in quaternionic functions u and v as follows:

$$\begin{aligned} \tfrac{1}{2}\mathcal{L} = (\bar{u}\partial_0 u + \bar{v}\partial_0 v)(i\sigma_3) - (\bar{u}\underline{\partial} v - \bar{v}\underline{\partial} u)\sigma_3 - \\ -qA_0(\bar{u}u + \bar{v}v) + iqA_k(\bar{u}\sigma_k v - \bar{v}\sigma_k u) + \\ +A_0 \Box A_0 - \underline{A}\Box\underline{A} + m(\bar{u}u - \bar{v}v) \end{aligned} \tag{4}$$

Equating to zero the derivatives

$$\frac{\partial\mathcal{L}}{\partial\bar{u}} = 0 \qquad \frac{\partial\mathcal{L}}{\partial\bar{v}} = 0$$

results in equations

$$\begin{aligned} \partial_0 u(i\sigma_3) - \underline{\partial} v\sigma_3 - qA^0 u + iq\underline{A}v + mu = 0 \\ \partial_0 v(i\sigma_3) + \underline{\partial} u\sigma_3 - qA^0 v - iq\underline{A}u - mv = 0 \end{aligned}$$

which are equivalent to the first two equations of (3), which are the Pauli algebra Dirac equations. This can be checked by multiplying the second equation by i and then adding and subtracting with the first equation.

In order to have a representation-independent form of the Lagrangian, the quaternionic functions, the real u and the imaginary iv, need to be written in terms of sigma matrices:

$$u = \begin{bmatrix} c & -d^* \\ d & c^* \end{bmatrix} = \begin{bmatrix} c_0 + ic_1 & -d_0 + id_1 \\ d_0 + id_1 & c_0 - ic_1 \end{bmatrix} = c_0\sigma_0 + id_1\sigma_1 - id_0\sigma_2 + ic_1\sigma_3$$

$$iv = \begin{bmatrix} a & b^* \\ b & -a^* \end{bmatrix} = \begin{bmatrix} a_0 + ia_1 & b_0 - ib_1 \\ b_0 + ib_1 & -a_0 + ia_1 \end{bmatrix} = ia_1\sigma_0 + b_0\sigma_1 + b_1\sigma_2 + a_0\sigma_3$$

4.1. Proof That the Two Forms of Lagrangian Are the Same

Proposition 4. *The two sums forming the interaction terms in the Lagrangian are equal:*

$$q\psi^*\bar{A}\,\psi + q\overline{\psi}A\,\overline{\psi}^* = \tfrac{1}{2}(\bar{A}\,J + A\,\bar{J} + \bar{J}\,A + J\,\bar{A}) \tag{5}$$

Proof. The probability current J^μ is calculated out of the real quaternionic and imaginary quaternionic components of the Pauli algebra spinor ψ as follows:

$$J^\mu = u^*\sigma_\mu(iv) + (iv)^*\sigma_\mu u$$

(Here the components of the conserved current J are understood not as numbers, but as scalar two by two matrices).

This can be checked by using the two-column completion of Ψ as described in (1) and in [1], and then rewriting the usual formula for the probability current [8] with 2×2 blocks iv and u.

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We need to calculate separately the scalar and the vector components of the current:

$$\begin{aligned} J_0 &= q[u^*u + (iv)^*(iv)]\sigma_0 \\ \underline{J} &= J_k\sigma_k = q[u^*\sigma_k(iv) + (iv)^*\sigma_k u]\sigma_k \end{aligned} \tag{6}$$

(Note that both $u^*u + (iv)^*(iv)$ and $u^*\sigma_k(iv) + (iv)^*\sigma_k u$ are real scalars)

To prove (5) we calculate separately the left side and the right side. The spinor $\psi = u + iv$ is decomposed as sum of a real quaternionic function u and an imaginary quaternionic function iv. We begin with the left side:

$$\begin{aligned} &L.S. = q\psi^*\bar{A}\,\psi + q\overline{\psi}A\,\overline{\psi}^* = \\ &q\,[u^* + (iv)^*](A_0 - \underline{A})\,(u + iv) + q\,[u^* - (iv)^*](A_0 + \underline{A})\,(u - iv) = \\ &= q\,(\bar{u} - i\bar{v})(A_0 - \underline{A})\,(u + iv) + q\,(\bar{u} + i\bar{v})(A_0 + \underline{A})\,(u - iv) = \\ &= 2qA_0(\bar{u}u + \bar{v}v)\sigma_0 - 2iqA_k(\bar{u}\sigma_k v - \bar{v}\sigma_k u)\sigma_k = \\ &= 2qA_0[u^*u + (iv)^*(iv)]\sigma_0 - 2qA_k[u^*\sigma_k(iv) + (iv)^*\sigma_k u]\sigma_k \end{aligned}$$

