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Yosef Akhtman *

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Schrödinger-Dirac Formalism in Finite Ring Continuum

Yosef Akhtman [1,2]

[1] AGH University of Science and Technology, Krakow, Poland; ya@gamma.earth; Tel.: +41-75-4179129
[2] Gamma Earth Sàrl, Morges, Switzerland

Abstract

We present a strictly finitist formulation of Schrödinger-type and Dirac-type dynamics in the Finite Ring Continuum, together with exact information counts for reversible and compressive shell maps. The construction uses a symmetry-complete prime field, its quadratic extension, and the Frobenius involution to define finite Hermitian state spaces and finitist Hamiltonians. On Euclidean shells, continuum time evolution is replaced by a Cayley update that preserves the Hermitian form exactly and therefore produces periodic trajectories. On the Lorentzian extension, we construct explicit gamma matrices, a finitist Dirac operator, its associated Klein-Gordon factorization, and a covariant lifted boost action. To connect the formalism with entropy and information theory without leaving strict finiteness, we measure finite maps by their image counts and exact loss factors. This separates reversible transformations, which preserve distinguishability exactly, from shell power maps, which merge states by a computable arithmetic factor. All results are finite, algebraic, and exact; no limits, differential calculus, or continuum structures are used.

Keywords: entropy; information loss; reversible computation; finite fields; finitist dynamics; Cayley evolution; Clifford algebra; discrete Dirac operator; finite Lorentz frames; finite ring continuum

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1. Introduction

The Finite Ring Continuum (FRC) program reconstructs arithmetic and geometry inside finite algebraic domains rather than by postulating an actual continuum [1,2]. In the foundational algebraic work, framed number classes were established over symmetry-complete prime fields $\mathbb{F}_p$ with $p = 4t + 1$ [1], while the Lorentzian paper showed that a genuine split-signature quadratic form requires the quadratic extension $\mathbb{F}_{p^2}$ [2]. On that basis, the present work develops finite wave dynamics and exact information counts.

Our aim is narrower than a reconstruction of all of quantum mechanics. We work strictly inside finite sets, finite fields, and finite matrices, and we ask for exact algebraic analogues of the two standard wave operators:

1. a Schrödinger-type time-step on Euclidean shells;
2. a Dirac-type first-order operator on the Lorentzian extension.

No limit, differential quotient, or continuum argument is used anywhere in the formalism.

Relation to Entropy and Information. In classical information theory, Hartley and Renyi isolate finite support size and distinguishability as basic information-theoretic quantities, with Shannon's theory providing the broader communication framework [3–5]. Here only the exact finite data behind those quantities are retained: image cardinalities, fiber cardinalities, and exact loss factors for finite maps. This avoids any real-valued encoding of these counts, as well as limits and measure-theoretic probability, while still obtaining precise statements about reversibility and information loss. This also places the results naturally next to reversible computation in the sense of Landauer and Bennett, and to conservative reversible logic in the sense of Fredkin and Toffoli [6–8]: bijective finite dynamics

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preserve information counts exactly, whereas many-to-one shell maps lose them by an exact arithmetic factor.

External Mathematical Context. Finite-state quantum kinematics, unitary operator bases, and discrete phase-space methods have a substantial literature of their own [9–11]. Finite-field variants of quantum theory have also been explored in several forms, including Galois-field formulations and finite spin models [12–14]. Discrete relativistic wave evolution has likewise been studied through unitary lattice and walk constructions [15,16]. The emphasis here is not to quantize over an arbitrary finite field, but to derive exact finitist wave dynamics from the specific Euclidean/Lorentzian split already established in the FRC shell hierarchy, and then to measure reversibility and compression by exact finite counts.

Mathematical Strategy. The central technical choice of the paper is to work with the finite coefficient field

$$K = \mathbb{F}_{\mathsf{p}}[c]/(c^2 - \nu) \cong \mathbb{F}_{\mathsf{p}^2},$$

where $\nu \in \mathbb{F}_{\mathsf{p}}^{\times}$ is a fixed nonsquare. The Frobenius map $z \mapsto z^{\mathsf{p}}$ supplies a genuine involution on K. This lets us define exact Hermitian forms on finite function spaces and thereby avoid the incorrect but tempting identification of finite dynamics with continuum probability calculus.

On Euclidean configuration spaces $X = \mathbb{F}_{\mathsf{p}}^{d}$, we define self-adjoint Hamiltonians from translation operators and finite potentials, then replace the formal continuum Schrödinger law by a rational Cayley step

$$(I - \alpha H)\psi_{n+1} = (I + \alpha H)\psi_n, \qquad \bar{\alpha} = -\alpha.$$

This step preserves the Hermitian form exactly whenever the denominator is invertible, and every orbit is exactly periodic because the relevant finite unitary group is finite.

On the Lorentzian shell K^4, we construct explicit gamma matrices adapted to the split quadratic form

$$Q_\nu(x_0, x_1, x_2, x_3) = -\nu x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

The resulting Dirac operator is a finitist first-order difference operator on Lorentz frames. We prove its Clifford relations, its factorization to the associated spinorial Klein-Gordon operator, and its covariance under an explicit lifted $1+1$ boost subgroup.

The information-theoretic layer is built from finite maps themselves. For a map $f : X \to Y$ on finite sets we use the image count $|f(X)|$ and the exact loss factor $|X|/|f(X)|$. This yields an exact combinatorial distinction between reversible operators, which have loss factor 1, and compressive maps, which merge arithmetic states.

Main Results. The precise contributions of the paper are as follows.

1. We define Hermitian function spaces K^X over finite Euclidean shells and prove that translation operators are unitary and the discrete Laplacian is self-adjoint.
2. We define finitist Hamiltonians and prove that their Cayley steps preserve the Hermitian form exactly. As a corollary, every trajectory is periodic.
3. We construct explicit gamma matrices over K satisfying the Clifford relations for the Lorentzian metric $\eta = \mathrm{diag}(-\nu, 1, 1, 1)$.
4. We define a frame-dependent finitist Dirac operator on Lorentz frames and prove the exact factorization
$$(\mathcal{D}_E - mI_{\mathcal{S}})(\mathcal{D}_E + mI_{\mathcal{S}}) = \mathrm{KG}_E^{\mathrm{spin}} - m^2 I_{\mathcal{S}}.$$
5. We define exact finite information counts and prove that the shell power map $x \mapsto x^{\varepsilon}$ on $\mathbb{F}_{\mathsf{p}}^{\times}$ has loss factor $\gcd(\varepsilon, \mathsf{p} - 1)$, while translations, scalings, Cayley propagators, and lifted boosts have loss factor 1.

