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Improvements for Inventory Models with Generalized Interarrival Times

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Abstract

Single-machine, single-product inventory models with generalized interarrival times have lacked a rigorously justified and computationally tractable optimization framework because a sign error in earlier derivations led to ad hoc feasibility restrictions and overly broad search domains. This study presents a corrected derivation that proves strict convexity of the minimum-cost objective and establishes existence and uniqueness of an interior optimum without auxiliary conditions, consolidating earlier fragmented results into a single theorem. Using these structural properties, the maximum-profit formulation is reduced to a one dimensional program with natural finite bounds that tightly bracket the optimizer, replacing earlier paired bounds on an effectively unbounded domain. Numerical experiments on a canonical benchmark show that the tightened admissible interval recovers the same optimum with substantially fewer evaluations, improving computational efficiency and implementation robustness. The analysis further clarifies the structural correspondence between cost minimization and profit maximization and provides a simple, reproducible solution path for capacity-constrained single machine systems under generalized interarrival times. The primary contributions are (i) a corrected optimality theory establishing strict convexity and a unique interior optimum without auxiliary conditions, (ii) a one-dimensional reformulation of the profit model with problem-native finite bounds, and (iii) demonstrably tighter admissible intervals that reduce evaluation counts while preserving optimality. Single-machine, single-product inventory models with generalized interarrival times have long lacked a rigorously justified and computationally tractable optimization framework, owing to a sign mis-specification in prior derivations that spawned ad hoc feasibility restrictions and expansive search domains. A corrected, succinct derivation establishes the strict convexity of the minimum-cost objective and proves the existence and uniqueness of an interior optimum without auxiliary conditions, unifying previously fragmented results into a general theorem. Leveraging these structural properties, the maximum-profit formulation is reduced to a single-variable program over natural, finite bounds that tightly bracket the optimizer, supplanting earlier paired bounds defined on an unbounded domain. Numerical evidence on a canonical benchmark shows that the admissible interval is markedly tighter yet attains the same optimum with fewer evaluations, thereby improving numerical efficiency and robustness of implementation. The analysis clarifies the correspondence between cost-minimization and profit-maximization formulations and provides an operationally simple, reproducible solution path for capacity-constrained single-machine systems under generalized interarrival times. Principal contributions are: (i) a corrected optimality theory establishing strict convexity and a unique interior optimum without auxiliary conditions; (ii) a dimensionality reduction of the profit model to a single-variable program with natural finite bounds; and (iii) demonstrably tighter admissible intervals that cut evaluations while preserving optimality.

Keywords: inventory model; generalized interarrival time; lot sizing; convexity; optimization

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1. Introduction

Single-machine, single-product inventory systems with generalized interarrival time distributions remain analytically fragile (Rangaswamy [1]). Two persistent limitations recur in the literature: (i) strict convexity and uniqueness of an interior optimum for the minimum-cost model have not been rigorously established because of a sign error in a key derivative, and (ii) the corresponding profit-maximization problem is commonly posed over expansive—sometimes effectively unbounded—search domains, motivating ad hoc feasibility restrictions and costly numerical search. These weaknesses can translate into unstable implementations, inflated evaluation budgets, and unclear optimality guarantees in capacity-constrained settings (Andaz et al. [2]).

Choi and Enns [3] develops minimum-cost and maximum-profit formulations for a capacity-constrained single machine under generalized interarrival time assumptions. For the single-product case, the reduction relies on graphical arguments and heuristic search, leaving strict convexity and interior optimality unverified and effectively placing the profit model on an unbounded search domain. Yang [4] revisits the minimum-cost case but introduces a sign error in a key derivative; the resulting analysis adds unnecessary feasibility restrictions and still leaves an optimality gap. In addition, the heuristic procedure in Choi and Enns [3] is not guaranteed to locate the true optimum.

Recent queuing–inventory studies with non-Poisson arrivals—including work on priority classes and stability, variable-speed replenishment control, and retrial dynamics (Otten and Daduna [5])—often rely on model-specific or policy-based analyses rather than providing closed-form convexity certificates for the canonical single-machine generalized interarrival-time setting, leaving a persistent structural gap (Amjath et al. [6]; Harikrishnan et al. [7]). In parallel, recent lot-sizing research prioritizes computational tractability via high-quality heuristics and uncertainty-aware planning; without provable curvature properties and finite search bounds, such methods can require expansive search domains, large evaluation budgets, and careful parameter tuning in capacity-constrained implementations (Baek [8]; Dziuba and Almeder [9]; Kohlmann and Sahling [10]).

