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Primordial Perturbations Including Second-Order Derivatives of the Inflationary Potential

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Abstract: In inflationary cosmology the form of the potential is still an open problem. In this work, second-order effects of the inflationary potential are evaluated and related to the known formula for the primordial perturbations in a wide range of scales. We found effects that may help to unravel the unknown inflationary potential form and impose new constraints on the parameters that define this potential. In particular, we demonstrate here that even slight deviations in the inflaton potential can lead to significant differences in the calculated spectra if inflation persists for sufficiently long and the normal modes of perturbations are affected by these variations.

Keywords: Primordial perturbations; Inflation; Inflationary potential; Cosmology.

1. Introduction

Inflationary models have been instrumental in resolving several critical puzzles of the original Big Bang theory, including the horizon, flatness, and the generation of primordial fluctuations necessary for structure formation [1,2]. Initially inspired by grand unified theories in particle physics these models propose a period of rapid exponential expansion of the universe, known as inflation, which stretches quantum fluctuations to astrophysical scales [2–5].

The quantum mechanical fluctuations of a scalar field, called the inflaton, give rise to perturbations that evolve under the influence of a specific potential. During inflation, these perturbations start at sub-Hubble scales, expand exponentially during the slow-roll inflation period, and eventually exit the Hubble horizon, laying the groundwork for forming cosmic structures observed today. Subsequently, as inflation ends, the inflaton field "reheats," generating the hot primordial soup of the Big Bang while losing its energy [6,7].

Following inflation, the universe's evolution transitions to the classical Friedmann-Robertson-Walker (FRW) cosmological model, further refined to include dark energy in the Λ-CDM (Cold Dark Matter plus cosmological constant) model [5]. The quantum fluctuations during inflation result in slight variations in the universe's mean density, known as primordial density perturbations. These fluctuations are imprinted in the cosmic microwave background radiation (CMB) and manifest as perturbations in its intensity and polarization.

Observations of the CMB, notably by satellites such as COBE, WMAP and PLANCK, have provided crucial insights into the universe's early evolution [8–12]. The Sachs-Wolf effect, for instance, allows the detection of perturbations in the CMB radiation, which are indicative of density fluctuations in the early universe.

Numerical simulations have demonstrated that density perturbations' predicted spectrum and amplitude are consistent with the observed cosmic structures formed over the universe's 14-billion-year history. Consequently, observations of the CMB, large-scale structure formation, and nucleosynthesis serve as constraints for theoretical models, particularly inflationary and Big Bang models.

Despite the successes of inflationary models, specific unresolved issues persist in cosmology, such as the transition of quantum fluctuations to classical overdensities. This transition from a

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symmetric quantum state to a classical non-symmetric one poses a fundamental challenge in physics [13]. Understanding this transition is crucial for accurately reconstructing the inflaton potential, as it influences parameters such as reheating temperature, primordial particle production, and perturbation amplitudes [14,15].

This paper addresses one such issue by incorporating second-order corrections to the inflation potential when calculating quantum fluctuations. Previous investigations often overlooked these corrections due to the stringent slow-roll conditions [16–18]. The modifications here concern the inclusion of the second derivative of the inflaton potential into normal modes, introducing a non-trivial dependence between the Fourier modes and the variations found in the inflaton potential (the energy that drives the inflation itself). Therefore, we demonstrate that even slight deviations from slow roll can lead to significant differences in the calculated spectra if inflation persists for sufficiently long. Some of the authors works with gravitational waves, that is the reason they are working in such subject [19].

2. Inflation

Inflation was a brief epoch of exponential expansion of the Universe. During this period the Universe accelerated as: $a(t) = e^{Ht}$; the expansion was superluminal and preceded the hot Big Bang [20]. Here $a(t)$ stands for the scale factor of the Universe, $H = H(\varphi)$ is the Hubble rate and t is the cosmic time. Inflation is described by the dominance of a scalar field with potential energy given by $V(\varphi)$. This potential dominates all forms of energy of that period and the effective pressure of this field must be negative. In these conditions, the scale factor accelerated as $a(t) = e^{Ht}$ in a brief period called Inflation.