Outline. Section 2 recalls the shell data required from the preceding papers. Section 3 defines the finite coefficient field, Hermitian forms, translations, and finite-difference operators. Section 4 develops the Cayley Schrödinger dynamics on Euclidean shells. Section 5 builds the Lorentzian Clifford algebra,

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the finitist Dirac operator, and the explicit lifted boosts. Section 6 adds the exact finite information counts and loss-factor results. Section 7 discusses reversibility, compression, and finite information counts, and Section 8 records finitist open problems.

2. Background: Shells, Extension, and Finitist Data

Symmetry-Complete Prime Shells. Let $\mathrm{p} = 4\mathrm{t} + 1$ be prime. By the results of paper 1 [1], the field $\mathbb{F}_\mathrm{p}$ is symmetry-complete: its multiplicative group is cyclic of order $4\mathrm{t}$, and therefore $\mathbb{F}_\mathrm{p}$ contains an element i_t with $i_\mathrm{t}^2 = -1$. The basic arithmetic symmetries are

$$T_a(x) = x + a, \qquad S_m(x) = mx, \qquad P_\varepsilon(x) = x^\varepsilon, \tag{1}$$

with $a \in \mathbb{F}_\mathrm{p}$, $m \in \mathbb{F}_\mathrm{p}^\times$, and $\varepsilon \in \mathbb{N}$ with $\varepsilon \geq 1$. The translations T_a and scalings S_m are always bijections. The power map P_ε is bijective exactly when $\gcd(\varepsilon, \mathrm{p}-1) = 1$; otherwise it merges residue classes.

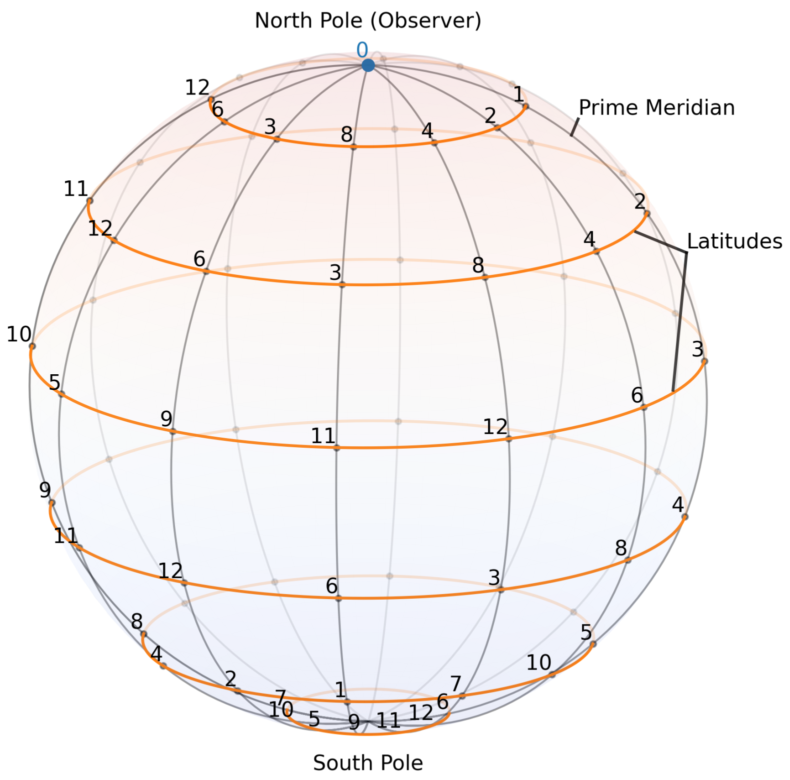

Figure 1. State diagram for the symmetry-complete field $\mathbb{F}_{13}$, viewed as a finite arithmetic shell with additive and multiplicative symmetry orbits.

The shell geometry developed in [1] interprets these actions as the discrete symmetry data of a Euclidean arithmetic shell. Only the finite algebra they generate is used below; no continuum or differential structure is assumed.

Quadratic Extension and Lorentzian Split. Paper 2 [2] shows that a genuine Lorentzian split cannot be realized inside $\mathbb{F}_\mathrm{p}$ itself. Fix a nonsquare $\nu \in \mathbb{F}_\mathrm{p}^\times$ and form the quadratic extension

$$K := \mathbb{F}_\mathrm{p}[c]/(c^2 - \nu) \cong \mathbb{F}_{\mathrm{p}^2}. \tag{2}$$

Standard background on finite fields and quadratic forms over fields may be found in [17,18]. The element c satisfies $c^2 = \nu$ and does not belong to $\mathbb{F}_\mathrm{p}$. Inside K one has the Lorentzian quadratic form

$$Q_\nu(x_0, x_1, x_2, x_3) = -\nu x_0^2 + x_1^2 + x_2^2 + x_3^2, \tag{3}$$

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with associated orthogonal group $O(Q_\nu, K)$ of split type (Witt index 1).

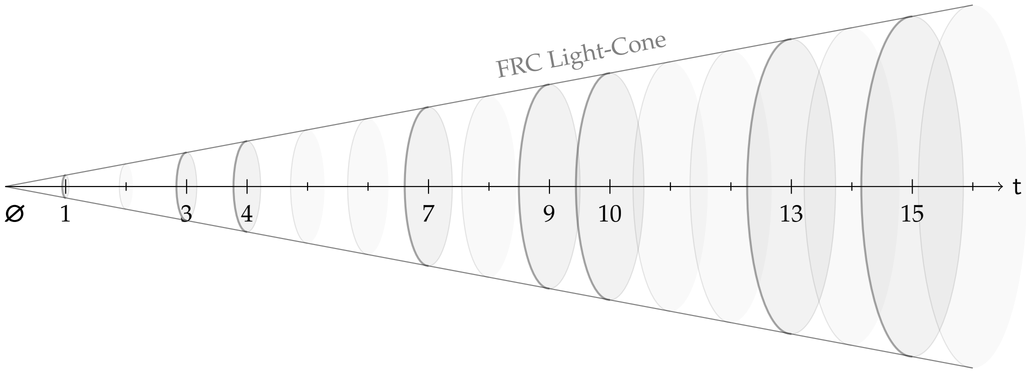

Figure 2. Chronon-indexed arithmetic shells of order $q = 4\mathrm{t} + 1$, with prime checkpoints emphasized.