This study develops a corrected optimality theory that rectifies the sign error identified in Yang [4]. The analysis replaces earlier graphical arguments with a concise proof and establishes strict convexity of the minimum-cost objective, thereby certifying existence and uniqueness of an interior optimizer without auxiliary feasibility constraints. Under standard regularity conditions (positive cost parameters and well-behaved interarrival distributions), the results consolidate previously fragmented claims into a single theorem and remove ad hoc restrictions induced by the sign error, aligning the analysis with recent emphasis on structural guarantees in non-Poisson queuing–inventory and uncertainty-aware lot-sizing models (Baek [8]; Hong and Scully [11]; Kohlmann and Sahling [10]). Using these structural properties, the profit-maximization model is reformulated as a one-dimensional program over natural finite bounds inherited from the cost model's marginal structure; the resulting bracketing interval follows from monotonicity and curvature, replacing paired bounds previously imposed over an effectively unbounded domain in Yang [4]. On the canonical benchmark from Choi and Enns [3], the tightened admissible interval recovers the exact optimum with substantially fewer evaluations, improving numerical efficiency and implementation robustness (Sereshti et al. [12]). The contributions are threefold: (i) a corrected optimality theory guaranteeing strict convexity and a unique interior optimum without auxiliary constraints; (ii) a one-dimensional reformulation of the profit model with problem-native finite bounds; and (iii) demonstrably tighter admissible domains that reduce evaluation counts while preserving optimality.

The remainder of this paper is organized as follows. Section 2 introduces the notation and assumptions. Section 3 revisits the minimum-cost model $T_C(Q)$ and presents a corrected derivation that (i) pinpoints the sign error in a key derivative, (ii) removes the two auxiliary feasibility conditions introduced in Yang [4], and (iii) consolidates previously fragmented results into a single general theorem. Section 3 also revisits the maximum-profit model $T_P(Q, D)$, clarifies its analytical structure, and reduces profit optimization to a one-dimensional program with natural finite bounds. Section 4 presents numerical results for the canonical benchmark in Choi and Enns [3] and discusses implications and extensions.

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2. Assumptions and Notation

Consistent with Choi and Enns [3] and Yang [4], the following assumptions and notation are adopted.

2.1. Notation (Parameters and Decision Variables)

A: the setup cost per lot.
C: the manufacturing cost per unit.
c_a: the coefficient of variation of lot interarrival times.
D: the throughput demand rate, with $0 < D < P$. D is a constant for the minimum-cost model, and a variable for the maximum-profit model.
H_{FG}: the holding-cost rate for finished goods.
H_{WIP}: the Holding-cost rate for work-in-process (WIP).
P: the processing rate.
Q: the lot size that is a decision variable.
τ: the setup time per lot.
$T_C(Q)$: the objective function for the minimum-cost model.
$T_P(Q, D)$: the objective function for the maximum-profit model.

2.2. Assumptions

(System) A single machine produces a single product; production occurs in lots of size Q>0.
(Arrivals) Lot interarrival times follow a generalized interarrival-time distribution with squared coefficient of variation $c_a^2 \geq 0$.
(Setup) Setup cost $A>0$ [$/lot] and setup time $\tau \geq 0$ [time] apply per lot.
(Rates) Processing rate $P > 0$ and throughput demand $D \in (0, P)$ [units/time] ensure stability.
(Holding costs) Finished-goods and work-in-process holding-cost rates satisfy H_{FG}, $H_{WIP} \geq 0$ [$/unit·time].
(Unit contribution) $C \geq 0$ denotes the unit contribution margin [$/unit].
(Objectives). $T_C(Q)$ and $T_P(Q, D)$ denote, respectively, the cost-minimization and profit-maximization objectives for the single-product case; unless stated otherwise, expectations are steady-state.

2.3. Abbreviations

$\mathrm{x(Q)} = Q[1 - (D/P)] - D\tau$, with $\mathrm{x(Q)} > 0$, used in Choi and Enns [3].
$\rho(Q) = (D\tau/Q) + (D/P)$, used in Choi and Enns [3]. (not the conventional utilization $\rho = (D/P)$).
$\Omega(Q) = C - (A/Q) - H_{WIP}[1 - (c_a^2/2)]\ [(Q/P) + \tau]$, with $\Omega(Q) > 0$.
$\Delta(Q) = (H_{WIP} c_a^2 Q^2 / 2\Omega(Q))[(Q/P) + \tau]$.
$D^{\#}(Q) = \left[Q - \sqrt{\Delta Q}\right] / [(Q/P) + \tau]$.
The parenthesization in $\Omega(Q)$, $\Delta(Q)$, and $D^{\#}(Q)$ follows from Yang [4] and is retained for dimensional consistency: Δ(Q) has units of unit s², so D#(Q) has units of units/time. These quantities are used later to state natural bounds and derive one-dimensional reformulations.

3. Methods

3.1. Background and Prior Formulations: Minimum-Cost Model

This section summarizes prior analyses of the single-product minimum-cost model using the notation and assumptions in Section 2. The purpose is to restate the reported formulations and claims without yet assessing validity; the corrected analysis is presented later.