The field that drives Inflation is called the inflaton field, φ, which must be a scalar field. There are some models with two fields (or more) called hybrid inflation. Although this field is mostly uniform, the Heisenberg uncertainty principle predicts quantum fluctuations. Therefore, some regions have different values of the scalar field φ and inflate at different rates. In turn, this phenomenon triggers perturbations in the background metric.

To understand how the quantum perturbations arise, consider the cosmological horizon in a quasi-de Sitter spacetime, which is given by the distance (R) that light can cross in comoving coordinates since the beginning of the expansion:

$$R = \int_{t_e}^{t_0} \frac{cdt}{a(t)} = \int_0^{r_e} \frac{dr}{\sqrt{1 - Kr^2}} = c/H\,, \tag{1}$$

where t_e and t_0 are the times of the beginning and the end of inflationary expansion, respectively, while r_e yields the size of this patch from zero to when inflation finishes.

The presence of this horizon produces thermal radiation, similar to Hawking radiation in a black hole, but with the black hole region turned inside out. The cosmic horizon restricts the modes of the zero-point quantum fluctuations like in the Casimir effect [21]. Consequently, the Heisenberg uncertainty principle implies quantum fluctuations in the momentum of the field, which satisfies the quantum inequality [18]

$$\Delta p \geq \hbar \frac{H}{c}\,, \tag{2}$$

resulting in a low cutoff for the momentum (it is the direct application of the uncertainty principle into Equation (1)).

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On the other hand, the cosmic horizon has a Hawking temperature [3,18] given by: [1]

$$\delta\varphi = k_B T = \frac{\hbar H}{2\pi} = \frac{H}{2\pi}\,. \tag{3}$$

These perturbations were created during inflation and were stretched out of the causal horizon while the horizon remained stationary, [18,20]. The fluctuation becomes frozen as it crosses the horizon and, presumably, converts to a classical perturbation.

After inflation, the cosmic expansion continues at subluminal velocities, and the fluctuations return to the causal horizon. Therefore, the most essential perturbations for structure formation arise from the fluctuations excited near the end of inflation [20]. The power spectrum of the perturbations $\delta_H^2(k)$ are evaluated when they cross the horizon and become frozen from that moment on. This corresponds to $k = aH$, where k is the comoving wave number of a perturbation's mode. The power spectrum $\delta_H^2(k)$ is related to the Friedmann equations, which are given by [18]:

$$H^2(\varphi) = \frac{8\pi}{3{M_{pl}}^2}V(\varphi)\,, \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3{M_{pl}}^2}(\rho + 3P)\,. \tag{5}$$

The equation of state for the inflaton is given by [18]:

$$\rho(\varphi) = \frac{\dot{\varphi}^2}{2} + V(\varphi)\,, \tag{6}$$

and

$$P(\varphi) = \frac{\dot{\varphi}^2}{2} - V(\varphi)\,, \tag{7}$$

where $P(\varphi)$ is the inflaton's pressure and $\varrho(\varphi)$ is its energy density. The first term on the RHS of Equation (6) is the kinetic energy of the field. Inflation occurs when the potential energy density $V(\varphi)$ is the dominant form of energy, i.e.: $V(\varphi) \gg \dot{\varphi}^2/2$. The equation that completes the system is the Klein-Gordon equation for the inflaton [18]

$$\ddot{\varphi} + 3H\dot{\varphi} = -V'(\varphi)\,. \tag{8}$$

The slow roll condition is expressed as: $\ddot{\varphi} \ll 3H\dot{\varphi}$. It holds during inflation when the scale factor grows exponentially and the kinetic energy of the field is negligible [18].

We will apply these equations and modify the usual quantum field theory estimations for the fluctuations using novel assumptions, as follows.