Frobenius Involution. The coefficient field K carries a canonical involution:

$$\bar{z} := z^{\mathsf{p}}, \qquad z \in K. \tag{4}$$

Because c satisfies an irreducible quadratic polynomial over $\mathbb{F}_{\mathsf{p}}$, the Frobenius involution acts by

$$\overline{a + bc} = a - bc, \qquad a, b \in \mathbb{F}_{\mathsf{p}}.$$

This is the unique nontrivial $\mathbb{F}_{\mathsf{p}}$-automorphism of K, and it is the basic finitist replacement for complex conjugation used below.

Finite Working Domain. Everything below is formulated using only finite fields, finite sets, and finite matrices:

1. Euclidean dynamics takes place on finite configuration spaces $X = \mathbb{F}_{\mathsf{p}}^d$ with coefficient field K.
2. Lorentzian dynamics takes place on the finite spacetime shell K^4 equipped with Q_ν.
3. Time evolution is encoded by exact algebraic recursions, not by differential equations or limiting procedures.

These are the background ingredients needed for the dynamics developed below.

3. Finite Hermitian Function Spaces and Difference Operators

Coefficient Field. Throughout the paper we write

$$K = \mathbb{F}_{\mathsf{p}}[c]/(c^2 - \nu)$$

for the quadratic extension introduced in Section 2. The fixed field of the Frobenius involution $z \mapsto \bar{z} = z^{\mathsf{p}}$ is exactly $\mathbb{F}_{\mathsf{p}}$, and the trace-zero sector is

$$K^- := \{\, \alpha \in K \mid \bar{\alpha} = -\alpha \,\} = \{\, bc \mid b \in \mathbb{F}_{\mathsf{p}} \,\}.$$

The distinguished element c belongs to K^-.

Hermitian Spaces Over Finite Sets. Let X be any finite set. We denote by

$$\mathcal{H}(X) := K^X$$

the space of all K-valued functions on X.

Definition 1 (Hermitian Form). *For $\psi, \phi \in \mathcal{H}(X)$ define*

$$\langle \psi, \phi \rangle_X := \sum_{x \in X} \overline{\psi(x)}\, \phi(x). \tag{5}$$

Lemma 1. *The form (5) is Hermitian and nondegenerate.*

Proof. Hermitian symmetry follows from

$$\overline{\langle \psi, \phi \rangle_X} = \sum_{x \in X} \psi(x)\, \overline{\phi(x)} = \langle \phi, \psi \rangle_X.$$

For nondegeneracy, let $\psi \neq 0$. Choose $x_0 \in X$ with $\psi(x_0) \neq 0$, and define ϕ by $\phi(x_0) = 1$ and $\phi(x) = 0$ for $x \neq x_0$. Then $\langle \psi, \phi \rangle_X = \overline{\psi(x_0)} \neq 0$. □

Definition 2 (Adjoint). *Let $A : \mathcal{H}(X) \to \mathcal{H}(X)$ be K-linear. Its adjoint is the unique K-linear operator $A^\dagger$ satisfying*

$$\langle A\psi, \phi \rangle_X = \langle \psi, A^\dagger \phi \rangle_X$$

for all $\psi, \phi \in \mathcal{H}(X)$.

Because the Hermitian form is nondegenerate and $\mathcal{H}(X)$ is finite-dimensional over the finite field K, every K-linear operator admits a unique adjoint.

Euclidean Configuration Spaces. For the Euclidean part of the paper we take

$$X_d := \mathbb{F}_p^d$$

with its additive group law. For $u \in X_d$ define the translation operator

$$(T_u \psi)(x) := \psi(x + u). \tag{6}$$

Proposition 1. *For every $u \in X_d$, the translation operator T_u is unitary on $\mathcal{H}(X_d)$ and satisfies*

$$T_u^\dagger = T_{-u}.$$

Proof. Because $x \mapsto x + u$ is a bijection of X_d, reindexing gives

$$\langle T_u\psi, T_u\phi \rangle_{X_d} = \sum_{x \in X_d} \overline{\psi(x+u)}\, \phi(x+u) = \sum_{y \in X_d} \overline{\psi(y)}\, \phi(y) = \langle \psi, \phi \rangle_{X_d}.$$

The adjoint relation follows from

$$\langle T_u\psi, \phi \rangle_{X_d} = \langle \psi, T_{-u}\phi \rangle_{X_d}.$$

□

Definition 3 (Frame-Dependent Differences). *Let $E = (e_1, \ldots, e_d)$ be an ordered basis of X_d. The forward, backward, and Laplace operators associated with E are*

$$\delta_\mu^{E,+} := T_{e_\mu} - I, \tag{7}$$

$$\delta_\mu^{E,-} := T_{-e_\mu} - I, \tag{8}$$

$$\Delta_E := \sum_{\mu=1}^{d} \left(T_{e_\mu} + T_{-e_\mu} - 2I \right). \tag{9}$$

Proposition 2. *For every ordered basis E of X_d, the Laplace operator Δ_E is self-adjoint on $\mathcal{H}(X_d)$.*

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Proof. By the previous proposition, $T_{e_\mu}^\dagger = T_{-e_\mu}$ and $T_{-e_\mu}^\dagger = T_{e_\mu}$. Hence each summand $T_{e_\mu} + T_{-e_\mu} - 2I$ is self-adjoint, and so is their sum. □

Finite Potentials. Let $V : X_d \to \mathbb{F}_\mathsf{p}$. Multiplication by V defines an operator

$$(M_V\psi)(x) := V(x)\psi(x). \tag{10}$$

Since $V(x) \in \mathbb{F}_\mathsf{p}$ is fixed by Frobenius, M_V is self-adjoint.

