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Jiazheng Liu *

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Bandlimited Green's Functions from Minkowski Isometry: A Geometric Derivation of Celestial Conformal Weights

Jiazheng Liu

Independent Researcher, China; lib88419@gmail.com

Abstract

We present a complete, parameter-free derivation of the bandlimited Green's function $G = \mathrm{sinc}(\sqrt{-\sigma^2})$ and the celestial conformal weights $\Delta_l = l + 1$ from a single input: four-dimensional Minkowski spacetime $(M, \eta_{\mu\nu})$. The derivation proceeds in three steps. First, the exactness of the exponential map $\exp_p : T_pM \to M$ in flat spacetime, combined with the requirement that discrete sampling be isometrically equivalent to the continuous field, uniquely determines—via the Whittaker interpolation theorem—the reproducing kernel $G = \mathrm{sinc}(\Omega\sqrt{-\sigma^2})$. Second, the null geodesic locus $\sigma^2 = 0$ emerges as the natural boundary through the reproducing kernel normalisation condition $K(x, x) = 1$; restriction to this null hypersurface induces a signature flip from Lorentzian $(-, +, +, +)$ to Euclidean $(+, +)$ on the transverse S^2. Third, the $SL(2, \mathbb{C})$ principal-series representation on the Euclidean celestial sphere, combined with the spherical Bessel decomposition of G, yields $\Delta_l = l + 1$ as a pure spectral theorem with no free parameters. The result is cross-validated by five independent routes: Kempf's operator-theoretic reconstruction [1,2], the present geometric construction, a boundary RKHS derivation, Pasterski–Shao–Strominger [3] from scattering amplitudes, and Gover–Shaukat–Waldron tractor calculus [4] providing the $SO(4, 2)$ group-theoretic skeleton explaining why all five routes converge. The scale Ω is structurally irrelevant: all physical conclusions depend only on the Minkowski metric. We identify the null-geodesic data set as a natural basis for geometric consistency checks, and note that if the universe is a quantum state, the multi-path convergence in principle circumvents the classical cosmic variance bound.

Keywords: bandlimited Green's function; celestial conformal weights; Minkowski isometry; reproducing Kernel Hilbert space; Whittaker interpolation theorem; signature flip; principal-series representation; cross-validation; cosmic variance

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1. Introduction

The relationship between bulk physics in Minkowski spacetime and boundary conformal field theory has been approached from many directions: string-theoretic AdS/CFT correspondence, celestial holography via soft theorems and BMS symmetry, and abstract operator-theoretic frameworks for quantum gravity at minimum length scales. A common feature is that the conformal dimension $\Delta_l = l + 1$ for massless fields of spin l appears as a robust output across frameworks that begin from entirely different physical assumptions.

The question we address is: what is the minimal geometric input from which this result follows as a theorem, with no free parameters and no approximations? We argue that the answer is simply the Minkowski metric $\eta_{\mu\nu}$, combined with the requirement that field sampling be isometrically consistent. No Planck-scale physics, no string theory, no soft theorem technology is invoked; these are independent confirmations of a geometrical fact.

Our construction proceeds through three logically independent steps, each of which is a mathematical theorem rather than a physical assumption:

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Step 1 (Uniqueness of G). The exponential map of flat spacetime is an exact global isometry. Requiring that discrete sampling of a field be isometrically equivalent to its continuous L^2 representation—via the Whittaker–Shannon–Kotelnikov interpolation theorem—uniquely determines the reproducing kernel to be the sinc function. The bandlimit Ω is a free scale; all structural conclusions are independent of it.

Step 2 (Natural boundary). The reproducing kernel normalisation condition $K(x,x) = 1$, which holds in every normalised reproducing kernel Hilbert space, forces $G|_{\sigma^2=0} = 1$. This identifies the null cone $\sigma^2 = 0$ as the natural boundary of the construction. Geometric restriction to this null hypersurface induces a signature flip: the transverse metric on S^2 is Euclidean. This is a theorem of Lorentzian geometry, not an assumption.

Step 3 (Conformal spectrum). The Euclidean celestial sphere S^2 carries the $SL(2,\mathbb{C}) \cong SO(3,1)$ action simultaneously as the Lorentz group of the bulk and the conformal group of the boundary. The spherical Bessel decomposition of G, combined with the principal-series representation theory and the peeling property, yields $\Delta_l = l + 1$ as a spectral theorem.

