Brief Report

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Brief Report

Mathematical Duality: The Ramsey Approach

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Abstract: The meta-mathematic approach to the mathematical duality is introduced. Systems of mathematical theorems/propositions related each to other by the mutually exclusive relations of "duality" and "non-duality" are addressed. Theorems/propositions are considered as the vertices of the graph, and the relations of duality appear as the links connecting the vertices. Thus, the bi-colored, complete graph emerges. The coloring procedure is exemplified with the theorems of projective geometry. It is demonstrated, that that the graph built of the quartet of vertices/theorems of projective geometry, which contains no mono-colored triangle is possible. The monochromatic triangle will necessarily appear in the graph containing five vertices/theorems of projective geometry. The emerging graph is different from the traditional Ramsey graph. The graph representation is easily generalized for any system (finite or infinite) of theorems/propositions in which the relations of duality are established.

Keywords: duality; theorems; graph; projective geometry; monochromatic triangle; infinite Ramsey theory; meta-mathematics

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1. Introduction

Duality is a multi-faceted, fundamental phenomenon, which takes a central place in the philosophy of mathematics (Krömer and Corfield, 2014, Catren and Cukierman, 2022, Krömer and Haffner, 2024, p.101). The notion of duality possesses a "fine structure", namely, we recognize: i) Stone duality, which establishes that a category of "algebraic objects" (Boolean algebras) is the categorical dual of a category of "geometrical" objects (Stone spaces) (Awodey and Forssell, 2013; Gehrke, 2016); ii) variational duality (Mond and Hanson, 1967); iii) projective duality, which emerges from the symmetries between points and lines in planar projective geometry and between points and planes in solid projective (Eder, 2021, Tevelev, 2005, pp. 1-16); iv) Poincare duality, that links the dual notions of homology groups and cohomology groups for manifolds (Spivak, 1967); v) de Morgan logical duality (Gastaldi, 2024); vi) Fourier duality (Dutkay and Jorgensen, 2012); vii) linear programming duality (Bachem and Kern, 1992).

Let us try to define accurately the notion of duality (the meaning and use of "duality" remain too broad and even obscure in the literature). The common sense based definition of duality means the possibility to associate distinguishable objects in pairs. From the more rigorous mathematical point of view duality transforms concepts, theorems or mathematical structures into other concepts, theorems or structures in a one-to-one fashion, often (but not always) by means of an involution transformation : if the dual of Ψ is Ω, then the dual of Ψ is A.

The Langlands program, relating disconnected areas of mathematics, such as the number theory and algebraic geometry, put the notion of duality in the focus of mathematical and philosophical discourse. It seems that historically first step in this direction was made by Fermat and Descartes in the XVII century, Fermat and Descartes, being titans of the nascent European science, independently invented analytic geometry, that is, the study of the correspondence/duality between algebraic expressions and geometric figures by means of given coordinate system (Catren and Cukierman, 2022).

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The notion of duality is also in the focus of modern physics (Savit, 1980, Mitra, 1986, Girvin, 1996, Rowe, Carvalho and Repka, 2012). And in the realm of physics duality also possesses a fine structure. We distinguish: i) Maxwell duality, emerging from the symmetry of $\vec{E}$ (electric) and $\vec{B}$ (magnetic) fields in the Maxwell equations; ii) fundamental duality of p- and q- variables in classical mechanics (Mitra, 1986, Goldstein, 1959); iii) dual variables in continuum mechanics (Giessen and Kollmann,1996); iv) power law duality in classical and quantum mechanics (Inomata and Junker, 2021); v) dual pairing of symmetry and dynamical groups in atomic and nuclear physics, (Carvalho and Repka 2012); vi) dual variational principles of physics (Gray, Karl and Novikov, 1996, Gray CG, Karl and Novikov 1999); vii) duality of waves and particles in quantum mechanics (Messiah, 2014). And the aforementioned list is not complete.

The duality of syntax and semantics was discussed in linguistics (Demey and Smessaert, 2016). The well-known example arising from linguistics addresses the duality between the adverbs "already" and "still" in human language: already outside means the same as not still inside, and therefore, not already outside means the same as still inside (where adverb "inside" is taken to be synonymous to "located not outside") (Demey and Smessaert, 2016). Similar examples demonstrate that dualities based on external/internal antithesis bring into existence a broad variety of dual logical and linguistic operators.