3.1.1. Baseline Objective and First-Order Condition (Choi–Enns [3])

Choi and Enns [3] proposes two models for a capacity-constrained single machine producing a single product in lots under generalized interarrival-time assumptions: the minimum-cost model $T_C(Q)$ and the maximum-profit model $T_P(Q, D)$. Holding costs induced by queuing delays are included, and the optimum is sought numerically via heuristic search. The minimum-cost objective (cost per unit time) is given

$$T_C(Q) = \frac{AD}{Q} + \frac{H_{FG}Q}{2} + H_{WIP}D\left\{\frac{c_a^2(D/Q)(\tau+Q/P)^2}{2[1-(D/Q)(\tau+Q/P)]} + \left(\tau + \frac{Q}{P}\right)\right\}, (1)$$

and the derivative with respect to Q is reported

$$\frac{\partial T_C(Q)}{\partial Q} = \frac{-AD}{Q^2} + \frac{H_{FG}}{2} + H_{WIP}D\left[\frac{1}{P} - \frac{c_a^2\rho(\tau+Q/P)(\rho/Q-D/PQ)}{2(1-\rho)^2} + \frac{c_a^2\rho}{P(1-\rho)} - \frac{c_a^2\rho(\tau+Q/P)}{2Q(1-\rho)}\right], (2)$$

using the auxiliary quantity $\rho(Q) = D\tau/Q - D/P$. The standard stability condition $0 \le D < P$ applies; remaining symbols follow Section 2.

3.1.2. Yang's Re-Expression and Convexity Claim for (Q)

Building on Choi and Enns [3], Yang [4] introduces an auxiliary term and rewrites the objective in a compact form to derive derivative relations used to claim convexity of $T_C(Q)$. The notation is defined

$$x(Q) = Q[1-(D/P)] - D\tau, (3)$$

(called the "mean lot service time" in Choi and Enns [3]) and is used here purely as a notational device. Yang [4] then rewrites Equation (1) as

$$T_C(Q) = \frac{AD}{Q} + H_{FG}\frac{Q}{2} + H_{WIP}D\left\{\frac{c_a^2}{2D}\left(\frac{Q^2}{x} - 2Q + x\right) + \tau + \frac{Q}{P}\right\}. (4)$$

Because $x(Q)$ is affined in Q, $d^2x/dQ^2 = 0$. Using this fact, Yang [4] reports the second-derivative expression

$$\frac{d^2}{dQ^2}T_C(Q) = \frac{2AD}{Q^3} + \frac{H_{WIP}c_a^2}{2}\left[\frac{d^2}{dQ^2}\frac{Q^2}{x}\right], (5)$$

and concludes via direct calculation that

$$\frac{d^2}{dQ^2}T_C(Q) = \frac{2AD}{Q^3} + \frac{H_{WIP}c_a^2D^2\tau^2}{x^3} > 0. (6)$$

is strictly positive. Therefore, $T_C(Q)$ is claimed to be strictly convex on the domain of interest (in particular, when $x(Q) > 0$).

Remark. The identity $\frac{d^2}{dQ^2}\left(\frac{Q^2}{x}\right) = \frac{2(D\tau)^2}{x^3}$ follows directly from the quotient rule with $x(Q) = Q(1-D/P) - D\tau$ and dx/dQ=1− D/P. Stating this identity explicitly makes the step from Equation (5) to Equation (6) transparent.

3.1.3. Feasible Domain and Boundary Behavior (x(Q) > 0, Ω(Q) > 0)

To ensure positivity of the auxiliary term and to define a feasible domain, Yang [4] imposes x(Q) > 0, which yields a lower bound on Q. Together with the condition Ω(Q) > 0 (defined in Section 2 and used later in the profit model), this restricts the search from an initially unbounded domain to a constrained interval. Yang [4] then evaluates endpoint derivatives at the induced boundaries to characterize monotonic behavior.

Specifically, Yang [4] noted that the positivity of the "mean lot service time" implies

$$Q > \frac{DP\tau}{P-D} (7)$$

so the feasible domain changes from $0 < Q < \infty$ to $Q \in (Q_{min}, \infty)$ with $Q_{min} = DP\tau/(P-D)$. After establishing convexity for $T_C(Q)$, Yang [4] evaluated the first derivative at the two endpoints, $Q \to Q_{min}$ and $Q \to \infty$, and emphasized that convexity alone is insufficient to guarantee an interior optimizer. He therefore aimed to show

$$\lim_{Q\to DP\tau/(P-D)} dT_C(Q)/dQ < 0, (8)$$

and

$$\lim_{Q\to\infty} dT_C(Q)/dQ > 0, (9)$$

within the restricted domain.

An alternative expression for the derivative permits further equivalences under the imposed conditions. In particular, Yang [4] rewrote Equation (2) as

$$dT_C(Q)/dQ = \frac{H_{FG}}{2} + \frac{H_{WIP}D}{P} - \frac{AD}{Q^2} - \frac{H_{WIP}c_a^2}{2}\left[\frac{D}{P}\left(1-\frac{Q^2}{x^2}\right) + \left(\frac{Q}{x}-1\right)^2\right], \quad (10)$$

Recalling Equation (3), if $Q \to \infty$, then $Q/x \to P/(P-D)$; consequently, Yang [4] reported

$$\lim_{Q\to\infty}\left[\frac{D}{P}\left(1-\frac{Q^2}{x^2}\right) + \left(\frac{Q}{x}-1\right)^2\right] = \frac{D^2}{P(P-D)}. \quad (11)$$

and hence verifying $\lim_{Q\to\infty} dT_C(Q)/dQ > 0$ is equivalent to checking

$$\frac{H_{FG}}{2} + \frac{H_{WIP}D}{P} > \frac{H_{WIP}c_a^2 D^2}{2P(P-D)}. \quad (12)$$