(Note that the expressions in square brackets are real scalars)

Now calculate the right side:

$$\begin{aligned} &R.S. = \tfrac{1}{2}(\bar{A}\,J + A\,\bar{J} + \bar{J}\,A + J\,\bar{A}) = \\ &= \tfrac{1}{2}[(A_0 - \underline{A})(J_0 + \underline{J}) + (A_0 + \underline{A})(J_0 - \underline{J}) + \\ &+ (J_0 - \underline{J})(A_0 + \underline{A}) + (J_0 + \underline{J})(A_0 - \underline{A})] = \\ &= 2(A_0J_0 - 2\underline{A}\cdot\underline{J} - 2\underline{A}\wedge\underline{J} - 2\underline{J}\wedge\underline{A}) = \\ &= 2(A_0J_0 - 2\underline{A}\cdot\underline{J}) \end{aligned}$$

(the last equality because the outer product is anticommutative)

Now use the expressions for the current in (6):

$$R.S. = 2qA_0[u^*u + (iv)^*(iv)]\sigma_0 - 2qA_k[u^*\sigma_k(iv) + (iv)^*\sigma_k u]\sigma_k$$

$L.S. = R.S.$ □

5. The Clifford Algebra Lagrangian

The Pauli algebra $M_2(\mathbb{C})$ is isomorphic to the Clifford algebra $\mathrm{Cl}_{1,2}$ so that every element can be expressed in its three generators:

$$\mathrm{e}_0^2 = 1, \qquad \mathrm{e}_1^2 = -1, \qquad \mathrm{e}_2^2 = -1 \tag{7}$$

The table below defines six matrix representations of $\mathrm{Cl}_{1,2}$ into $M_2(\mathbb{C})$, of which the first is chosen to be the standard one.

The automorphisms of $M_2(\mathbb{C})$ are labelled by their action on the three cyclic subgroups of order 4 of the finite quaternion group Q_8. These automorphisms form a group isomorphic to the group of permutations of three letters, see [10,11].

Here the finite quaternion group is (including their opposite signs):

$$Q_8 = \{\sigma_0,\ -i\sigma_1,\ -i\sigma_2,\ -i\sigma_3\} \leftrightarrow \{1,\ \mathbf{i},\ \mathbf{j},\ \mathbf{k}\ \}$$

This action can be seen in the columns corresponding to the generators

$$\mathrm{e}_1,\ \mathrm{e}_2,\ \mathrm{e}_1\mathrm{e}_2$$

The subscripts ijk in the following table show the isomorphism between the outer automorphisms of Q_8 and the group of permutations of three letters.

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The morphisms between the representations are obtained by combining the inverse of one representation with another.

The automorphism of $M_2(\mathbb{C})$ converting the DE_{123} into DE_{231} for example will be

$$\varphi_{231} \circ \varphi_{123}^{-1}$$

Here is the table for the six representations $\varphi_{ijk} : \mathrm{Cl}_{1,2} \to M_2(\mathbb{C})$:

φ_{ijk}	e_0	e_1	e_2	e_1e_2	e_0e_1	e_0e_2	$e_0e_1e_2$
φ_{123}	σ_3	$-i\sigma_1$	$-i\sigma_2$	$-i\sigma_3$	σ_2	$-\sigma_1$	$-i\sigma_0$
φ_{231}	σ_1	$-i\sigma_2$	$-i\sigma_3$	$-i\sigma_1$	σ_3	$-\sigma_2$	$-i\sigma_0$
φ_{312}	σ_2	$-i\sigma_3$	$-i\sigma_1$	$-i\sigma_2$	σ_1	$-\sigma_3$	$-i\sigma_0$
φ_{213}	σ_3	$-i\sigma_2$	$-i\sigma_1$	$i\sigma_3$	$-\sigma_1$	σ_2	$i\sigma_0$
φ_{132}	σ_2	$-i\sigma_1$	$-i\sigma_3$	$i\sigma_2$	$-\sigma_3$	σ_1	$i\sigma_0$
φ_{321}	σ_1	$-i\sigma_3$	$-i\sigma_2$	$i\sigma_1$	$-\sigma_2$	σ_3	$i\sigma_0$

The action of the bar-star automorphism and of the bar and of "star" (Hermitian conjugate) anti-automorphisms on the generators of the Clifford algebra can be seen in the following table:

operation	e_0	e_1	e_2	e_1e_2	e_0e_1	e_0e_2	$e_0e_1e_2$
bar-star	-1	1	1	1	-1	-1	-1
bar	-1	-1	-1	-1	-1	-1	1
star	1	-1	-1	-1	1	1	-1