We emphasise that this framework is an *ideal model*: it establishes the geometric skeleton—which sectors exist, what their conformal dimensions are, and why the bulk-boundary correspondence is exact—but does not independently derive the non-Abelian gauge structure or the particle content within each sector. For the identification of $l = 1$ with photons and gluons, and $l = 2$ with gravitons, we follow Pasterski–Shao–Strominger [3].

The paper is organised as follows. Section 2 establishes notation and recalls the relevant properties of reproducing kernel Hilbert spaces. Section 3 derives the bandlimited Green's function from the Minkowski isometry condition. Section 4 identifies the natural boundary and proves the signature flip. Section 5 derives the conformal spectrum. Section 6 presents the five-path cross-validation. Section 7 discusses the scope and limitations of the ideal model. Section 8 discusses the quantum-state method for circumventing cosmic variance. Section 9 concludes.

2. Notation and Preliminaries

2.1. Minkowski Spacetime

We work in four-dimensional Minkowski spacetime $(M, \eta_{\mu\nu})$ with metric signature $(-,+,+,+)$. For two spacetime events $x, y \in M$, the squared interval is

$$\sigma^2(x,y) = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = -\Delta t^2 + \Delta r^2 + r^2\, d\Omega^2, \qquad \Delta x^\mu = x^\mu - y^\mu. \tag{1}$$

This is the unique quadratic Lorentz scalar constructed from $\eta_{\mu\nu}$ and Δx^μ. The null cone based at y is $\Lambda_y = \{x \in M : \sigma^2(x,y) = 0\}$.

A critical property of flat spacetime is that the exponential map $\exp_p\colon T_pM \to M$ is a *global isometry* for all $p \in M$. This is not an approximation—it holds exactly and globally, in contrast to curved spacetime where $\exp_p$ is only a local diffeomorphism. This exactness is the key geometric input of our construction.

2.2. Reproducing Kernel Hilbert Spaces

A Hilbert space $\mathcal{H}$ of functions on a set X is a *reproducing kernel Hilbert space (RKHS)* if evaluation functionals $f \mapsto f(x)$ are bounded for all $x \in X$. By the Riesz representation theorem, there exists a unique *reproducing kernel* $K\colon X \times X \to \mathbb{C}$ such that

$$f(x) = \langle f, K(x,\cdot)\rangle_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H},\ x \in X. \tag{2}$$

Theorem 1 (Aronszajn [5]). *Every positive definite kernel uniquely determines an RKHS, and every RKHS has a unique reproducing kernel.*

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Two properties of the reproducing kernel are essential for our purposes:

(R1) Normalisation: $K(x,x) = \|K(x,\cdot)\|^2_{\mathcal{H}}$. For a *normalised* RKHS in which $\|K(x,\cdot)\| = 1$ for all x, one has $K(x,x) = 1$.

(R2) Uniqueness: Given $\mathcal{H}$, the reproducing kernel K is uniquely determined. There is no freedom of choice.

2.3. Paley–Wiener Space and the Whittaker Theorem

The *Paley–Wiener space* $\mathrm{PW}_\Omega \subset L^2(\mathbb{R})$ is the Hilbert space of square-integrable functions whose Fourier transform is supported in $[-\Omega, \Omega]$:

$$\mathrm{PW}_\Omega = \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subseteq [-\Omega, \Omega]\}. \tag{3}$$

Theorem 2 (Whittaker–Shannon–Kotelnikov)**.** *The sampling map*

$$\mathcal{W} \colon \mathrm{PW}_\Omega \to \ell^2(\mathbb{Z}), \qquad f \mapsto \{f(n/2\Omega)\}_{n\in\mathbb{Z}} \tag{4}$$

is a unitary isomorphism. The unique inverse is given by the sinc reconstruction formula

$$f(x) = \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{2\Omega}\Big)\,\mathrm{sinc}\Big(\Omega\Big(x - \frac{n}{2\Omega}\Big)\Big), \qquad \mathrm{sinc}(z) := \frac{\sin z}{z}. \tag{5}$$

The reproducing kernel of PW_Ω is therefore $K(x,y) = \mathrm{sinc}(\Omega(x-y))$. This is *the unique* kernel compatible with the isometric sampling structure. Any other kernel would break the isometry.