Using the numerical example in Chou and Enns [3] with $H_{FG} = 0.2$, $H_{WIP} = 0.05$, $D = 10$, $P = 12$, and $c_a = 0.5$, Yang [4] computed

$$\frac{H_{FG}}{2} + \frac{H_{WIP}D}{P} \approx 0.14, \quad (13)$$

and

$$\frac{H_{WIP}c_a^2 D^2}{2P(P-D)} = \frac{5}{192} \approx 0.026. \quad (14)$$

Comparing Equations (13) and (14), the inequality in Equation (12) holds for the benchmark parameters. Together with Equation (8), Yang [4] concluded that $dT_C(Q)/dQ$ increases from a negative value at $Q \to Q_{min}$ to a positive value as $Q \to \infty$, implying a unique solution to $dT_C(Q)/dQ = 0$ on (Q_{min}, ∞); the corresponding theorem was then stated.

Note: The role of $\Omega(Q)$>0 is primarily associated with the profit model; it is listed here for completeness and forward reference. A detailed examination of the limit in Equation (11) and the equivalence in Equation (12) is deferred to Section 4, where a corrected analysis is provided.

3.1.4. Reported Theorems and Stated Limitations

Based on the convexity claim for $T_C(Q)$ and the endpoint evaluations in Equations (8) and (9), Yang [4] reported two theorems. The first addresses the regime in which the derivative changes sign on (Q_{min}, ∞); the second covers the complementary regime.

Theorem 1 (Yang [4])**.** If $\frac{H_{FG}}{H_{WIP}}\frac{P}{D} + 2 > \frac{D}{P-D}c_a^2$, there exists a unique solution Q^*∈(Q_{min}, ∞) to $dT_C(Q)/dQ = 0$, and this Q^* is the optimal lot size for the minimum-cost model. *Equivalently*, the above condition is the same as Equation (12) obtained by algebraic rearrangement.

Theorem 2 (Yang [4]). If $\frac{H_{FG}}{H_{WIP}}\frac{P}{D} + 2 \le \frac{D}{P-D}c_a^2$, then $dT_C(Q)/dQ < 0$ for all $Q > Q_{min} = \frac{DP\tau}{(P-D)}$. Consequently, $T_C(Q)$ is strictly decreasing on (Q_{min}, ∞) and attains its minimum as $Q \to \infty$. Since Q denotes a lot size, Yang [4] stated that the implementable solution is the largest feasible lot size under storage and capital-budget constraints.

Stated limitations. The reported conclusions depend on (i) the endpoint behavior in **Equations (8)** and **(9)** and (ii) the limit evaluation in **Equation (11)** used to derive **Equation (12)**. The derivation of **Equation (11)** contains a step that warrants closer scrutiny; Section 3.2 revisits this point and provides a corrected treatment. Accordingly, the reported uniqueness result is conditional on the stated inequality and the domain assumptions $x(Q)$ >0, where relevant later, $\Omega(Q) > 0$.

3.2. Corrected Optimality Analysis for the Minimum-cost Model

3.2.1. Sign-Corrected Limit as Q →∞

Re-examining the endpoint limit used in Equation (11), let $x(Q) = Q(1 - D/P) - D\tau$ and note that $Q/x \to P/(P-D)$ as $Q\to\infty$. Then

$$\lim_{Q\to\infty}\left[\frac{D}{P}\left(1-\frac{Q^2}{x^2}\right) + \left(\frac{Q}{x}-1\right)^2\right],$$

$$= \frac{D}{P}\left[1-\left(\frac{P}{P-D}\right)^2\right] + \left[\left(\frac{P}{P-D}\right)-1\right]^2,$$

$$= \frac{D}{P}\left(\frac{D^2-2DP}{(P-D)^2}\right) + \frac{D^2}{(P-D)^2}, \quad = \frac{D^2}{P(P-D)^2}[(D-2P)+P],$$

$$= \frac{-D^2}{P(P-D)}, \quad (15)$$

which differs in sign from the positive value reported in Equation (11).

3.2.2. Consequence for the Endpoint Derivative and Redundancy of the Extra Condition

Substituting Equation (15) into the alternative derivative representation of Equation (10) yields

$$\lim_{Q\to\infty} \frac{d}{dQ} T_C(Q) = \frac{H_{FG}}{2} + H_{WIP}\frac{D}{P} + H_{WIP}\frac{c_a^2 D^2}{2P(P-D)} > 0. \quad (16)$$

Hence, the positivity of the endpoint derivative at Q→∞ holds unconditionally (under standard parameter positivity), rendering the comparison in Equations (13), and (14) and the auxiliary inequality in Equation (12) unnecessary for establishing an interior optimizer. In particular, Equation (9) is automatically satisfied.

