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Article

New Classes and Stability Analysis of Deviation Tensors in Interdimensional Weak Null-Preserving Maps

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Abstract

We extend the theory of interdimensional weak null-preserving maps by providing a complete local classification of the deviation tensor T based on its rank and kernel. We define weak k -plane null-preserving maps, examine their canonical decomposition, and analyze the local stability of T under small perturbations. Explicit examples illustrate the new classes of local behaviors. These results offer a rigorous and original contribution to the study of null structures in pseudo-Riemannian geometry.

Keywords: interdimensional maps; weak null-preserving; deviation tensor; k -plane preservation; Pseudo-Riemannian geometry

1. Introduction

The study of null structures in pseudo-Riemannian geometry provides fundamental insight into the causal relationships and local geometric rigidity of manifolds [5,8,10]. Weak null-preserving maps offer a minimal yet powerful framework to investigate how null directions are preserved under smooth mappings without requiring full conformality or isometry [2–4]. These maps have applications in mathematics and theoretical physics, for instance in embedding lower-dimensional spacetimes into higher-dimensional models, analyzing causal projections, and studying geometric constraints arising from Lorentzian structures [6,7,14].

Classical results mostly focused on equidimensional mappings, showing that continuous causal-preserving maps are essentially conformal [2] and that the class of continuous timelike curves determines the topology of spacetime [3]. Hawking, King, and McCarthy [4] introduced alternative topologies compatible with causal structures, and later studies [9,11–13] extended these ideas to more general pseudo-Riemannian manifolds, examining differentiability, null geodesic structures, and causal embeddings.

Recently in [1], we introduced interdimensional weak null-preserving maps $F : M^n \rightarrow N^m$, where the source and target Pseudo-Riemannian Manifolds have different dimensions. In this framework, the deviation tensor T quantifies the failure of a map to be conformal along null directions, and a canonical decomposition describes T locally. While that work established existence and basic properties, it did not provide a systematic classification of T nor study its behavior under perturbations, leaving open questions regarding the structure of T and its robustness under smooth deformations.

In this paper, we provide some extensions of the theory. We classify the deviation tensor T in terms of rank and kernel structure for interdimensional weak null-preserving maps, introduce weak k -plane null-preserving maps that preserve entire k -dimensional null subspaces, and analyze the local stability of T under small C^2 perturbations of the mapping. This approach not only extends the previous work by offering a complete local description of T , but also introduces new mathematical structures and constraints that were absent in the earlier study, demonstrating both the reduction of maximal rank under k -plane preservation and the quantitative bounds on stability.

These results deepen the understanding of interdimensional causal mappings, provide new tools for studying the interplay between null structures and mappings of different dimensions, and lay the

foundation for future investigations in geometric analysis, Lorentzian geometry, and applications in theoretical physics [15].

We recall definitions from [1].

Definition 1. Let (M, g) be a pseudo-Riemannian manifold of dimension $n \geq 3$. The null cone at $p \in M$ is

$$N_p := \{v \in T_p M \setminus \{0\} \mid g_p(v, v) = 0\}.$$

Definition 2. Let (M^n, g_M) and (N^m, g_N) be pseudo-Riemannian manifolds. A C^2 map $F : M^n \rightarrow N^m$ is interdimensional weak null-preserving if:

$$dF_p(N_p^M) \subset N_{F(p)}^N, \quad \forall p \in M^n.$$

Definition 3. For a weak null-preserving map F , the deviation tensor T is defined locally by:

$$T := F^* g_N - \Omega^2 g_M,$$

where Ω is a smooth positive function determined by the canonical decomposition.

2. Classification of Deviation Tensor

In this section, we provide a complete local classification of the deviation tensor T according to its rank and kernel structure.

Theorem 1. Let $F : M^n \rightarrow N^m$ be an interdimensional weak null-preserving map with $n > m$, and let T be its deviation tensor. Then, for each point $p \in M^n$, T can be locally decomposed as

$$T = \sum_{i=1}^r \lambda_i \theta_i \otimes \theta_i,$$

where $\lambda_i \in \mathbb{R}$, θ_i are 1-forms vanishing on $\ker(dF_p)$, and $0 \leq r \leq n - m$. Moreover:

1. $\text{rank}_p(T) = r$.
2. $\ker(T) \supset N_p M \cap \ker(dF_p)$.
3. For $r = 0$, T vanishes along all null directions tangent to the kernel.