Now use φ_{123}^{-1} to rewrite all the vector expressions in terms of generators of the Clifford algebra, according to the first line in the table. The σ_0 being identity is omitted.

$$\begin{aligned} \underline{\partial} &= -e_0e_2\, \partial_1 + e_0e_1\, \partial_2 + e_0\, \partial_3 \\ \underline{A} &= -e_0e_2\, A^1 + e_0e_1\, A^2 + e_0\, A^3 \\ \underline{J} &= -e_0e_2\, J^1 + e_0e_1\, J^2 + e_0\, J^3 \end{aligned} \tag{8}$$

The other elements of the Lagrangian that depends on the representation are $i\sigma_3$ and σ_3, and according to the first line of the table, they should be replaced with $-e_1e_2$ and with e_0.

Now one can write the representation-independent Lagrangian, by substituting (8) into (4). In this article the substitution is done only for the case of the free Dirac equation .

If the six representations are applied to all the expressions in the Lagrangian and in the resulting equations, then the morphisms between representations will transform solutions into solutions. Both the new Pauli algebra spinors and the corresponding new equations will be different but the underlying eight equations of the Dirac equation and its Hermitian conjugate will be exactly the same.

This raises the question, will the solutions to the transformed equations be the same as before, or will they be genuinely different? This question is answered in Proposition 5. Only the free Dirac equation is needed there, so here is its representation-independent form:

$$(\partial_0 - e_0e_2\, \partial_1 + e_0e_1\, \partial_2 + e_0\, \partial_3)\psi + e_0e_1e_2\, \mathrm{m}\, \overline{\psi}^* e_0 = 0$$

6. The Six Sectors of Fermion Fields Are Distinct

The method used to show this is the mass inversion symmetry, described in [1]. For convenience this short argument is reproduced here. Given a solution ψ of the free Dirac equation define $\psi' = \psi\sigma_3$. Then multiply this equation, which appears first, on the right by σ_3, and recall that $\overline{\sigma_3}^* = -\sigma_3$:

$$\begin{array}{ll} \partial\psi = i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & \partial\psi\sigma_3 = i\,\mathrm{m}\,\overline{\psi}^*\sigma_3\sigma_3 \\ \partial(\psi\sigma_3) = -i\,\mathrm{m}\,\overline{(\psi\sigma_3)}^*\sigma_3 & \partial\psi' = i(-\,\mathrm{m}\,)\overline{\psi'}^*\sigma_3 \end{array}$$

So ψ' is also a solution of the Dirac equation but with the mass of opposite sign.

Proposition 5. *Only massless spinors can be solutions to two versions of the free Dirac equation belonging to two different sectors.*

Proof. These are the six versions of the free Dirac equation obtained by applying the six representations to the representation-independent free Dirac equation :

$$\begin{array}{llcl} 123 & (\partial_0 + \sigma_1\,\partial_1 + \sigma_2\,\partial_2 + \sigma_3\,\partial_3)\psi - i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & = & 0 \\ 231 & (\partial_0 + \sigma_2\,\partial_1 + \sigma_3\,\partial_2 + \sigma_1\,\partial_3)\psi - i\,\mathrm{m}\,\overline{\psi}^*\sigma_2 & = & 0 \\ 312 & (\partial_0 + \sigma_3\,\partial_1 + \sigma_1\,\partial_2 + \sigma_2\,\partial_3)\psi - i\,\mathrm{m}\,\overline{\psi}^*\sigma_1 & = & 0 \\ 213 & (\partial_0 - \sigma_2\,\partial_1 - \sigma_1\,\partial_2 + \sigma_3\,\partial_3)\psi + i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & = & 0 \\ 132 & (\partial_0 - \sigma_1\,\partial_1 - \sigma_3\,\partial_2 + \sigma_2\,\partial_3)\psi + i\,\mathrm{m}\,\overline{\psi}^*\sigma_1 & = & 0 \\ 321 & (\partial_0 - \sigma_3\,\partial_1 - \sigma_2\,\partial_2 + \sigma_1\,\partial_3)\psi + i\,\mathrm{m}\,\overline{\psi}^*\sigma_2 & = & 0 \end{array}$$

Suppose that the spinor ψ is a solution of two different equations. We can always use the morphisms between representations to make the first of two equations the top line 123. Now let the second equation be any other except 213 (this case will be dealt with separately). For example suppose the second equation is 231. Multiply both equations on the right by σ_3:

$$\begin{array}{llcl} 123 & (\partial_0 + \sigma_1\,\partial_1 + \sigma_2\,\partial_2 + \sigma_3\,\partial_3)\psi' + i\,\mathrm{m}\,\overline{\psi'}^*\sigma_3 & = & 0 \\ 231 & (\partial_0 + \sigma_2\,\partial_1 + \sigma_3\,\partial_2 + \sigma_1\,\partial_3)\psi' - i\,\mathrm{m}\,\overline{\psi'}^*\sigma_2 & = & 0 \end{array}$$

So ψ' is a solution for the first equation with mass $-\,\mathrm{m}$ and for the second with mass m, which is only possible if the mass is zero. This argument applies to all lines except 213 where we do not have the anticommutation.