3.2.3. Existence and Uniqueness of the Interior Optimizer (Consolidated Theorem)

From Equation (6) (or the strengthened curvature in Equation (19)), $d^2T_C(Q)/dQ^2 > 0$ on the feasible domain, so $dT_C(Q)/dQ$ is strictly increasing. Together with $\lim_{Q\to Q_{\min}} dT_C(Q)/dQ < 0$ in Equation (8) and $\lim_{Q\to\infty} dT_C(Q)/dQ > 0$ in Equation (16), there exists a unique $Q^\# \in (Q_{\min}, \infty)$ such that $\frac{dT_C(Q)}{dQ}(Q^\#) = 0$, and $Q^\#$ minimizes $T_C(Q)$. Therefore, the case described by "Theorem 2" (monotone decrease with minimum at $Q\to\infty$) cannot occur under the corrected endpoint behavior; the two statements can be consolidated as follows.

Theorem A (consolidated). There exists a unique solution $Q^\# \in (Q_{\min}, \infty)$ to $dT_C(Q)/dQ = 0$; this $Q^\#$ is the optimal lot size for the single-product minimum-cost model.

Theorem A provides a consolidated statement of the model's structural properties.

3.3. Succinct Re-Expression and Direct Convexity Proof

A direct, notation-light re-expression of $T_C(Q)$ avoids introducing $x(Q)$ and yields the following equivalent form:

$$T_c(Q) = \frac{AD}{Q} + H_{FG}\frac{Q}{2} + H_{WIP}D\left(\tau + \frac{Q}{P}\right),$$
$$+H_{WIP}\frac{c_a^2 D^2}{2}\left[\frac{Q}{P(P-D)} + \frac{\tau(2P-D)}{(P-D)^2} + \frac{\tau^2 P^3}{(P-D)^2\left((P-D)Q - DP\tau\right)}\right], \quad (17)$$

so that

$$\frac{d}{dQ}\mathrm{T_C(Q)} = \frac{-AD}{Q^2} + \frac{H_{FG}}{2} + H_{WIP}\frac{\mathrm{D}}{\mathrm{P}} + H_{WIP}\frac{c_a^2 D^2}{2P(P-D)},$$
$$-H_{WIP}\frac{c_a^2 D^2}{2}\left[\frac{\tau^2 P^3}{(p-D)\left((P-D)Q - DP\tau\right)^2}\right], \quad (18)$$

and

$$\frac{d^2}{dQ^2}\mathrm{T_C(Q)} = \frac{2AD}{Q^3} + H_{WIP}\frac{c_a^2 D^2 \tau^2 P^3}{[(P-D)Q - DP\tau]^3} > 0. \quad (19)$$

Equations (17-19) provide as self-contained convexity proof and monotonicity characterization, demonstrating that introducing $x(Q)$ is not required for tractable analysis.

3.4. Prior Analysis and Variable Reduction of the Maximum-Profit Model $T_P(Q, D)$

This section restates the reported reduction of the maximum-profit problem to a one-variable program centered on the critical curve and finite Q-bounds. The Section 3.4 develops a simplified, structure-aware optimization route that leverages convexity and natural bounds to further reduce computational effort.

3.4.1. Baseline Maximum-profit Formulation and Notation (Choi–Enns [3])

This section outlines the reported analytic procedure for the maximum-profit model $T_P(Q, D)$. Choi and Enns [3] proposed, for a single machine producing a single product in lots under generalized interarrival time assumptions, the objective

$$T_P(Q,D) = CD - \frac{A}{Q}D - H_{FG}\frac{Q}{2} - H_{WIP}\left(\frac{Q}{P} + \tau\right)D \; - H_{WIP}\frac{c_a^2}{2}\left(\frac{Q^2}{x} - 2Q + x\right), \quad (20)$$

where $\mathrm{x = Q - [(Q/P) + \tau]D}$, with $\mathrm{x > 0}$, and C is the profit contribution multiplier.

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They reported first-order conditions with respect to Q and D, and sketched a graph suggesting concavity in (Q, D), but did not provide a complete analytic characterization of the optimizer.

3.4.2. First-Order Condition with Respect to D and Definition of Ω(Q)

Differentiating Equation (20) with respect to D, Yang [2] obtained

$\frac{\partial}{\partial D}T_P(Q,D) = \Omega(Q) - H_{WIP}\frac{c_a^2 Q^2\left(\frac{Q}{P}+\tau\right)}{2\left[Q-\left(\frac{Q}{P}+\tau\right)D\right]^2}$, (21)

with

$\Omega(Q) = C - (A/Q) - H_{WIP}[1-(c_a^2/2)]\ [(Q/P)+\tau]$, (22)

which serves as a key quantity in subsequent steps.