Definition 4 (Finitist Hamiltonian). *Let $\kappa \in \mathbb{F}_\mathsf{p}$. The Hamiltonian attached to the basis E, the potential V, and the scalar κ is*

$$H_{E,V,\kappa} := -\kappa\Delta_E + M_V. \tag{11}$$

Corollary 1. *The Hamiltonian $H_{E,V,\kappa}$ is self-adjoint on $\mathcal{H}(X_d)$.*

Proof. The operator Δ_E is self-adjoint, the operator M_V is self-adjoint, and $\kappa \in \mathbb{F}_\mathsf{p}$ is fixed by Frobenius. □

Definition 5 (Finite Unitary Group). *For a finite set X, write*

$$U(\mathcal{H}(X), \langle\cdot,\cdot\rangle_X) := \{\, U \in \mathrm{GL}(\mathcal{H}(X)) \mid U^\dagger U = I \,\}.$$

Frame Covariance. The difference operators are defined relative to a finite frame rather than relative to any ambient external space. This makes covariance a finite change-of-basis statement.

Lemma 2. *Let $A \in \mathrm{GL}_d(\mathbb{F}_\mathsf{p})$, let $E = (e_1, \dots, e_d)$ be an ordered basis of X_d, and let $E' = (Ae_1, \dots, Ae_d)$. Define the pullback*

$$(A_*\psi)(x) := \psi(A^{-1}x).$$

Then, for each μ,

$$T_{Ae_\mu} A_* = A_* T_{e_\mu}, \qquad \delta_\mu^{E',\pm} A_* = A_*\delta_\mu^{E,\pm}, \qquad \Delta_{E'} A_* = A_*\Delta_E.$$

Proof. For translations,

$$(T_{Ae_\mu} A_*\psi)(x) = (A_*\psi)(x + Ae_\mu) = \psi(A^{-1}x + e_\mu) = (A_* T_{e_\mu}\psi)(x).$$

The claims for the difference operators and the Laplacian follow immediately from their definitions. □

The Euclidean and Lorentzian dynamics below will use only these finitist operators and their exact algebraic identities.

4. Schrödinger-Type Cayley Dynamics on Euclidean Shells

From Formal Wave Law to Finite Step. In a strictly finitist setting the natural dynamical object is an exact one-step recursion rather than a differential equation. Let $X_d = \mathbb{F}_\mathsf{p}^d$, let E be an ordered basis of X_d, and let $H = H_{E,V,\kappa}$ be a self-adjoint Hamiltonian of the form (11). The Cayley update below is algebraically close in spirit to Crank–Nicolson discretization, but is used here as an exact finite recursion on a finite state space rather than as a continuum approximation scheme [19].

Definition 6 (Cayley Step). *Let $\alpha \in K^-$, so that $\bar\alpha = -\alpha$. Whenever $I - \alpha H$ is invertible on $\mathcal{H}(X_d)$, define*

$$U_{\alpha,H} := (I - \alpha H)^{-1}(I + \alpha H). \tag{12}$$

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Equivalently, the step is the exact recursion

$$(I - \alpha H)\psi_{n+1} = (I + \alpha H)\psi_n. \tag{13}$$

Definition 7 (Admissible Parameters)**.** *For a fixed self-adjoint Hamiltonian H, define*

$$\mathcal{A}(H) := \{\, \alpha \in K^- \mid I - \alpha H \text{ is invertible on } \mathcal{H}(X_d) \,\}.$$

Because K^- is finite, the admissible set $\mathcal{A}(H)$ is an exactly computable finite subset of K^-.

Theorem 1 (Exact Hermitian Preservation)**.** *Let H be self-adjoint and let $\alpha \in K^-$ be such that $I - \alpha H$ is invertible. Then the Cayley operator $U_{\alpha,H}$ preserves the Hermitian form:*

$$\langle U_{\alpha,H}\psi, U_{\alpha,H}\phi\rangle_{X_d} = \langle \psi, \phi\rangle_{X_d} \tag{14}$$

for all $\psi, \phi \in \mathcal{H}(X_d)$.

Proof. Since $H^\dagger = H$ and $\bar{\alpha} = -\alpha$,

$$(I - \alpha H)^\dagger = I + \alpha H, \qquad (I + \alpha H)^\dagger = I - \alpha H.$$

Therefore

$$U_{\alpha,H}^\dagger = (I + \alpha H)^\dagger \left((I - \alpha H)^{-1}\right)^\dagger = (I - \alpha H)(I + \alpha H)^{-1}.$$

Because both factors are polynomials in H, they commute. Hence

$$U_{\alpha,H}^\dagger U_{\alpha,H} = (I - \alpha H)(I + \alpha H)^{-1}(I - \alpha H)^{-1}(I + \alpha H) = I.$$

The identity (14) follows. □

Corollary 2 (Finite Periodicity)**.** *Every Cayley orbit is exactly periodic. More precisely, for every $\psi_0 \in \mathcal{H}(X_d)$ there exists $N \geq 1$ such that*

$$\psi_N = \psi_0 \qquad \text{for} \qquad \psi_n := U_{\alpha,H}^n \psi_0.$$

Proof. The space $\mathcal{H}(X_d)$ has finite dimension $|X_d|$ over the finite field K, hence the finite unitary group

$$U(\mathcal{H}(X_d), \langle\cdot,\cdot\rangle_{X_d})$$

is finite. Therefore the element $U_{\alpha,H}$ has finite order. □

Finite Schrödinger Dynamics. The operator $U_{\alpha,H}$ is the finitist Schrödinger propagator of this formalism. Its defining data are all finite:

1. a finite Euclidean configuration space X_d;
2. a self-adjoint Hamiltonian built from translations and a finite potential;
3. a trace-zero coefficient α;
4. the exact invertibility test for $I - \alpha H$.

No appeal is made to infinitesimal time steps, spectral completions, or probability densities.