For the lines 123 and 213 suppose that $\mathrm{m} \neq 0$ and consider also their bar-star equations:

$$\begin{array}{lcl} (\partial_0 + \sigma_1\,\partial_1 + \sigma_2\,\partial_2 + \sigma_3\,\partial_3)\psi - i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & = & 0 \\ (\partial_0 - \sigma_2\,\partial_1 - \sigma_1\,\partial_2 + \sigma_3\,\partial_3)\psi + i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & = & 0 \\ (\partial_0 - \sigma_1\,\partial_1 - \sigma_2\,\partial_2 - \sigma_3\,\partial_3)\overline{\psi}^* - i\,\mathrm{m}\,\psi\sigma_3 & = & 0 \\ (\partial_0 + \sigma_2\,\partial_1 + \sigma_1\,\partial_2 - \sigma_3\,\partial_3)\overline{\psi}^* + i\,\mathrm{m}\,\psi\sigma_3 & = & 0 \end{array} \quad (9)$$

Subtracting the first two lines and the last two lines we get

$$\begin{array}{lcl} \frac{1}{2}(\sigma_1 + \sigma_2)(\partial_1 + \partial_2)\,\psi & = & i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 \\ \frac{1}{2}(\sigma_1 + \sigma_2)(\partial_1 + \partial_2)\,\overline{\psi}^* & = & -i\,\mathrm{m}\,\psi\sigma_3 \end{array}$$

Combining we get $\frac{1}{2\,\mathrm{m}^2}(\partial_1 + \partial_2)^2\psi = \psi$ and with a change of coordinates we have an ODE

$$\frac{1}{2\,\mathrm{m}^2}\frac{\partial^2\psi}{\partial s^2} - \psi = 0$$

with real roots of its characteristic equation. So ψ is an exponential function and this is incompatible with it being normalizable. □

7. The Lack of Electromagnetic Interaction Between Sectors

We consider the six symmetric versions of the equation (2) that links the fermion field to the electromagnetic field. In the version labeled by the permutation 123 the linking matrix is $i\sigma_3$. A look at the fourth column of the table for φ_{ijk} shows that the representation independent expression is $-\mathrm{e}_1\mathrm{e}_2$. Its values under the six representations are all different, indicating a specific coupling within each sector.

8. The Group of Discrete Symmetries

Proposition 6. *Discrete symmetries of fermion fields correspond to the group of automorphisms of the first Pauli group* G_1.

$\mathrm{Inn}(G_1) = \{1, C, M, CM\}$, *the charge conjugation and the mass inversion symmetries.*

$\mathrm{Out}(G_1) = C_2 \times \Sigma_3$, *parity involution cyclic group direct product with the group of permutations.*

The map $\chi(e_1) = e_2$, $\chi(e_2) = e_1e_2$ *is the triality automorphism of order* 3.

Proof. The proof consists of identifying the discrete symmetries with elements of the automorphism group $\mathrm{Aut}(G_1)$.

The inner automorphisms of the first Pauli group G_1 are the conjugations by σ_1 which is the charge conjugation symmetry C, by σ_3 which is the mass inversion symmetry M, and by their product $i\sigma_2$, which is $CM = MC$ [1].

$$1 \longrightarrow \mathrm{Inn}\, G_1 \longrightarrow \mathrm{Aut}\, G_1 \longrightarrow \mathrm{Out}\, G_1 \longrightarrow 1$$
$$1 \longrightarrow C_2 \times C_2 \longrightarrow \mathrm{Aut} G_1 \longrightarrow C_2 \times \Sigma_3 \longrightarrow 1$$

where in the outer automorphisms the cyclic group of order two is generated by the parity transformation "bar-star" and the group of permutations permutes between spatial derivatives and the three sigma matrices.

The parity transformation inverts the signs of the three spatial derivatives and the three components of the vector potential. Recalling that there are two versions of the Pauli algebra Dirac equation related by the bar-star automorphism, this transformation essentially replaces the spinor ψ with its bar-star image $\overline{\psi}^*$ and vice versa. Since the choice between them is arbitrary, there are only six sectors of fermion fields and not twelve. The group of automorphisms of G_1 is described in [3]. □

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