3.4.3. Δ(Q), the Critical Curve $D^{\#}(Q)$, and Positivity Conditions

To simplify expressions, Yang [4] defined

$\Delta(Q) = \frac{H_{WIP}c_a^2 Q^2}{2\Omega(Q)}\left(\frac{Q}{P}+\tau\right)$. (23)

and—under Ω(Q)>0—expressed the D-critical relation as

$D^{\#}(Q) = \frac{Q-\sqrt{\Delta Q}}{(Q/P)+\tau}$. (24)

Ensuring $D^{\#}(Q) > 0$, imposes

$Q - \sqrt{\Delta(Q)} > 0$. (25)

From the restriction of Equation (22), $\Omega(Q) > 0$, Yang [4] derived the following lemma.

Lemma 1 of Yang [2]. When $\alpha_1 > 0$ and $\alpha_3 > 0$, there is a lower bound and an upper bound, say Q_1 and Q_2 of Q to guarantee that $\Omega(Q) > 0$, with $\alpha_1 = C - H_{WIP}\left(1-\frac{c_a^2}{2}\right)\tau$, $\alpha_2 = (2-c_a^2)\frac{H_{WIP}}{P}$, $\alpha_3 = \alpha_1^2 - 2\alpha_2 A$, $Q_1 = \frac{\alpha_1-\sqrt{\alpha_3}}{\alpha_2}$, and $Q_2 = \frac{\alpha_1+\sqrt{\alpha_3}}{\alpha_2}$.

For the restriction of Equation (25) as $Q - \sqrt{\Delta(Q)} > 0$, Yang [4] opbtained the next lemma.

Lemma 2 of Yang [2]. When $\beta_1 > 0$ and $\beta_3 > 0$, there is a lower bound and an upper bound of Q, say Q_3 and Q_4, to guarantee that $D^{\#}(Q) > 0$, where $\beta_1 = C - H_{WIP}\tau$, $\beta_2 = 2\frac{H_{WIP}}{P}$, $\beta_3 = \beta_1^2 - 2\beta_2 A$, $Q_3 = \frac{\beta_1-\sqrt{\beta_3}}{\beta_2}$ and $Q_4 = \frac{\beta_1+\sqrt{\beta_3}}{\beta_2}$.

3.4.4. Bounds on Q (Lemma 1 and Lemma 2 of Yang [4])

From $\Omega(Q) > 0$ and $Q - \sqrt{\Delta(Q)} > 0$, Yang [4] stated two lemmas that provide two pairs: a lower and an upper bound for Q: With $\alpha_1 = C - H_{WIP}\left(1-\frac{c_a^2}{2}\right)\tau$, $\alpha_2 = (2-c_a^2)\frac{H_{WIP}}{P}$, $\alpha_3 = \alpha_1^2 - 2\alpha_2 A$, $Q_1 = \frac{\alpha_1-\sqrt{\alpha_3}}{\alpha_2}$, $Q_2 = \frac{\alpha_1+\sqrt{\alpha_3}}{\alpha_2}$, and with $\beta_1 = C - H_{WIP}\tau$, $\beta_2 = 2\frac{H_{WIP}}{P}$, $\beta_3 = \beta_1^2 - 2\beta_2 A$, $Q_3 = \frac{\beta_1-\sqrt{\beta_3}}{\beta_2}$ and $Q_4 = \frac{\beta_1+\sqrt{\beta_3}}{\beta_2}$.

When $\alpha_1 > 0$, and $\alpha_3 > 0$ and $\beta_1 > 0$, and $\beta_3 > 0$, the feasible interval for Q becomes

$max\{Q_1, Q_3\} < Q < min\{Q_2, Q_4\}$, (26)

to guarantee that $\Omega(Q) > 0$ and $D^{\#}(Q) > 0$.

3.4.5. Monotonicity in D and Reduction to $T_P(Q, D^{\#}(Q))$

Using Equations (21-25), Yang [4] showed that for fixed Q, $\frac{\partial}{\partial D}T_P > 0$, when $D < D^{\#}(Q)$ and $\frac{\partial}{\partial D}T_P < 0$ when $D > D^{\#}(Q)$, hence $D^{\#}(Q)$ is a local maximizer in D: if $D < D^{\#}(Q)$,

$\frac{\partial T_P(Q,D)}{\partial D} > 0$, (27)

and if $D > D^{\#}(Q)$,

$\frac{\partial T_P(Q,D)}{\partial D} < 0$. (28)

Therefore, the two-variable program reduces to a single-variable problem,

$T_p\left(Q, D^{\#}(Q)\right) = CD^{\#}(Q) - \frac{A}{Q}D^{\#}(Q) - H_{FG}\frac{Q}{2} - H_{WIP}\left(\frac{Q}{P}+\tau\right)D^{\#}(Q) - H_{WIP}\frac{c_a^2(Q-x)^2}{2x}$, (29)

where $x = Q - \left((Q/P)+\tau\right)D^{\#}(Q)$.