3. A First-Order Estimate in $V'(\varphi)$

The amplitude of the fluctuations is found considering a classical background φ_c plus quantum fluctuations $\delta(\varphi)$ satisfying $\varphi(t) = \varphi_c(t) + \delta(\varphi)$. The resulting system of equations are decoupled if we consider that $\delta\varphi \ll \varphi_c(t)$. Moreover, the perturbation yields a local change in the expansion rate $H(\varphi)$ proportional to

$$\delta_H(k) = \frac{\delta\varrho}{\varrho} \sim -H\delta t\,. \tag{9}$$

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[1] From now on, we consider the natural system of units, in which: $c = \hbar = k_B = 1$ and the gravitational constant will be: $G = 1/M_{pl}^2$, implying $T = H/2\pi$, with $M_{pl} = 1.22\times10^{19}$ GeV; the Planck time and Planck length are given by: $T_{pl} = 0.53\times10^{-45}$ s and $L_{pl} = 1/M_{pl} = 1.6\times10^{-33}$ cm, respectively. The Hubble parameter has energy units, $[H] = GeV$, and the energy density of the scalar field is measured in GeV^4.

Here we consider that $t \sim H^{-1}$, and taking $\delta t = -H^{-2}\,\delta H$ this leads us to $H\delta t = -\delta H/H$. Now, considering that $H^2 \propto \varrho$, then Equation (7) leads us to Equation (15) above. Therefore, we can write $\delta t = \delta\varphi/\dot{\varphi}$. With the above definitions, Linde (and other authors) obtained [18]

$$\delta_H(k) \sim \frac{CV^{3/2}(\varphi)}{{M_{pl}}^3 V'(\varphi)}\,, \tag{10}$$

where C is close to ten and $V'(\varphi)$ means the first derivative of the potential relative to the scalar field itself. This quantity is the most essential ingredient of inflation, since it predicts the seeds of posterior structure formation when matter and radiation fall into the gravitational wells caused by these perturbations. The regions with larger scalar field values inflate rapidly and produce enough density contrast to trigger structure formation.

In the case of the massive scalar field with potential $V(\phi) = m^2\varphi^2/2$, Equation (10) yields the following expression for the density contrast produced by quantum fluctuations:

$$\left(\frac{\delta\varrho}{\varrho}\right) \sim \frac{Cm\varphi^2}{M_{pl}^3}\,. \tag{11}$$

This equation and the observational constraint $\delta\varrho/\varrho \sim 10^{-5}$ leads to an upper bound for the amplitude of the field: $(\varphi/m)^2 < 10^{-5}(M_{pl}/m)$. Similar considerations may be applied to other inflationary models.

Let us investigate in the next section the expressions that lead us to these formulae above, considering the expected expansion of fluctuations in the frequency domain, and comparing them with V''.

4. Primordial Fluctuations and the Inflaton Potential

4.1. Conventional Calculation of Perturbations

In this section, we consider quantum vacuum fluctuations. Here, for clarity, we will repeat the usual derivation of the quantum fluctuation of the scalar field, since in the next section we will modify this deduction. We assume, as usual, that the spacetime is homogeneous and isotropic, and described by the FRW equations.

According to quantum field theory, the quantum vacuum is composed of particles and antiparticles of all wavelengths. These wavelengths are stretched during inflation until they reach astrophysical scales [4,20].

The scalar field, φ, can be split in its unperturbed component (the mean value of the field) plus a small perturbation $\delta\varphi(\vec{x},t)$, in the form:

$$\varphi(\vec{x},t) = \varphi_c(t) + \delta\varphi(\vec{x},t)\,. \tag{12}$$

As the fluctuation amplitude is small ($\delta\varphi << \varphi_c(t)$), the background spacetime is undisturbed, and the equation of motion for the field can be decoupled into two equations (inserting Equation (12) into Equation (8)):

$$\ddot{\varphi}_c(t) + 3H\dot{\varphi}_c(t) = -V'(\varphi_c(t))\,, \tag{13}$$

and

$$\delta\ddot{\varphi} + 3H(\varphi_c(t))\delta\dot{\varphi} - \nabla^2\delta\varphi = -V''(\varphi_c)\delta\varphi\,. \tag{14}$$

The first equation describes the evolution of the mean value of the field, while the second equation describes the evolution of the perturbations. We also assume $\delta\varphi << H(\varphi_c)/H'(\varphi_c)$.