3. The Bandlimited Green's Function from Minkowski Isometry

3.1. The Core Argument

We now state and prove the central construction. The sole input is Minkowski spacetime $(M, \eta_{\mu\nu})$.

Definition 1. *A* field *on Minkowski spacetime is a function* $\varphi \colon M \to \mathbb{C}$ *belonging to a Hilbert space* $\mathcal{H} \subseteq L^2(M)$*. A* discrete sampling *of* φ *is the sequence* $\{\varphi(x_n)\}_n \in \ell^2$ *for a countable set of spacetime points* $\{x_n\} \subset M$.

Definition 2 (Isometric Sampling Condition)**.** *The sampling map* $\mathcal{S} \colon \mathcal{H} \to \ell^2$, $\varphi \mapsto \{\varphi(x_n)\}$ *is required to be a* unitary isomorphism*: it preserves inner products,*

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \langle \mathcal{S}(\varphi), \mathcal{S}(\psi) \rangle_{\ell^2} = \sum_n \varphi(x_n)\,\overline{\psi(x_n)}. \tag{6}$$

Theorem 3 (Uniqueness of the Bandlimited Green's Function)**.** *Let* $(M, \eta_{\mu\nu})$ *be Minkowski spacetime, and suppose the Isometric Sampling Condition* (6) *holds. Then:*

1. *The field space* $\mathcal{H}$ *is uniquely isomorphic to a Paley–Wiener space* PW_Ω *for some bandlimit* $\Omega > 0$.
2. *The reproducing kernel of* $\mathcal{H}$ *is uniquely determined to be*

$$G(x,y) = \mathrm{sinc}\Big(\Omega\sqrt{-\sigma^2(x,y)}\Big) = \frac{\sin(\Omega\sqrt{-\sigma^2})}{\Omega\sqrt{-\sigma^2}}. \tag{7}$$

3. *All structural properties of* G*—including the identification of its natural boundary and the conformal spectrum—are independent of* Ω.

Proof. (i) The tangent space T_pM at any $p \in M$ is isometrically isomorphic to $\mathbb{R}^4$ as an inner product space. Since $\exp_p \colon T_pM \to M$ is a global isometry in flat spacetime, we have $L^2(M) \cong L^2(T_pM) \cong L^2(\mathbb{R}^4)$. The Isometric Sampling Condition (6) applied in each temporal and spatial direction reduces,

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by the Whittaker theorem, to the requirement that the field space be PW_Ω in each coordinate. The Lorentz-invariant extension combines these into the four-dimensional Paley–Wiener space over M.

(ii) By property (R2) of RKHS, the reproducing kernel of PW_Ω is unique. In one dimension it is $K(x,y) = \mathrm{sinc}(\Omega(x-y))$. The unique Lorentz-invariant extension, replacing $x-y$ by the Lorentz scalar $\sqrt{-\sigma^2(x,y)}$, gives (7).

(iii) The parameter Ω acts as a global scale. The null cone $\{\sigma^2 = 0\}$, the normalisation $G|_{\sigma^2=0} = \mathrm{sinc}(0) = 1$, and the spherical Bessel structure of G are all independent of Ω. Without loss of generality we set $\Omega = 1$ in all subsequent structural computations. □

Remark 1. *The derivation in Theorem 3 does* not *begin from the statement that the Dirac delta $\delta^4 \notin L^2(\mathbb{R}^4)$. That observation (from our earlier work) motivates the correction of a category error, but the present derivation is a positive construction: it derives $G = \mathrm{sinc}$ as the unique kernel satisfying the isometric sampling condition, without any reference to distributional theory.*

3.2. Spherical Bessel Decomposition

With $\Omega = 1$, the Green's function $G(x,y)$ admits the Lorentz-covariant expansion

$$G(x,y) = \frac{1}{2\pi^2}\sum_{l=0}^{\infty}(2l+1)\, j_l(\rho)\, j_l(\rho')\, P_l(\cos\theta), \tag{8}$$

where $\rho = |r|$, $\rho' = |r'|$ are the radial coordinates, θ is the angle between x and y, j_l are spherical Bessel functions of the first kind, and P_l are Legendre polynomials.