Example 1 (Prime $p = 13$)**.** *Let $\mathsf{p} = 13$, choose the nonsquare $\nu = 2$, and write $K = \mathbb{F}_{13}[c]/(c^2 - 2)$. Take $d = 1$, $X_1 = \mathbb{F}_{13}$, the basis $E = (1)$, the zero potential $V = 0$, and $\kappa = 1$. The Hamiltonian is*

$$H = -\Delta_E.$$

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Choose $\alpha = c \in K^-$. Direct exact computation in the 13-dimensional K-vector space $\mathcal{H}(X_1)$ shows that $I - cH$ is invertible and that the corresponding Cayley operator satisfies

$$U_{c,H}^{13} = I.$$

Equivalently, exact multiplication of the 13×13 matrix of $U_{c,H}$ over K returns the identity on the thirteenth power. For the delta state $\psi_0(x) = \delta_{x,0}$ one therefore has

$$\langle \psi_n, \psi_n \rangle_{X_1} = \langle \psi_0, \psi_0 \rangle_{X_1} = 1 \qquad \text{and} \qquad \psi_{13} = \psi_0.$$

This is an exact finite computation, not an approximation. The supplementary source file reproduces this calculation directly.

Whenever the Cayley denominator is invertible, the resulting Schrödinger-type evolution preserves the Hermitian form exactly and is periodic.

5. Dirac Operator on the Lorentzian Extension

Lorentzian Configuration Space. Fix the finite spacetime shell

$$Y := K^4$$

with ordered standard basis $E_0 = (e_0, e_1, e_2, e_3)$ and Lorentzian quadratic form (3). A *Lorentz frame* is an ordered basis $E = (u_0, u_1, u_2, u_3)$ of Y for which the Gram matrix of Q_ν in that basis is exactly $\eta = \mathrm{diag}(-\nu, 1, 1, 1)$. Equivalently, $E = \Lambda E_0$ for some $\Lambda \in O(Q_\nu, K)$. For each Lorentz frame $E = (u_0, u_1, u_2, u_3)$ define the directional difference operators

$$(\nabla_\mu^E \psi)(x) := \psi(x + u_\mu) - \psi(x), \qquad \psi \in \mathcal{S}(Y) := (K^4)^Y, \tag{15}$$

Since Y is an additive group, all translations commute, and therefore so do the operators ∇_μ^E. Unitary lattice models and quantum-walk analogues of relativistic wave equations provide an external comparison point for this finite difference setting [15,16].

Explicit Gamma Matrices. Let

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^2 = \begin{pmatrix} 0 & -i_t \\ i_t & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli identities $\sigma^j \sigma^k + \sigma^k \sigma^j = 2\delta_{jk} I_2$ hold over K because $i_t^2 = -1$ already in $\mathbb{F}_p$.

Define

$$\beta = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad \rho^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \quad (k = 1, 2, 3),$$

and then set

$$\gamma^0 := i_t\, c\, \beta, \qquad \gamma^k := i_t\, \rho^k \quad (k = 1, 2, 3). \tag{16}$$

Proposition 3 (Clifford Relations). *The matrices (16) satisfy*

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I_4, \qquad \eta = \mathrm{diag}(-\nu, 1, 1, 1). \tag{17}$$

Here $\eta^{\mu\nu}$ denotes the entries of the fixed diagonal matrix $\eta = \mathrm{diag}(-\nu, 1, 1, 1)$; no index-raising convention is being used.

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Proof. From the Pauli identities one has $(\rho^k)^2 = -I_4$ and $\rho^j\rho^k + \rho^k\rho^j = 0$ for $j \neq k$. Also $\beta^2 = I_4$ and $\beta\rho^k + \rho^k\beta = 0$. Since $(i_{\mathrm{t}}c)^2 = -\nu$ and $i_{\mathrm{t}}^2 = -1$, it follows that

$$(\gamma^0)^2 = -\nu I_4, \qquad (\gamma^k)^2 = I_4,$$

while all mixed anticommutators vanish. □

Frame-Dependent Gamma Families. For any Lorentz frame $E = \Lambda E_0$ with $\Lambda \in O(Q_\nu, K)$, define

$$\gamma_E^\mu := (\Lambda^{-1})^\mu{}_\nu \gamma^\nu. \tag{18}$$

This is the exact finite analogue of transforming the gamma matrices with the inverse frame matrix. Because Λ preserves Q_ν, the family $\{\gamma_E^\mu\}$ satisfies the same Clifford relations (17) as the standard family $\{\gamma^\mu\}$.

Definition 8 (Spinorial Klein-Gordon Operator). *For a Lorentz frame E, let $\Box_E$ denote the scalar operator on K^Y given by*

$$\Box_E := \eta^{\mu\nu}\nabla_\mu^E\nabla_\nu^E = -\nu(\nabla_0^E)^2 + \sum_{k=1}^{3}(\nabla_k^E)^2. \tag{19}$$

Its componentwise extension to spinor fields is

$$\mathrm{KG}_E^{\mathrm{spin}} := \Box_E \otimes I_4.$$

Definition 9 (Finitist Dirac Operator). *For a Lorentz frame E define*

$$\mathcal{D}_E := \sum_{\mu=0}^{3} \gamma_E^\mu \nabla_\mu^E \tag{20}$$

and write the finitist Dirac equation as

$$(\mathcal{D}_E - mI_{\mathcal{S}})\psi = 0, \qquad m \in \mathbb{F}_{\mathrm{p}}. \tag{21}$$

where $I_{\mathcal{S}}$ is the identity operator on $\mathcal{S}(Y)$.