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3.4.6. Finite Search Domain and Numerical Considerations

Under Lemmas 1–2 of Yang [4], the search domain contracts from the infinite set
$0 < Q < \infty$, (30)
to the finite interval
$max\{Q_1, Q_3\} < Q < min\{Q_2, Q_4\}$. (31)
Yang did not quantify the number of stationary points of $T_p(Q, D^{\#}(Q))$ in Q, and regarded the problem as analytically intractable, thus recommending numerical search over Equation (31). For reference, the numerical data in Choi and Enns [3], and Yang [4] imply $\alpha_j > 0$, and $\beta_i > 0$ and validate the bounds Equation (26). In Section 3.4, structural results from the minimum-cost analysis is used to shrink the search in D from $\mathbf{0 \leq D0D}$ to the natural interval $\mathbf{0 \leq D < P}$ and to work with problem-native finite bounds that are tighter in practice (e.g., improving from $7 < Q < 2381$ to $0 \leq D < 12$).

3.4. Cost–Profit Decomposition and One-Dimensional Reformulation of the Maximum-Profit Model

This section provides a structure-aware simplification of Yang's analytical workflow for the maximum-profit model $T_P(Q, D)$. By comparing the profit objective in Equation (20) with the re-expression of the minimum-cost objective in Equation (4), it follows that
$T_p(Q, D) = CD - T_C(Q)$. (32)
Consequently, for fixed throughput demand D, maximizing $T_P(\cdot, D)$ over Q is equivalent to minimizing $T_C(\cdot)$, and
$\frac{\partial}{\partial Q} T_P(Q, D) = -\frac{d}{dQ} T_C(Q)$. (33)
Since $T_C(Q)$ is strictly convex and admits a unique interior minimizer on its feasible domain (Theorem A), $T_P(\cdot, D)$ is strictly concave in Q and thus possesses a unique maximizer $Q^*(D)$ for each feasible D. This reduces the original two-variable maximization to a one-variable program $\max_D T_P(Q^*(D), D)$. A natural upper bound for D follows from the feasibility condition $x(Q, D) = Q - [(Q/P) + \tau]D > 0$ (see Equation (20)), which implies
$D < \frac{Q}{(Q/P)+\tau} < \frac{Q}{Q/P} = P$. (34)
Hence, the search range for D contracts from the infinite half-line $D \geq 0$ to the natural finite interval $\mathbf{0 \leq D < P}$, consistent with the stability assumption stated earlier. In the numerical example of Choi and Enns [3], and Yang [4], this transforms a wide Q-interval into a compact D-interval, enabling an efficient one-dimensional search.

4. Results

4.1. Results

This section evaluates the solution procedure on the canonical benchmark used by Choi and Enns [3] and subsequently by Yang [4], adopting the same parameter set $(A, C, H_{FG}, H_{WIP}, c_a, \tau, P) = (70, 10, 0.2, 0.05, 0.5, 1, 12)$. The feasible demand range follows the natural bound $0 \leq D < P$ (see Equation (34)), i.e., $0 \leq D < 12$. A coarse-to-fine one-dimensional search over D is employed while, for each fixed D, the unique $Q^*(D)$ is obtained by solving $\frac{dT_C(Q)}{dQ} = 0$ (Equation (18)) with T_C as Equation (17); the profit is then evaluated as $T_p(Q^*(D), D)$ using Equation (20).

Stage 1 (coarse grid). Since the revenue term CD grows linearly in D while costs remain bounded by feasibility, a coarse screening over $3 \leq D \leq 11$ with step size 1 is adopted. For each $D \in \{3, 4, \ldots, 10, 11\}$, Equation (18) is solved to obtain $Q^*(D)$ and $T_p(Q^*(D), D)$ is computed via Equation (20). The coarse grid attains its maximum at
$T_p(Q^*(10), 10) = \max\{T_p(Q^*(D), D) : D \in \{3,4, \ldots, 10,11\}\}$. (35)
Stage 2 (refinement). The search interval is narrowed to $9 \leq D \leq 11$. Evaluations on a 0.1 grid, $D \in \{9, 9.1, \ldots, 11\}$, yield
$T_p(Q^*(9.7), 9.7) = \max\{T_p(Q^*(D), D) : D \in \{9, 9.1, \ldots, 11\}\}$. (36)

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Stage 3 (fine refinement). Focusing on $9.6 \le D \le 9.8$ with step size 0.01, i.e., $D \in \{9.6, 9.61, \dots, 9.8\}$, the maximizer is found at

$$T_p(Q^*(9.74), 9.74) = \max\{T_p(Q^*(D), D) : D \in \{9.6, 9.61, \dots, 9.8\}\}. \quad (37)$$

Table 1. Key results for the maximum-profit model.

Stage	D	$Q^*(D)$	$T_p(Q^*(D), D)$	Description
1	9	70.9725	68.1464	
2	9.6	82.3691	70.6476	
2	9.7	85.4614	70.7628	2nd stage best
3	9.73	86.4777	70.7722	
3	9.74	86.7260	70.7725	Optimal solution
3	9.75	87.1793	70.7713	
2	9.8	89.0211	70.7429	
1	10	97.7936	70.1817	1st stage best
1	11	214.4753	35.3898	

The final selection is therefore $D^* = 9.74$, $Q^*(9.74) = 86.7260$ and $T_p(Q^*(D^*), D^*) = 70.7725$, consistent with Yang's reported optimum on the same benchmark. Key evaluation results are summarized in Table 1 (values of D, the corresponding $Q^*(D)$, and $T_p(Q^*(D), D)$).