The sectors $l = 0, 1, 2, \ldots$ correspond to the scalar, vector, and tensor angular-momentum channels. Following the physical identification of PSS [3], the $l = 0$ sector is the scalar field propagator, $l = 1$ corresponds to gauge bosons (photons, gluons), and $l = 2$ to gravitons. The present framework derives the *spectrum* of these sectors; the non-Abelian gauge structure within the $l = 1$ sector is outside the scope of this ideal model.

4. The Natural Boundary and the Signature Flip

4.1. The Null Cone as Natural Boundary

Proposition 1. *For the reproducing kernel G defined in (7), the null cone $\Lambda = \{(x,y) : \sigma^2(x,y) = 0\}$ is characterised by the reproducing kernel normalisation condition $K(x,x) = 1$.*

Proof. Since $\mathrm{sinc}(0) = \lim_{z\to 0}\sin(z)/z = 1$, we have $G(x,y) = 1$ whenever $\sigma^2(x,y) = 0$. The point $\sigma^2 = 0$ is a removable singularity, not a geometric singularity. The value $G = 1$ coincides with the normalised reproducing kernel diagonal: $K(x,x) = \|K_x\|^2 = 1$ in the normalised RKHS. Thus the null cone is the *locus of self-evaluation* in the reproducing kernel structure. □

The reproducing kernel normalisation $K(x,x) = 1$ is a universal property of normalised RKHS; the fact that it forces $G = 1$ precisely on the null cone $\sigma^2 = 0$ is a consequence of the Lorentz invariance of the construction. The null cone is the *natural boundary* determined by the geometry itself.

4.2. Signature Flip: Lorentzian Bulk to Euclidean Boundary

Theorem 4 (Signature Flip)**.** *The restriction of the Minkowski metric to the null cone Λ induces a Euclidean signature metric on the transverse two-sphere S^2.*

Proof. The null cone $\Lambda_y = \{x : \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = 0\}$ is a null hypersurface in $(M, \eta_{\mu\nu})$. Its normal vector $n^\mu = \partial^\mu \sigma^2$ satisfies $\eta_{\mu\nu} n^\mu n^\nu = 0$ (it is null). The induced metric on Λ_y is degenerate in the null direction and positive-definite in the two transverse directions. Explicitly, with $r = t$ on the null cone:

$$ds^2\Big|_\Lambda = \left(-dt^2 + dr^2 + r^2\, d\Omega^2\right)\big|_{r=t} = r^2\, d\Omega^2, \tag{9}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the round metric on S^2 with Euclidean signature $(+,+)$. The null direction $dt = dr$ drops out. The transverse geometry is Euclidean. □

Remark 2. *This signature transition*

$$(-,+,+,+) \longrightarrow (+,+) \tag{10}$$

is a theorem of Lorentzian geometry and requires no additional physical input. It should be contrasted with AdS/CFT, where the bulk (AdS, Lorentzian) and boundary (Minkowski, Lorentzian) share the same signature, and the holographic correspondence requires a non-trivial physical assumption. Here, the signature flip is automatic: the boundary is Euclidean because the bulk is Lorentzian and the boundary is a null hypersurface.

Corollary 1. *The symmetry group $SL(2,\mathbb{C}) \cong SO(3,1)$ acts simultaneously as*

- *the Lorentz group of the Minkowski bulk, and*
- *the conformal group of the Euclidean boundary S^2.*

This algebraic coincidence is the group-theoretic reason why a bulk-boundary correspondence exists; it follows from the Lorentzian geometry and requires no additional correspondence principle.

5. Conformal Weights from the Euclidean Celestial Sphere

5.1. Principal-Series Representations of $SL(2,\mathbb{C})$

The principal-series unitary representations of $SL(2,\mathbb{C})$ acting on $L^2(S^2)$ are labeled by (Δ, s) where $\Delta = 1 + i\lambda$ for $\lambda \in \mathbb{R}$, and $s \in \mathbb{Z}/2$ is the helicity. For massless fields satisfying the wave equation in Minkowski spacetime, the *peeling property* [6] constrains the asymptotic behaviour:

$$\varphi(r,\theta,\varphi) \sim r^{-(l+1)} \quad \text{as } r \to \infty, \text{ on the null cone.} \tag{11}$$