Proof. Fix a point $p \in M^n$. Since $dF_p : T_p M^n \rightarrow T_{F(p)} N^m$ has rank at most m , it follows that

$$\dim \ker(dF_p) \geq n - m.$$

Let $\{v_1, \dots, v_{n-m}\}$ be a maximal set of linearly independent null vectors in $\ker(dF_p)$. By the definition of the deviation tensor T , we have $T(k, k) = 0$ for all null vectors $k \in N_p M$, which implies

$$T(v_i, \cdot) = 0, \quad i = 1, \dots, n - m.$$

Next, choose a complementary subspace $W \subset T_p M^n$ such that

$$T_p M^n = \ker(dF_p) \oplus W.$$

On W , $T|_W$ is a symmetric bilinear form on a pseudo-Riemannian space of dimension at most m . By the spectral theorem for symmetric tensors, there exist scalars $\{\lambda_i\}_{i=1}^r$ and 1-forms $\{\theta_i\}_{i=1}^r$ on W , with $r \leq \dim W \leq n - m$, such that

$$T|_W = \sum_{i=1}^r \lambda_i \theta_i \otimes \theta_i.$$

Extending each θ_i to $T_p M^n$ by setting $\theta_i(v) = 0$ for all $v \in \ker(dF_p)$, we obtain the decomposition

$$T = \sum_{i=1}^r \lambda_i \theta_i \otimes \theta_i$$

over the full tangent space.

From this construction, it follows that

$$\ker(T) \supset N_p M \cap \ker(dF_p) \quad \text{and} \quad \text{rank}_p(T) = r \leq n - m.$$

In the special case $r = 0$, T vanishes along all null directions tangent to the kernel, completing the classification. $\square$

Corollary 1.

$r = 0$: T is entirely degenerate; F is conformal along null directions tangent to $\ker(dF_p)$.

$r = n - m$: T has maximal deviation; any null vector outside $\ker(dF_p)$ contributes non-trivially.

3. Weak k -Plane Null-Preserving Maps

Definition 4. $F : M^n \rightarrow N^m$ is weak k -plane null-preserving if for any $p \in M^n$, every k -dimensional null subspace $\Pi \subset T_p M$ satisfies:

$$dF_p(\Pi) \subset N_{F(p)}^N.$$

Proposition 1. Every weak k -plane null-preserving map is weak null-preserving. Conversely, if $k < n - m$, weak k -plane preservation imposes stricter constraints on the deviation tensor T , reducing its possible rank.

Proof. Let $F : M^n \rightarrow N^m$ be a weak k -plane null-preserving map. By definition, for each point $p \in M^n$ and every k -dimensional null subspace $V_p \subset T_p M^n$, the differential maps the subspace into a null subspace of $T_{F(p)} N^m$:

$$dF_p(V_p) \subset N_{F(p)}^N.$$

Since V_p is spanned by k linearly independent null vectors $\{v_1, \dots, v_k\}$, the image under dF_p of each v_i is null in $T_{F(p)} N^m$, i.e.,

$$g_N(dF_p(v_i), dF_p(v_i)) = 0, \quad i = 1, \dots, k.$$

This immediately implies that F preserves each individual null direction contained in V_p . Because this holds for all k -dimensional null subspaces at p , and all points $p \in M^n$, we conclude that F is weak null-preserving.

Conversely, suppose $k < n - m$. Preservation of every k -dimensional null subspace V_p imposes additional linear constraints on the deviation tensor T . Indeed, for each V_p , the vectors in the image $dF_p(V_p)$ are null, so T must vanish on all vectors in the span of $dF_p(V_p)$. Denote the kernel of T at p by:

$$\ker(T_p) \supset \bigcup_{V_p} dF_p(V_p),$$

where the union is over all k -dimensional null subspaces of $T_p M^n$.

As a consequence, the maximal rank of T is reduced relative to the general weak null-preserving case, because its kernel now contains additional directions corresponding to these k -planes. Hence, weak k -plane null-preservation restricts the possible rank of T further than general weak null-preservation. $\square$

4. Stability Analysis under Perturbations

Proposition 2. Let $F_\epsilon = F + \epsilon H$ be a C^2 perturbation of a map $F : M^n \rightarrow N^m$, with ϵ sufficiently small and $H : M^n \rightarrow N^m$ smooth. Then the deviation tensor T_ϵ associated to F_ϵ satisfies:

$$T_\epsilon = T + O(\epsilon),$$

and its rank satisfies:

$$\text{rank}(T_\epsilon) \leq \text{rank}(T) + \dim(\text{image}(dH)).$$

Proof. By definition, the deviation tensor for F_ϵ is:

$$T_\epsilon := F_\epsilon^* g_N - \Omega_\epsilon^2 g_M.$$

Using the smoothness of F_ϵ and linearity of the pullback, we have:

$$F_\epsilon^* g_N = F^* g_N + \epsilon dH^* g_N + O(\epsilon^2),$$

where $dH^* g_N$ denotes the linearized change of the pullback metric induced by H .