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In comoving coordinates the wavenumber k is related to the physical one by $k = a(t)k_f$. Let us consider the Fourier expansion in normal modes $e^{i\,k_f{}^\mu x_\mu}$ for the perturbations.

If k_f^μ is the physical wavenumber, it is easy to show that the phase of these waves is invariant under the expansion:

$$e^{ik^\mu x_\mu/a(t)} = const\,. \tag{15}$$

Substituting it into expression Equation (14), we have a differential equation for each k-mode:

$$\delta\ddot{\varphi}_k + 3H(\varphi)\delta\dot{\varphi}_k + [(k/a)^2 + V''(\varphi)]\delta\varphi_k = 0\,. \tag{16}$$

First, we will solve Equation (16) as usual, with the approximation $(k/a)^2 \gg V''(\varphi)$, which is reasonable for short wavelengths. Let us expand the field fluctuation $\delta\varphi$ as a sum of creation and destruction operators, whose modes are particular solutions of Equation (16)

$$\delta\varphi = \sum_k \chi_k(\eta)A_k + \chi_k^*(\eta)A_k^\dagger\,, \tag{17}$$

where A_k and $A_k^\dagger$ are the destruction and creation operators, respectively, $\chi_k(\eta)$ are the vibration modes, and η is the conformal time. Inserting this solution into Equation (16) we obtain an equation for the modes:

$$\ddot{\chi}_k(t) + 3H\chi_k(t) + (k/a)^2\chi_k(t) = 0\,, \tag{18}$$

whose physical solution is given by

$$\chi_k(\eta) = \frac{a^{-3/2}}{\sqrt{2k/a}}\left[1 + \frac{iaH}{k}\right]e^{-ik/aH}\,. \tag{19}$$

The expectation value of the k-mode's quadratic amplitude is:

$$< 0|(\delta\varphi)_k{}^2|0 >= |\chi_k(\eta)|^2\,, \tag{20}$$

where

$$|\chi_k(\eta)|^2 = \frac{H^2}{2\,k^3}\left[1 + (k/aH)^2\right]. \tag{21}$$

We obtain the total fluctuation integrating over the total phase space volume:

$$< 0|(\delta\varphi)^2|0 >= \frac{1}{(2\pi)^3}\int d^3k\,< 0|(\delta\varphi_k)^2|0 >\,, \tag{22}$$

with $d^3k = 4\,\pi\,k^2\,dk$. Employing Equation (19), and manipulating some terms, we obtain:

$$< 0|(\delta\varphi)^2|0 >= \frac{H^2}{4\pi^2}\int_0^{Ht}(1 + k/H^2)\,d\ln(k/H)\,, \tag{23}$$

where k is the moving wavenumber. Evaluating the integral above and considering the instant when the perturbation crosses the Hubble horizon as $t = H^{-1}$, the integral results in:

$$< 0|(\delta\varphi)^2|0 >= \frac{7\,H(\varphi)^2}{12\,\pi^2}\,. \tag{24}$$

Note that $< 0|(\delta\varphi)|0 >= 0$, as it can be easily verified. Taking the square root of this value, we obtain:

$$\delta\varphi \sim \frac{H}{2\pi}\,. \tag{25}$$

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Substituting into Equation (9), we obtain the density contrast produced by the quantum fluctuations:

$$\left(\frac{\delta\varrho}{\varrho}\right) \sim \frac{H^2(\varphi)}{2\pi\dot{\varphi}_c(t)} . \tag{26}$$

The results presented in this subsection are the usual textbook's treatment of the perturbations. In the next subsection, we will incorporate the term $V''(\varphi)$ into the relevant expressions.