Theorem 2 (Dirac-to-Klein-Gordon Factorization). *For every Lorentz frame E on Y,*

$$(\mathcal{D}_E - mI_{\mathcal{S}})(\mathcal{D}_E + mI_{\mathcal{S}}) = \mathrm{KG}_E^{\mathrm{spin}} - m^2 I_{\mathcal{S}}. \tag{22}$$

Proof. Because the directional differences commute,

$$\mathcal{D}_E^2 = \sum_{\mu,\nu=0}^{3} \gamma_E^\mu\gamma_E^\nu\nabla_\mu^E\nabla_\nu^E = \frac{1}{2}\sum_{\mu,\nu=0}^{3}(\gamma_E^\mu\gamma_E^\nu + \gamma_E^\nu\gamma_E^\mu)\nabla_\mu^E\nabla_\nu^E = \eta^{\mu\nu}\nabla_\mu^E\nabla_\nu^E I_4 = \mathrm{KG}_E^{\mathrm{spin}}.$$

The identity

$$(\mathcal{D}_E - mI_{\mathcal{S}})(\mathcal{D}_E + mI_{\mathcal{S}}) = \mathcal{D}_E^2 - m^2 I_{\mathcal{S}}$$

then yields (22). □

Explicit Lifted $1+1$ Boosts. Write

$$M := \gamma^0\gamma^1.$$

By the Clifford relations, $M^2 = \nu I_4$.

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Definition 10 (Lifted Boost Parameter). *Let $x, y \in \mathbb{F}_p$ and assume*

$$\delta := x^2 - \nu y^2 \neq 0.$$

Define

$$S(x,y) := xI_4 + yM. \tag{23}$$

Proposition 4. *The matrix $S(x,y)$ is invertible with*

$$S(x,y)^{-1} = \delta^{-1}(xI_4 - yM).$$

Moreover, with

$$A := \frac{x^2 + \nu y^2}{x^2 - \nu y^2}, \qquad B := \frac{-2xy}{x^2 - \nu y^2}, \tag{24}$$

one has

$$S(x,y)^{-1}\gamma^0 S(x,y) = A\gamma^0 + \nu B\gamma^1, \tag{25}$$

$$S(x,y)^{-1}\gamma^1 S(x,y) = B\gamma^0 + A\gamma^1, \tag{26}$$

$$S(x,y)^{-1}\gamma^2 S(x,y) = \gamma^2, \qquad S(x,y)^{-1}\gamma^3 S(x,y) = \gamma^3. \tag{27}$$

Proof. The inverse formula follows from $M^2 = \nu I_4$:

$$(xI_4 + yM)(xI_4 - yM) = (x^2 - \nu y^2)I_4 = \delta I_4.$$

Next,

$$M\gamma^0 = \nu\gamma^1, \qquad \gamma^0 M = -\nu\gamma^1, \qquad M\gamma^1 = \gamma^0, \qquad \gamma^1 M = -\gamma^0.$$

Substituting these identities into $S^{-1}\gamma^\mu S = \delta^{-1}(xI_4 - yM)\gamma^\mu(xI_4 + yM)$ yields (25)–(27). □

Corollary 3. *The matrix*

$$\Lambda(x,y) := \begin{pmatrix} A & B & 0 & 0 \\ \nu B & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{28}$$

belongs to $O(Q_\nu, K)$. In particular, $A^2 - \nu B^2 = 1$.

Proof. The identity $A^2 - \nu B^2 = 1$ follows from (24) by direct calculation, and that identity is exactly the orthogonality condition for the displayed matrix. □

Proposition 5 (Boost Subgroup). *Let*

$$G_\nu := \{\, \Lambda(x,y) \mid x, y \in \mathbb{F}_p,\ x^2 - \nu y^2 \neq 0 \,\}.$$

Then G_ν is a subgroup of $O(Q_\nu, \mathbb{F}_p) \subset O(Q_\nu, K)$.

Proof. Write $z = x + yc$ and $w = u + vc$ in $K^\times$ with norms $N(z) = x^2 - \nu y^2 \neq 0$ and $N(w) = u^2 - \nu v^2 \neq 0$. Their product is

$$zw = (xu + \nu yv) + (xv + yu)c.$$

The corresponding spin lifts satisfy the exact product law

$$S(x,y)S(u,v) = S(xu + \nu yv, xv + yu).$$

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By the conjugation formulas (25)–(27), the Lorentz matrix attached to the product spin lift is exactly $\Lambda(xu + \nu yv, xv + yu)$. Since composition of conjugations equals conjugation by the product, one obtains

$$\Lambda(x,y)\Lambda(u,v) = \Lambda(xu + \nu yv, xv + yu).$$

Because

$$(xu + \nu yv)^2 - \nu(xv + yu)^2 = (x^2 - \nu y^2)(u^2 - \nu v^2) \neq 0,$$

the right-hand side again belongs to G_ν. Also $\Lambda(1,0) = I_4$ and $\Lambda(x,y)^{-1} = \Lambda(x,-y)$. Therefore G_ν is a subgroup. □

Frame Covariance. Let $\Lambda = \Lambda(x,y)$ and let $E' = \Lambda E_0$. For any spinor field $\psi \in \mathcal{S}(Y)$ define the pullback

$$(\Lambda_*\psi)(z) := \psi(\Lambda^{-1}z).$$

Because E' is obtained by pushing the standard basis through Λ, one has

$$\nabla_\mu^{E'}\Lambda_* = \Lambda_*\nabla_\mu^{E_0} \tag{29}$$

for every μ.

Lemma 3. *Let $\Lambda = \Lambda(x,y) \in G_\nu$ and $S = S(x,y)$. Then*

$$S^{-1}\gamma_{E'}^\mu S = \gamma^\mu \qquad (\mu = 0,1,2,3).$$

Proof. By definition,

$$\gamma_{E'}^\mu = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu.$$

The conjugation formulas (25)–(27) say exactly that $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$. Multiplying by $(\Lambda^{-1})^\rho{}_\mu$ and summing over μ gives $S^{-1}\gamma_{E'}^\rho S = \gamma^\rho$. □

Theorem 3 (Explicit Covariance of the Finitist Dirac Operator)**.** *Let $\Lambda = \Lambda(x,y)$ and $S = S(x,y)$ be as above. Then, for every spinor field $\psi \in \mathcal{S}(Y)$,*

$$\mathcal{D}_{E'}(S\,\Lambda_*\psi) = S\,\Lambda_*(\mathcal{D}_{E_0}\psi). \tag{30}$$

Consequently, if ψ satisfies the Dirac equation $(\mathcal{D}_{E_0} - mI_{\mathcal{S}})\psi = 0$, then $S\,\Lambda_*\psi$ satisfies $(\mathcal{D}_{E'} - mI_{\mathcal{S}})(S\,\Lambda_*\psi) = 0$.