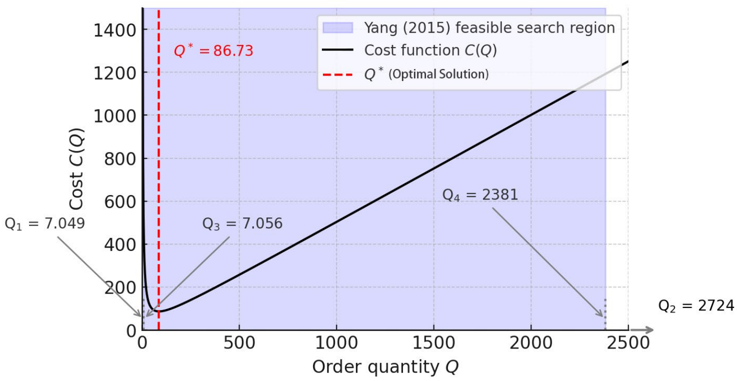

Figure 1. When $D = 9.74$, the minimum-cost.

Figure 1 depicts $T_C(Q)$ at $D = 9.74$ highlighting the unique minimizer $Q^*(D)$.

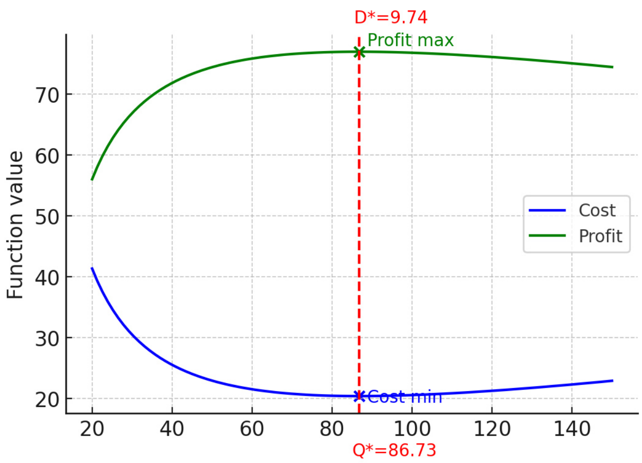

Figure 2. The relationship between the minimum-cost and maximum-profit.

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Figure 2 overlays $T_C(Q)$ and $T_P(Q, 9.74)$ to visualize how the minimum-cost solution maps to the maximum-profit optimum through the identity $T_P = C\ D - T_C$.

4.2. Search Space Reduction and Computational Implications

Yang's procedure operates over a wide *Q*-interval obtained from Lemmas 1–2,

$max\{Q_1, Q_3\} = 7.056 < Q < min\{Q_2, Q_4\} = 2381,$ (38)

where $Q_1 = 7.049$, $Q_2 = 2724$, $Q_3 = 7.056$, and $Q_4 = 2381$. In contrast, the present structure-aware routine searches over the natural finite demand interval $0 \le D < P = 12$, and, as demonstrated by the coarse-to-fine evaluation (Equations (35)–(37)), quickly concentrates near the maximizer. This reduction from a very broad *Q*-domain to a compact *D*-domain materially lowers the number of objective evaluations while preserving the optimal solution, thereby improving numerical efficiency and implementation robustness.

Comparing the feasible Q domain in Equation (38) (Yang [4]) with the natural feasible D-domain in the following,

$0 \le D < P = 12.$ (39)

shows that the proposed search domain is substantially tighter. This tightening reduces the number of objective evaluations required in practice while preserving the optimal solution.

5. Conclusions

This study develops a corrected optimality theory for single-machine, single-product inventory models under generalized interarrival-time assumptions. A sign--corrected endpoint analysis, combined with a succinct derivation, establishes strict convexity of the minimum-cost objective and certifies the existence and uniqueness of an interior optimizer without auxiliary feasibility constraints. These results consolidate previously separate statements into a single general theorem and provide portable optimality certificates across parameter regimes. Exploiting the identity $T_P(Q, D) = C\ D - T_C(Q; D)$, the maximum-profit problem reduces to a one-dimensional optimization in *D* over the natural finite interval $0 \le D < P$. The framework replaces paired bounds on an unbounded *Q*-domain with problem-native limits, attains the same optimum as prior work on the canonical benchmark, and does so with fewer evaluations, thereby clarifying the structural correspondence between cost minimization and profit maximization. The procedure is straightforward to implement: for each *D*, solve $\frac{dT_C(Q)}{dQ} = 0$ to obtain $Q^*(D)$; then perform a one-dimensional search over *D* over [0,*P*). In practice, this reduces search effort, improves numerical stability, and provides a transparent path from modeling assumptions to actionable decisions in capacity-constrained single-machine settings. Limitations and avenues for further work include extensions to multi-product and multi-machine systems; sequencing and setup-dependent environments; stochastic processing rates and non-stationary arrivals; robust or distributionally robust formulations under parameter uncertainty; and learning-based estimation of interarrival variability to inform the proposed bounds.

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