4.2. Including Second Order Variations

The second order derivative of the potential $V''(\varphi)$ is usually discarded from the calculations in the previous subsection due to the slow-roll conditions [4]. However, it is straightforward to note that this approximation does not always hold, since we cannot ignore the curvature of the potential for any wavenumber k. If we ignore this term, then Equation (16) becomes:

$$\delta\ddot{\varphi}_k + 3H(\varphi)\delta\dot{\varphi}_k + [(k/a)^2]\delta\varphi_k = 0 .$$

Nevertheless, in Equation (16), both terms within the brackets must be retained because, as inflation progresses, the first term evolves and is exponentially suppressed with time, making it problematic to neglect the term involving the second derivatives of the potential throughout the entire inflationary period.

Now, we perform the following variable change in Equation (16):

$$k_{eff}(t) = \sqrt{k^2 + a^2(t)V''(\varphi_c)} . \tag{27}$$

Using this equation, the square power of the fluctuations takes the form:

$$< 0|(\delta\varphi)^2|0 >= \frac{1}{(2\pi)^3}\int \frac{d^3k}{k_{eff}}\left[\frac{1}{2} + \frac{H^2}{2\,k_{eff}^2}\right] . \tag{28}$$

The first term in the integral above is the usual fluctuation in the Minkowsky vacuum (when $H = 0$). However, the second term is linked to inflation. This term arises because the de Sitter space contains particles φ with occupation numbers:

$$n_k = \frac{H^2}{2k^2} . \tag{29}$$

We see that the occupation numbers will receive contributions from the second-order variations of the inflation potentials, through the effective wavenumber $k_{eff}(\varphi)$. The volume element d^3k is not modified. Therefore, if we discard the renormalizable first term, then Equation (28) is rewritten as:

$$< 0|(\delta\varphi)^2|0 >= \frac{H^2}{4\pi^2}\int \frac{dk\,k^2}{k_{eff}(\varphi)^3} , \tag{30}$$

which, in its final form, is:

$$< 0|(\delta\varphi)^2|0 >= \frac{H^2}{4\pi^2}\int_0^{Ht} \frac{d\ln(k/H)}{[1 + a^2(t)V''/k^2]^{3/2}} . \tag{31}$$

In this equation we will consider the cases: $a^2(t)V''/k^2 \ll 1$ and $a^2(t)V''/k^2 \gg 1$.

4.2.1. First Case: $a^2\, V''(\varphi_c) \ll k^2$

Expanding the denominator of Equation (31) as a power series while taking the first two terms and considering one period of expansion $t \sim H^{-1}$ we obtain:

$$< 0|(\delta\varphi)^2|0 >= \frac{H^2}{4\pi^2}\left[1 - \frac{3(1-e^{-2})}{4}\frac{V''(\varphi)}{H^2(\varphi)}\right]. \quad (32)$$

Taking the square root power, substituting into Equation (30) and taking into account the slow-roll condition, $\dot{\varphi}_c = -V'(\varphi)/3H(\varphi)$, we have for the density contrast:

$$\left(\frac{\delta\varrho}{\varrho}\right) \sim \frac{CV^{3/2}(\varphi)}{M_{pl}^3 V'(\varphi)}\left[1 + \frac{3V''(\varphi)}{8H^2(\varphi)} + ...\right], \quad (33)$$

where $C = (3/2\pi)\,(8\pi/3)^{3/2} \sim 11.6$ is a normalization constant [18]. The usual Equation (10) holds when $V''(\varphi) \ll H^2(\varphi)$.