5.2. Derivation of $\Delta_l = l + 1$

Theorem 5 (Conformal Weights)**.** *For a massless field of angular momentum quantum number $l = 0, 1, 2, \ldots$, the conformal dimension of the corresponding primary operator on the Euclidean celestial sphere is*

$$\boxed{\Delta_l = l + 1.} \tag{12}$$

Proof. The spherical Bessel decomposition (8) identifies the angular-momentum sectors. On the null cone $r = t \to \infty$, the asymptotic behaviour of the spherical Bessel function is

$$j_l(kr) \sim \frac{\sin(kr - l\pi/2)}{kr} \quad \text{as } r \to \infty, \tag{13}$$

so the radial envelope of the l-th sector decays as r^{-1}. The full field, being a product of two Bessel functions (one for each point), decays as $r^{-1} \cdot r^{-l} = r^{-(l+1)}$ in the appropriate limit, consistent with the peeling property (11). Under the $SL(2,\mathbb{C})$ action, operators with asymptotic decay $r^{-\Delta}$ transform as primary operators of conformal dimension Δ. Therefore $\Delta_l = l + 1$. The result is independent of Ω because it depends only on the power-law exponent of the Bessel asymptotics, which is a pure integer. □

The physical identification of the sectors follows PSS [3]; see Table 1.

Table 1. Massless field sectors, conformal weights, and physical identification (following PSS [3]).

Sector l	Δ_l	Physical field
$l = 0$	$\Delta_0 = 1$	Massless scalar
$l = 1$	$\Delta_1 = 2$	Photon, gluon
$l = 2$	$\Delta_2 = 3$	Graviton

The present framework provides the *geometric derivation* of this spectrum. PSS [3] provides the *particle identification* within each sector. The non-Abelian gauge structure within the $l = 1$ sector (distinguishing photons from gluons) lies outside the scope of this ideal model.

6. Five-Path Cross-Validation

The result $\Delta_l = l + 1$ is confirmed by five independent derivation paths. Their convergence is not coincidental: Section 6.5 explains it through a common group-theoretic skeleton. Each path begins from a different physical premise and uses different mathematical machinery.

6.1. Path A: Kempf's Operator-Theoretic Reconstruction

Kempf [1,2] considers coordinate operators X_i that are *simple symmetric* (linear with real expectation values but without self-adjoint extensions). This operator-theoretic structure—motivated by the existence of a minimum position uncertainty—admits exactly three short-distance geometries: continuous, discrete (lattice), and *fuzzy*. Only the fuzzy geometry preserves external symmetries while providing a UV cutoff.

Kempf [2] proves that for uniform minimum uncertainty ΔX_{min}, the reconstruction kernel is *exactly* the sinc function:

$$G(x, x_n) = \text{sinc}\left(\frac{\pi(x - x_n)}{2\Delta X_{\text{min}}}\right). \tag{14}$$

This is an operator-theoretic theorem, not a physical assumption. Setting $\Delta X_{\text{min}} = t_P$ (Planck length) gives $\Omega = \pi/t_P$, but as established in Theorem 3(iii), the structural conclusions are independent of this choice.

Crucially, Kempf [1], §6 predicts: degrees of freedom cut off beyond the Planck scale naturally reappear as internal degrees of freedom, with gauge group structure related to the unitary extension group. The present framework provides the explicit geometric realisation: the l-sectors and their gauge structure correspond precisely to these truncated degrees of freedom reappearing as internal degrees of freedom.

6.2. Path B: The Present Geometric Construction

As derived in Sections 3–5:

$$(M, \eta_{\mu\nu}) \;\Rightarrow\; \text{sinc} \;\Rightarrow\; \sigma^2 = 0 \;\Rightarrow\; S^2\,(\text{Euclidean}) \;\Rightarrow\; \Delta_l = l + 1.$$

Single input. No free parameters.

6.3. Path C: Boundary RKHS Derivation

Taking as input only the wave equation $\Box_g \varphi = 0$ on a Lorentz manifold and finite radiation energy $E < \infty$, the space $\mathcal{R}$ of smooth radiative solutions is an RKHS. The wave equation forces smoothness, so $\mathcal{R} \subset C^\infty$ and $L^2(S^2)$ is insufficient. By Aronszajn's theorem, the two-point function is unique. The $SL(2, \mathbb{C})$ principal-series representation of the solution space, combined with the peeling property, yields $\Delta_l = l + 1$. This derivation uses standard GR inputs only; the bandlimited Green's function is not invoked.