Proof. Using the previous lemma and (29),

$$\begin{aligned}
\mathcal{D}_{E'}(S\Lambda_*\psi) &= \sum_{\mu=0}^{3}\gamma_{E'}^\mu\nabla_\mu^{E'}(S\Lambda_*\psi) = \sum_{\mu=0}^{3}\gamma_{E'}^\mu S\Lambda_*(\nabla_\mu^{E_0}\psi) \\
&= S\sum_{\mu=0}^{3}S^{-1}\gamma_{E'}^\mu S\,\Lambda_*(\nabla_\mu^{E_0}\psi) = S\sum_{\mu=0}^{3}\gamma^\mu\Lambda_*(\nabla_\mu^{E_0}\psi) \\
&= S\Lambda_*\Big(\sum_{\mu=0}^{3}\gamma^\mu\nabla_\mu^{E_0}\psi\Big) = S\Lambda_*(\mathcal{D}_{E_0}\psi).
\end{aligned}$$

The Dirac equation transforms accordingly. □

Example 2 (Prime $p = 13$)**.** *Let $\mathsf{p} = 13$ and $\nu = 2$. Choosing $x = y = 1$ gives*

$$A = 10, \qquad B = 2,$$

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and therefore the exact boost

$$\Lambda(1,1) = \begin{pmatrix} 10 & 2 & 0 & 0 \\ 4 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O(Q_2, \mathbb{F}_{13}) \subset O(Q_2, \mathbb{F}_{13^2}).$$

The corresponding spin lift is $S(1,1) = I_4 + \gamma^0\gamma^1$. The covariance identity (30) is then an exact finite matrix identity over $\mathbb{F}_{13^2}$, verified by direct multiplication of the finite matrices and translation operators involved. The supplementary source file reproduces the matrix checks explicitly.

The quadratic extension therefore supplies an explicit Clifford algebra for the Lorentzian metric, a finitist Dirac operator built from commuting difference operators, an exact Klein-Gordon factorization, and a concrete lifted boost subgroup acting covariantly on frame-dependent finite differences. No continuum spin geometry is invoked.

6. Finite Information Counts, Reversibility, and Compression

Strictly Finitist Information Measures. To connect the present dynamics with the entropy literature without leaving strict finiteness, we do not assign real-valued entropies to finite sets. Instead we retain the exact finite combinatorics from which classical Hartley quantities and the order-zero Renyi point of view are derived [3–5].

Definition 11 (Image Count and Loss Factor)**.** *Let $f : X \to Y$ be a map of finite sets. The image count of f is*

$$I_X(f) := |f(X)|.$$

Its exact loss factor is

$$L_X(f) := \frac{|X|}{|f(X)|} \in \mathbb{Q}.$$

The image count records how many states remain distinguishable after the map is applied, while the loss factor records by what exact arithmetic factor distinguishability is reduced.

Proposition 6. *Let $f : X \to X$ be a self-map of a finite set. The following are equivalent:*

1. *f is bijective;*
2. *$I_X(f) = |X|$;*
3. *$L_X(f) = 1$.*

Proof. If f is bijective, then $f(X) = X$, so $I_X(f) = |X|$ and $L_X(f) = 1$. If $I_X(f) = |X|$, then the image has the same size as the domain, so a self-map on a finite set must be surjective and therefore bijective. The equivalence with $L_X(f) = 1$ is immediate from the definition. □

Exact Loss Under Power Maps. The shell arithmetic already provides a natural many-to-one family, namely the power maps on the multiplicative shell $\mathbb{F}_p^\times$.

Theorem 4 (Power-Map Count Formula)**.** *Let $X = \mathbb{F}_p^\times$ and let*

$$P_\varepsilon(x) = x^\varepsilon \qquad (\varepsilon \geq 1).$$

Write

$$d := \gcd(\varepsilon, p - 1).$$

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Then

$$I_X(P_\varepsilon) = \frac{\mathsf{p}-1}{d}, \qquad L_X(P_\varepsilon) = d.$$

Moreover, every fiber over a point of $P_\varepsilon(X)$ has exactly d elements.

Proof. The multiplicative group $\mathbb{F}_\mathsf{p}^\times$ is cyclic of order $\mathsf{p}-1$. Choose a generator g. Every element of $\mathbb{F}_\mathsf{p}^\times$ has the form g^k with $k \in \mathbb{Z}/(\mathsf{p}-1)\mathbb{Z}$, and

$$P_\varepsilon(g^k) = g^{\varepsilon k}.$$

Thus P_ε corresponds exactly to multiplication by ε on the cyclic group $\mathbb{Z}/(\mathsf{p}-1)\mathbb{Z}$. The image of that map has size $(\mathsf{p}-1)/d$, and the kernel has size d. Therefore the image of P_ε has size $(\mathsf{p}-1)/d$, each fiber has size d, and the loss factor is d. □

Corollary 4 (Shell Symmetries). *For the basic shell symmetries from Section 2 one has:*

1. $I_{\mathbb{F}_\mathsf{p}}(T_a) = \mathsf{p}$ *and* $L_{\mathbb{F}_\mathsf{p}}(T_a) = 1$ *for every translation* $T_a(x) = x + a$*;*
2. $I_{\mathbb{F}_\mathsf{p}}(S_m) = \mathsf{p}$ *and* $L_{\mathbb{F}_\mathsf{p}}(S_m) = 1$ *for every scaling* $S_m(x) = mx$ *with* $m \in \mathbb{F}_\mathsf{p}^\times$*;*
3. $I_{\mathbb{F}_\mathsf{p}^\times}(P_\varepsilon) = (\mathsf{p}-1)/d$ *and* $L_{\mathbb{F}_\mathsf{p}^\times}(P_\varepsilon) = d$ *for* $d = \gcd(\varepsilon, \mathsf{p}-1)$.

Proof. Translations and nonzero scalings are bijections of $\mathbb{F}_\mathsf{p}$, so the first two claims follow from the previous proposition. The third is exactly Theorem 4. □

Zero Loss for Reversible Dynamics. The finitist wave operators constructed in Sections 4 and 5 are reversible operators on finite state spaces.