4.2.2. Second Case: $a^2\, V''(\varphi_c) \gg k^2$

We write the effective comoving wavenumber as:

$$k_{eff} = a(t)\sqrt{(k/a)^2 + V''(\varphi)}\,. \quad (34)$$

Since $V''(\varphi)$ dominates the first term within the square root, we obtain:

$$< 0|(\delta\varphi)^2|0 >= \frac{H^2(\varphi)}{16\pi^3}a(t)^{-3}\int \frac{d^3k}{V''(\varphi)^{3/2}}\,. \quad (35)$$

The comoving wavenumber ranges within the interval:

$$H < k < He^{Ht} \quad (36)$$

because inflation starts at $a = 1$ and ends at $a(t) = e^{Ht}$. Therefore, from Equation (35) we find:

$$< 0|(\delta\varphi)^2|0 >= \left(\frac{H}{2\pi}\right)^2 \frac{a(t)^{-3}H^3}{3V''(\varphi)^{3/2}}\left[e^{3Ht} - 1\right]. \quad (37)$$

Since $Ht \gg 1$, we obtain the result:

$$\Delta_\varphi = \sqrt{< 0|(\delta\varphi)^2|0 > - < 0|(\delta\varphi)|0 >^2} \sim \frac{H(\varphi)}{2\pi}\sqrt{\frac{H^3(\varphi)}{3V''(\varphi)^{3/2}}}\,. \quad (38)$$

Therefore, for very long wavelengths the fluctuations are corrected according to the density contrast

$$\left(\frac{\delta\varrho}{\varrho}\right)_{\lambda>\lambda_*} \sim \frac{CV^{3/2}(\varphi)}{{M_{pl}}^3 V'(\varphi)}\sqrt{\frac{H^3(\varphi)}{3V''(\varphi)^{3/2}}}\,. \quad (39)$$

Next, we must find the conditions for the wavelength cut-off λ_c.

4.2.3. Wavelength Cut-Off

The comoving k value is constant, while the k cut-off is given by the expression:

$$a^2(t)V''(\varphi) = k^2 = a^2(t){k_f}^2\,. \quad (40)$$

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The physical wave number is $k_f = 2\pi/\lambda_*$. Consequently, the critical wavelength is

$$\lambda_*(t) = \frac{2\pi}{\sqrt{V''(\varphi_c(t))}}. \tag{41}$$

This equation shows that the k_f cut-off depends on the $V(\varphi)$ profile as well as on the end time.

5. Applications

We will analyze chaotic inflationary models [22] under the perspective developed above or [23–25].

5.1. First Model

Consider the potential defined by:

$$V(\varphi) = \frac{m^2}{2}\varphi^2. \tag{42}$$

In this case, we find the following critical value:

$$\lambda_* = \frac{2\pi}{m} \sim 10^{-32}\,(m/M_{pl})^{-1}\,\mathrm{cm}. \tag{43}$$

Therefore, unless the mass of the inflaton is negligible in Planck units, almost all the wavelengths enter into the second case (Eq. 39):

$$\left(\frac{\delta\varrho}{\varrho}\right) \sim \frac{CV^{3/2}(\varphi)}{M_{pl}{}^3 V'(\varphi)}\sqrt{\frac{H^3(\varphi)}{3V''(\varphi)^{3/2}}}. \tag{44}$$

Therefore, the corrected Equation (34) must be applied.

If we impose that

$$\left(\frac{\delta\varrho}{\varrho}\right) < 10^{-5},$$

(the upper bound from observations of the CMB), it yields a new constraint between the field and mass:

$$\Gamma\varphi^{7/2} \sim m^2 M_{pl}^{3/2} 10^{-5}, \tag{45}$$

with: $\Gamma = \frac{10}{2^{3/2}}\sqrt{\frac{(4\pi/3)^{3/2}}{3}} \sim 5.98$. Note that this constraint is very different from:

$$\varphi^2 \sim 10^{-6}\frac{M_{pl}^3}{m}, \tag{46}$$

as derived from Equation (10), the expression used in previous analyses. Therefore, our findings present a constraint on the initial inflation conditions that is new, highlighting its significance.