6.4. Path D: Pasterski–Shao–Strominger from Scattering Amplitudes

PSS [3] show that massless flat-space scattering amplitudes, when expressed in a basis of conformal primary wavefunctions on the celestial sphere, exhibit exact conformal covariance under $SL(2, \mathbb{C})$. Massless particles of spin l are mapped to primary operators with conformal dimension $\Delta_l = l + 1$. This derivation starts from the S-matrix and uses the momentum-eigenstate/conformal-primary-state change of basis; it makes no reference to the Green's function or RKHS structure.

We adopt the PSS particle identification (Table 1) since our framework establishes the geometry of sectors but does not derive the gauge structure within each sector independently.

6.5. Path E: Gover–Shaukat–Waldron and the $SO(4,2)$ Skeleton

Gover, Shaukat, and Waldron [4] develop tractor calculus as a systematic framework for Weyl-invariant theories. Their central result is that the tractor connection is a Weyl-invariant $\mathfrak{so}(d,2)$-connection. In $d = 4$, this is $\mathfrak{so}(4,2) \cong \mathfrak{so}(6,\mathbb{C})$. The tractor equation of motion

$$\mathcal{G}^{M_1\cdots M_s} = I\cdot DV^{M_1\cdots M_s} - s\,D^{(M_1}I\cdot V^{M_2\cdots M_s)} = 0 \tag{15}$$

unifies massless fields of all spins $s = 0, 1, 2, \ldots$ as sections of tractor bundles with $SO(4,2)$ symmetry. For massless fields, the mass-weight relation yields conformal dimensions consistent with $\Delta_l = l + 1$.

The role of [4] in our framework is *not* to provide an independent proof of $\Delta_l = l + 1$, but to explain *why* the four paths above necessarily converge: all four operate within the $SO(4,2)$ representation-theoretic framework that the tractor connection encodes. Different paths are different projections of the same $SO(4,2)$ structure.

Specifically, the tractor auxiliary slots V^+, V^- correspond to the internal degrees of freedom predicted by Kempf [1], §6. The l-sector decomposition of our G corresponds to the tractor bundle decomposition. The algebraic coincidence $SL(2,\mathbb{C}) \cong SO(3,1)$ acting as both the bulk Lorentz group and the boundary conformal group is the $\mathfrak{so}(4,2)$ content in disguise.

6.6. Summary of Cross-Validation

Table 2. Five independent paths to $\Delta_l = l + 1$.

Path	Starting point	Key tool	Output	Role
A: Kempf	Operator algebra, min. uncertainty	Simple symmetric operators	sinc, $\Delta_l = l + 1$	Algebraic confirmation
B: This work	$(M, \eta_{\mu\nu})$ only	Whittaker isometry uniqueness	$\Delta_l = l + 1$	Geometric foundation
C: RKHS boundary	$\Box\varphi = 0$, $E < \infty$	Aronszajn + peeling	$\Delta_l = l + 1$	GR confirmation
D: PSS 2017	Scattering amplitudes	Conformal primary wavefunctions	$\Delta_l = l + 1$	Particle identification
E: GSW tractor	Weyl invariance, conformal geom.	$\mathfrak{so}(4,2)$ tractor bundle	$SO(4,2)$ structure	Explains convergence

7. Scope of the Ideal Model

The present framework is an *ideal model* in the precise sense that it establishes exact geometric conclusions from minimal input. We clarify what the model covers and what lies beyond it.

What the framework establishes.

1. The unique reproducing kernel of Minkowski spacetime with isometric sampling is $G = \mathrm{sinc}(\sqrt{-\sigma^2})$.
2. The natural boundary is the null cone, with Euclidean transverse geometry S^2.
3. The conformal dimensions of massless fields of angular-momentum sector l are $\Delta_l = l + 1$.
4. All conclusions are independent of the bandlimit scale Ω.
5. The null-geodesic data set $\{\varphi|_{\sigma^2=0}\}$ provides a natural basis for geometric consistency checks across theories.

What lies beyond the framework.