We suggest the novel approach to the problem of mathematical duality based on the modern graph theory, or more rigorously speaking, on the Ramsey theory of the bi-colored, complete graphs (Li and Lin, 2020). Thus, the synthesis of ideas emerging from the theory of mathematical duality and ideas rooted in the graph theory becomes possible. Frank Plumpton Ramsey was a brightly talented British philosopher and logician, who made major essential contributions to both mathematics and philosophy before his untimely death at the age of 26. He was a colleague of Ludwig Wittgenstein in Cambridge, and translated "Tractatus Logico-Philosophicus" into English. It was reported, that transmission of some ideas of the pragmatist tradition to Ludwig Wittgenstein occurred through the intermediary of F. P. Ramsey, with whom he had numerous fruitful and inspiring discussions at Cambridge in 1929 (Marion, 1995, Marion, 2012).

Actually complete logical graphs appeared first in the papers devoted to the Aristotelian square of oppositions and to the John Buridan's octagons of opposition (Smessaert and Demey 2014; Demey 2018). Aristotelian square of oppositions and the John Buridan octagon of opposition may be seen as four-colored complete graphs built of four and eight vertices correspondingly (Smessaert and Demey 2014; Demey 2018).

2. Ramsey Scheme for the Dual Mathematical Structures

Now we propose the procedure enabling conversion of mathematical schemes into the bi-colored, complete graphs. We illustrate the idea with the set of theorems related to the projective geometry; however, the suggested scheme enables conversion of any (finite or infinite) set of propositions into the graph (Eder, 2021). In projective geometry, duality is a logical formalization of the symmetry of the roles fulfilled by points and lines in the definitions and theorems of projective planes. In its simplest meaning the projective duality is interpreted as follows: in a projective plane a statement involving points, lines and incidence between them that is obtained from another such statement by interchanging the words "point" and "line" and making appropriate grammatical corrections, is considered the plane dual statement of the first statement (Eder, 2021). The plane dual statement of "Two points are on a unique line" is "Two lines meet at a unique point". Forming the plane dual of a statement is known as dualizing the statement.

For example, the Desargues theorem is dual to the converse/dual of Desargues' theorem; the Pascal's theorem is dual/converse to the Brianchon theorem; the Menelaus theorem is dual to the Ceva theorem. Let us illustrate the projective duality with famous Desargues' theorem (Courant and Robbins, 1996, pp. 185-192, Eder, 2021):

Desargues' theorem: if corresponding vertices of two triangles (labeled ABC and $A'B'C'$ in **Figure 1**) are located on three lines that meet in a point (which is called the perspective center

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(homology center, or pole, appearing as point O in Figure 1), then corresponding sides of the triangles meet in three points that lie on a line (points *K*, *L* and *M* in Figure 1).

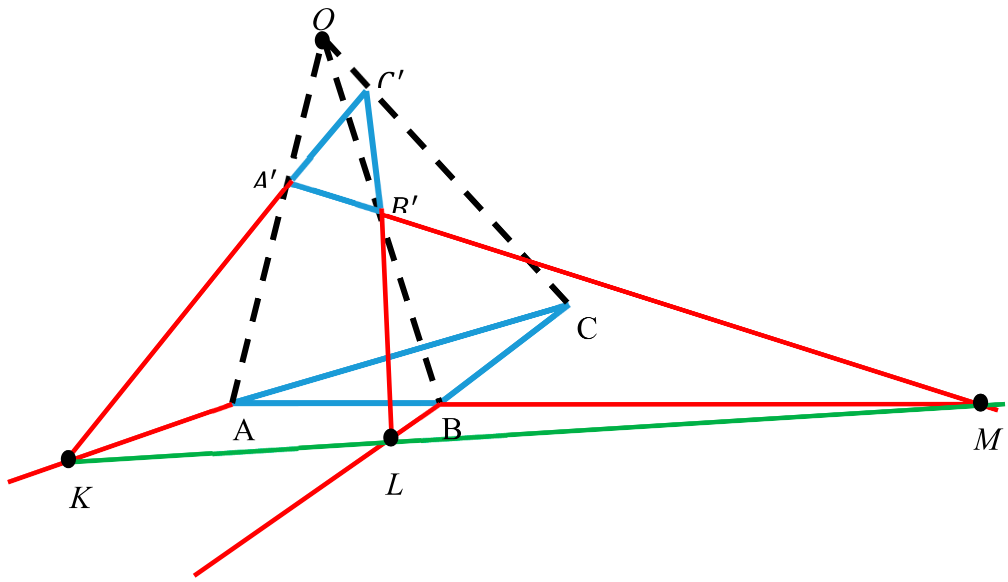

Figure 1. Sketch illustrating Desargues' theorem. Point *O* is a perspective center. Points *K*, *L* and *M* are located on the same line.