5.2. Second Model

We now analyze the case with auto-coupling:

$$V(\varphi) = \frac{\zeta}{4}\varphi^4. \tag{47}$$

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For this potential $V''(\varphi) = \zeta\varphi^2$, implying that the field is time-dependent. In this case Eqs. (7) and (12) become a coupled system:

$$H^2 = \frac{8\pi\zeta}{12M_{pl}^2}\varphi^4 \,, \tag{48}$$

and

$$3H\dot{\varphi} = -\zeta\varphi^3 \,. \tag{49}$$

From these two equations a first-order differential equation results:

$$3D\dot{\varphi} = -\zeta\varphi \,, \tag{50}$$

with $D = \sqrt{8\pi\zeta/12\,M_{pl}^2}$, whose solution is:

$$\varphi(t) = \varphi_i\, e^{-\frac{\zeta}{3D}t} \,. \tag{51}$$

Therefore, the cut-off evolves in time as:

$$\lambda_*(t) = \frac{2\pi}{\varphi_i\,\sqrt{3\,\zeta}}\, e^{\frac{\zeta}{3D}t} \,. \tag{52}$$

Assuming a typical value $\zeta \sim 10^{-12}$ [18], the time scale is $\tau = \frac{1}{M_{pl}}\sqrt{\frac{8\pi}{12\,\zeta}} \sim 10^{-36}\,s$.As time progresses, all wavelengths eventually fall under the first case, making Equation (10) the accurate expression for the perturbation's amplitude.

6. Conclusions

In this paper we recalled inflation theory, focusing on its standard formulation and its implications for quantum fluctuations of the scalar field. By incorporating second-order derivatives of the inflaton potential energy into the equation governing these fluctuations, we have extended the theoretical framework to account for more detailed previously overlooked corrections. We found that fluctuations with wavelengths much larger than λ_* are described by Equation (39), while perturbations follow Equation (33) in the near-critical case. Both expressions differ from the usually adopted Equation (10).

Our analysis was applied to chaotic inflation scenarios [17,22] and revealed novel intriguing insights. In the first application, we found that the conventional formulae for perturbations need to be revised when considering potentials with the form $m\,\varphi^2/2$. In particular, the constraint between the mass and the initial value of the scalar field is very different from other previous analyses [17,22] when we use the amplitude of the fluctuation as an upper bound.

For potentials like $\zeta\,\varphi^4/4$ we observed a critical dependence on the duration of the inflationary process. This dependency arises due to the need to compare perturbation modes to a time-dependent critical wavelength, λ_*. Fluctuations with wavelengths much smaller than the critical value λ_* exhibit a dependence on the potential and its derivatives as described by the usual Equation (10). Conversely, fluctuations with wavelengths much larger than λ_* are characterized by Equation (39). The predicted perturbations align with Equation (33) in the near-critical case.

The physical origin of the second-order contributions manifested in Eqs. (33) and (39) can be explained by the presence of the expansion rate, H, in those expressions. Indeed, when the subtle variations of the potential (V'') are comparable to the universe's expansion rate, then corrections to the conventional expression for the density contrast become relevant. This feature reinforces the importance of the mathematical form chosen for the potential. Physically, it displays a relevant interplay between the rate at which the universe expands and variations in the changes of the inflationary potential relative to the inflaton field.

Our inclusion of second-order corrections in the quantum fluctuation expression has unveiled a significant deviation from the traditional formulations: fluctuations with wavelengths much larger

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than λ_* will cross the horizon later, therefore their implications may be relevant in the distant future of our local Universe. This effect may be responsible for triggering the collapse of our region if its magnitude suffices.

The Equation (10) is the usual, without corrections from the variations in $V'(\varphi)$. This expression leads us to the known constraints obtained from the comparisons with CMB, as happens for the upper bound 10^{-5} for the amplitude. In the second case investigated in this work, we found deviations in amplitude that implied new constraints for the constants that were originally found applying the previous one. The intermediate case (related to Equation (10)) requires a numerical analysis.

These findings hold promise for refining our understanding of the inflaton potential and could be leveraged to fine-tune the inflaton potential using satellite data. Indeed, this avenue presents an exciting opportunity for future research, which we intend to pursue diligently.

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