1. The non-Abelian gauge structure (colour charge, $SU(3)$ for gluons) within the $l = 1$ sector.
2. The mass generation mechanism and the massive sector (which appears at timelike infinity i^+ rather than null infinity).
3. The three-generation structure of fermions.
4. Specific coupling constants.

These are not failures of the framework—they are questions about *which fields populate the sectors*, not about the geometric structure of the sectors themselves. The ideal model correctly predicts the spectrum; a fuller theory is needed to populate it.

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8. Cosmic Variance and the Quantum-State Method

8.1. The Classical Limitation

Classical cosmological inference faces the *cosmic variance* bound: for the CMB angular power spectrum C_l, the irreducible variance is

$$\mathrm{Var}(C_l) \geq \frac{2}{2l+1} C_l^2. \tag{16}$$

This bound is *fundamental*: there is only one universe, so ensemble averaging is impossible. At low multipoles ($l = 2, 3, \ldots$) the bound is especially severe. No improvement in measurement technology can circumvent it within the classical framework.

8.2. The Quantum-State Framework

If the universe is a quantum state $|\Psi\rangle$, then different theoretical frameworks constitute different *projections* onto this state:

$$\hat{P}_i |\Psi\rangle, \quad i = A, B, C, D, E \text{ (the five paths).} \tag{17}$$

The classical cosmic-variance argument assumes access to only one realisation of one stochastic process. The quantum-state framework is different: $|\Psi\rangle$ is the complete information about the universe, and the five projections $\hat{P}_i |\Psi\rangle$ provide five *independent* windows onto it.

Theorem 6 (Convergence from Unique Ground State)**.** *If the underlying physical entity is unique (one universe, one quantum state $|\Psi\rangle$), then independent projections $\hat{P}_i |\Psi\rangle$ using projection operators corresponding to different theoretical frameworks must yield consistent results. Inconsistency implies that at least one framework is incompatible with the actual geometry.*

This turns the five-path convergence into a *constraint*: any quantum gravity theory that does not reproduce $\Delta_l = l + 1$ on the null-geodesic data set fails the geometric consistency condition.

8.3. Signal-to-Noise Improvement

The effective signal-to-noise ratio of N independent projections scales as

$$\mathrm{SNR}_{\mathrm{eff}} = \mathrm{SNR}_{\mathrm{single}} \times \sqrt{N}. \tag{18}$$

With $N = 5$ paths whose noise sources are *uncorrelated* (operator algebra, differential geometry, RKHS theory, S-matrix theory, and tractor calculus are independent mathematical disciplines), the improvement is genuine.

More importantly, the five-path convergence provides a qualitatively different type of evidence from cosmic-variance-limited single-path inference: the geometric consistency of the null-geodesic data set replaces statistical averaging over an ensemble of universes.

9. Conclusions

We have derived the bandlimited Green's function $G = \mathrm{sinc}(\sqrt{-\sigma^2})$ and the celestial conformal weights $\Delta_l = l + 1$ from a single geometric input—Minkowski spacetime—with no free parameters. The derivation rests on two pillars:

1. **Whittaker isometric uniqueness**: The requirement that discrete field sampling be isometrically equivalent to continuous L^2 representation, combined with the exactness of the Minkowski exponential map, uniquely determines the reproducing kernel to be the sinc function.
2. **Geometric signature flip**: The null cone is the natural boundary of the construction (via reproducing kernel normalisation), and restriction to it induces a Euclidean signature on the transverse S^2, making the boundary conformal group $SL(2, \mathbb{C})$ equal to the bulk Lorentz group as an algebraic theorem.

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Five independent paths converge to $\Delta_l = l + 1$: the Kempf operator-theoretic path, the present geometric construction, the boundary RKHS derivation, the PSS scattering-amplitude derivation, and the Gover–Shaukat–Waldron $SO(4,2)$ tractor framework. The Gover framework explains the convergence: all five paths are projections of the same $SO(4,2)$ structure.

The framework is an ideal model: it establishes the geometric spectrum of sectors exactly, while the particle identification within sectors (photons vs. gluons in $l = 1$) follows PSS [3]. The null-geodesic data set provides a geometric consistency criterion for quantum gravity theories, and the multi-path convergence provides—in the quantum-state picture—an in-principle method to circumvent classical cosmic variance bounds.

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