Similarly, the conformal factor Ω_ϵ is smooth in ϵ , so:

$$\Omega_\epsilon^2 = \Omega^2 + O(\epsilon).$$

Subtracting $\Omega_\epsilon^2 g_M$ from $F_\epsilon^* g_N$ gives:

$$T_\epsilon = (F^* g_N - \Omega^2 g_M) + \epsilon dH^* g_N + O(\epsilon^2) = T + O(\epsilon).$$

For the rank estimate, observe that the leading-order perturbation $\epsilon dH^* g_N$ has image contained in the span of $dH(T_p M^n)$. Hence, the maximal possible increase in the rank of T_ϵ relative to T is at most:

$$\dim(\text{image}(dH)) = \dim(\text{span}\{dH(v) \mid v \in T_p M^n\}),$$

giving:

$$\text{rank}(T_\epsilon) \leq \text{rank}(T) + \dim(\text{image}(dH)).$$

This completes the proof. $\square$

5. Examples of Original Classes

Example 1. Consider the map $F : M^4 \rightarrow N^2$ with $M^4 = \mathbb{R}^{1,3}$, $N^2 = \mathbb{R}^{1,1}$ defined by:

$$F(t, x, y, z) = (t + x, y + z).$$

Here, $n - m = 2$, and one can compute the deviation tensor T explicitly using:

$$T = F^* g_{N^2} - \Omega^2 g_{M^4}.$$

By construction, T attains maximal rank $r = 2$, corresponding to the two independent directions outside $\ker(dF)$. The kernel of the differential dF is spanned by vectors $(x, -t, 0, 0)$ and $(0, 0, y, -z)$, along which T is degenerate, i.e.,

$$T(v, \cdot) = 0, \quad \forall v \in \ker(dF).$$

This example illustrates the case where the deviation from conformality is maximal and aligns with the upper bound $r = n - m$.

Example 2. Consider the map $F : M^5 \rightarrow N^3$ with $M^5 = \mathbb{R}^{1,4}$, $N^3 = \mathbb{R}^{1,2}$ defined by

$$F(t, x, y, z, w) = (t + \phi(x, y), x, y),$$

where $\phi \in C^\infty(\mathbb{R}^2)$ is smooth. Let us choose ϕ such that the 2-dimensional null planes spanned by ∂_z and ∂_w are mapped to null directions in N^3 . In this setting, the deviation tensor T has rank $r = 1 < n - m = 2$, demonstrating a reduction in rank due to the imposed weak 2-plane preservation condition. The kernel of T contains the directions corresponding to the preserved null planes:

$$\ker(T) \supset \text{span}\{\partial_z, \partial_w\}.$$

This example shows explicitly how weak k -plane preservation restricts the possible deviation of F , reducing the maximal rank of T in comparison with the general weak null-preserving case.

Example 3. Let $F : M^4 \rightarrow N^3$ with $M^4 = \mathbb{R}^{1,3}$, $N^3 = \mathbb{R}^{1,2}$, be defined by:

$$F(t, x, y, z) = (t + \phi(x), x, y),$$

with $\phi \in C^\infty(\mathbb{R})$ smooth. The deviation tensor T for F has rank $r = 1$, with kernel containing directions corresponding to ∂_z and combinations of ∂_x determined by ϕ .

Now consider a small C^2 perturbation :

$$F_\epsilon = F + \epsilon H, \quad H(t, x, y, z) = (\psi_1(t, x), \psi_2(y, z), 0),$$

with $\epsilon \ll 1$ and ψ_1, ψ_2 smooth. By Proposition 4.1, the perturbed deviation tensor satisfies:

$$T_\epsilon = T + O(\epsilon), \quad \text{rank}(T_\epsilon) \leq r + \dim(\text{image}(dH)).$$

Explicitly, if $\dim(\text{image}(dH)) = 1$, then the rank of T_ϵ increases at most by one:

$$r \leq \text{rank}(T_\epsilon) \leq r + 1.$$

This example illustrates the quantitative bound on the rank of T under small perturbations and shows how the structure of the kernel evolves, providing insight into the local stability of the deviation tensor under smooth deformations of the map F .

6. Conclusions

In this work, we have presented a systematic and original study of interdimensional weak null-preserving maps between pseudo-Riemannian manifolds. We provided a complete local classification of the deviation tensor T in terms of its rank and kernel structure, introduced the concept of weak k -plane null-preserving maps, and analyzed the stability of T under small C^2 perturbations of the mapping.

These results extend the foundational work of [1] by offering not only a rigorous local description of T but also by introducing new mathematical structures and analyses that were absent in the previous study:

- The previous work established the existence of interdimensional weak null-preserving maps and provided a canonical decomposition of T , but it did not classify T systematically; here, we provide a complete local classification in terms of rank and kernel, including the extreme cases of degenerate and maximal-rank behaviors.
- We define and study weak k -plane null-preserving maps, a novel class of maps that impose stricter constraints on T by preserving entire k -dimensional null subspaces, thereby reducing the possible rank. This concept and its consequences were not considered previously.

- We analyze the local stability of T under small C^2 perturbations, providing quantitative bounds on changes in rank and kernel structure. The previous work did not investigate how T behaves under smooth deformations of the map.

Beyond these contributions, our results lay a solid foundation for future research directions, including the global classification of deviation tensors, exploration of curvature-related constraints, and applications in the study of interdimensional causal embeddings in both mathematical and physical contexts. These developments may inform the understanding of higher-dimensional causal structures, Lorentzian geometry, and geometric rigidity phenomena in theoretical physics [5,8,15]. Overall, this study offers new tools and perspectives for analyzing the interplay between null structures and interdimensional mappings, opening pathways for both theoretical investigations and practical applications in geometry and relativity.

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