Proposition 7. *Let Ω be a finite set, let $U : \Omega \to \Omega$ be bijective, and let $A \subseteq \Omega$. Then*

$$|U(A)| = |A|.$$

Equivalently, the restriction of U to A has loss factor 1.

Proof. Because U is injective, distinct elements of A have distinct images. Therefore $U : A \to U(A)$ is a bijection and the cardinalities agree. □

Corollary 5 (Zero Loss for Cayley Dynamics). *Let H be a self-adjoint Hamiltonian on $\mathcal{H}(X_d)$ and let $\alpha \in \mathcal{A}(H)$. Then the Cayley propagator*

$$U_{\alpha,H} : \mathcal{H}(X_d) \to \mathcal{H}(X_d)$$

has loss factor 1*. More generally, for every finite set $A \subseteq \mathcal{H}(X_d)$ one has*

$$|U_{\alpha,H}(A)| = |A|.$$

Proof. The operator $U_{\alpha,H}$ is invertible by construction, hence bijective on the finite set $\mathcal{H}(X_d)$. Apply Proposition 7. □

Corollary 6 (Zero Loss for Lifted Boost Transport). *Let $\Lambda \in G_v$ and let S be its spin lift. The transport operator*

$$\mathcal{T}_{\Lambda,S} : \mathcal{S}(Y) \to \mathcal{S}(Y), \qquad \mathcal{T}_{\Lambda,S}(\psi) := S\,\Lambda_*\psi$$

has loss factor 1*. More generally, for every finite set $B \subseteq \mathcal{S}(Y)$ one has*

$$|\mathcal{T}_{\Lambda,S}(B)| = |B|.$$

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Proof. The pullback Λ_* is bijective because Λ is invertible on Y, and multiplication by S is bijective because S is invertible. Hence $\mathcal{T}_{\Lambda,S}$ is bijective on $\mathcal{S}(Y)$. Apply Proposition 7. $\square$

Consequences. The information picture is exact and finite:

1. shell translations, scalings, Cayley propagators, and lifted boosts are zero-loss maps;
2. power maps lose information by the exact arithmetic factor $\gcd(\varepsilon, \mathsf{p}-1)$;
3. reversible dynamics and compressive arithmetic are distinguished by exact image counts, not by heuristic analogies.

These statements isolate reversibility and compression entirely through exact image counts and fiber structure.

7. Discussion: Reversibility and Compression

Finite Information Counts. Two kinds of exact finite statements appear together:

1. reversible finitist wave dynamics on Euclidean and Lorentzian shells;
2. exact information counts for finite shell maps, including explicit loss factors for compressive power maps.

The relevant quantity is not a continuum entropy functional but the exact count of distinguishable states remaining after a finite map is applied. Hartley's classical quantity and the order-zero Renyi perspective are built from such finite counts; only that exact finite arithmetic is retained here [3–5].

Reversible Dynamics Versus Compressive Maps. Sections 4, 5, and 6 separate two phenomena that are often conflated in heuristic discussions:

1. Cayley propagators and lifted boosts are reversible. They are bijections of finite state spaces and therefore have loss factor 1.
2. Power maps on the multiplicative shell can be compressive. Their loss factor is exactly $\gcd(\varepsilon, \mathsf{p}-1)$.

This finite distinction places the paper naturally next to the information-theoretic literature on reversibility and erasure. The algebraic shell symmetries supply exact finite examples of the same structural dichotomy emphasized by Landauer and Bennett: reversible transformations preserve distinguishability, while many-to-one maps lose it [6–8].

Relation to External Finite-Field Quantum Work. This framework also connects with external finite-field quantum literature [9–16]. The point of contact is the use of finite coefficient fields and exact finite operators. The main difference is the guiding question. Here the central objects are not abstract finite-field Hilbert-space surrogates alone, but wave operators tied to the Euclidean/Lorentzian split of the FRC shell hierarchy and evaluated by exact information counts.

Scope of the Exact Counts. The exact counts established here do *not* by themselves define:

1. a thermodynamic entropy production law;
2. a probabilistic entropy on finite wave amplitudes;
3. a packet-level shell-transfer theorem;
4. a particle interpretation of the loss factors.

They distinguish reversible propagation from compressive shell maps at the level of exact finite cardinalities.

8. Conclusions and Outlook

Conclusions. Three exact finitist structures emerge.

1. Euclidean shells carry Hermitian-preserving Cayley dynamics built from self-adjoint finite Hamiltonians.

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2. The Lorentzian extension carries an explicit Dirac operator with exact Clifford relations, exact spinorial Klein-Gordon factorization, and exact lifted boost covariance.
3. Finite shell maps carry exact information counts and loss factors: reversible maps have loss factor 1, while power maps lose distinguishability by the arithmetic factor $\gcd(\varepsilon, \mathsf{p}-1)$.

These results are proved entirely inside finite algebra. No appeal is made to limits, differential geometry, continuous spectra, or measure-theoretic entropy.

Immediate Extensions. The natural next steps remain finite and concrete.

1. Classify the self-adjoint finitist Hamiltonians on $\mathbb{F}_{\mathsf{p}}^{d}$ and the exact periods of their Cayley propagators.
2. Extend the explicit boost lift of Section 5 to larger finitist subgroups of $O(Q_\nu, K)$ and compare their information preservation properties.
3. Study exact loss factors for larger families of shell endomorphisms beyond the simple power maps treated here.
4. Relate these information counts to shell transfer and invariant extraction.

Further Entropy Questions. A natural next step is to build stronger finitist entropy formalisms on top of the exact counts proved here. Possible directions include:

1. finite partition entropies derived from shell equivalence classes;
2. exact entropy balance laws for reversible dynamics followed by compressive coarse-graining;
3. composite-shell information counts using the Chinese Remainder structure from paper 1.

Each of these directions can be pursued without abandoning strict finiteness.

Final Remark. Wave dynamics, reversibility, and information loss can therefore be treated in a unified setting while staying entirely inside finite arithmetic. The resulting framework is exact, algebraic, and strictly finitist throughout.

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