Dual of Desargues' theorem: If corresponding sides of two triangles meet in three points that lie on a line, then corresponding vertices of the triangles lie on three lines that meet in a point. After dualizing the Desargues theorem, we obtain with the theorem about the projective plane just like Desargues' theorem itself. In this particular case, we speak about self-duality and the dual proposition is the converse of the Desargues theorem. i.e. axial perspectivity is translated into central perspectivity and *vice versa*.

The Pascal theorem of projective geometry states, that "if the vertices of hexagon lie alternatively on two straight lines, the points of where opposite sides meet are collinear. The Brianchon theorem which is dual to the Pascal one, in turn, says: "if the sides of the hexagon pass alternatively through two points, the lines joining opposite vertices are concurrent (Courant and Robbins, 1996, p. 191).

Now we consider the set of four theorems or projective geometry $T = \{T_1, \dots, T_4\}$ containing the Pascal theorem, denoted PT and the Brianchon theorem, abbreviated BT, dual one to another; the Menelaus theorem, labeled MT, and the Desargues theorem labeled DT, which are not dual one to another; thus: $T = \{PT, BT,\ MT, DT\}$. These theorems/propositions are considered as the vertices as the graph. The propositions are connected with the black link, when they are dual one to another, and they are connected with the red link, when they are not dual one to another, as illustrated with Figure 2.

Figure 2. Coloring of the suggested graph built of theorems is illustrated. Vertices represents the theorems of the projective geometry. **A**. Pascal (PT) and Brianchon theorem (BT) are dual one to another, and they are connected with the black link. Pascal (PT) and Desargues theorem (DT) are not dual one to another, and they are connected with the red link.

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Dual theorems, connected with the black links, are considered as the “friends” in the terms of the Ramsey theory, and not dual theorems, connected with a red link are treated, in turn, as “strangers” (Katz and Reimann, 2018, pp. 1-34).

Now consider the entire quartet of theorems $T = \{PT, BT, MT, DT\}$ represented by the graph shown in Figure 3.

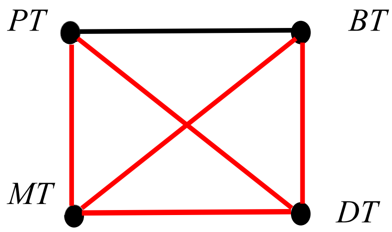

Figure 3. Bi-colored complete graph, formed by the quartet of theorems: $T = \{PT, BT, MT, DT\}$.

It is recognized from **Figure 3** that the set of theorems $T = \{PT, BT, MT, DT\}$ form the bi-colored, complete graph, when we describe it in the terms of the graph theory (Katz and Reimann, 2018). We latently adopted that a single dual theorem appears for a given one, and this suggestion is very important for our future treatment. Two mono-colored red triangles (namely PT, DT, MT and BT, DT, MT) are recognized in **Figure 3**. However, it is in principle possible to prepare the bi-colored, complete graph built of four vertices/theorems of projective geometry, which does not contain any mono-colored triangle. Consider the quartet of theorems $T = \{PT, BT, MT, CT\}$, where CT denotes the Ceva theorem. The corresponding graph is shown in Figure 4.

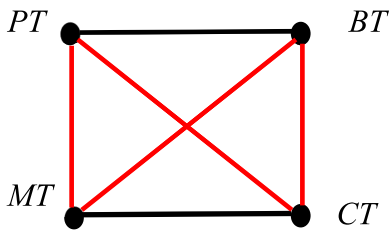

Figure 4. Bi-colored, complete graph, formed by the quartet of theorems $T = \{PT, BT, MT, CT\}$. No mono-colored triangle is recognized in the graph.

No mono-colored triangle appears in the complete, bi-colored graph, shown in **Figure 4**. However, such a monochromatic triangle will necessarily appear in the graph representing five theorems of projective geometry, related each to other with the relations of duality/non-duality. Consider the set of five vertices/theorems $T = \{PT, BT, MT, CT, DT\}$, and the corresponding graph, shown in Figures 5A-B.

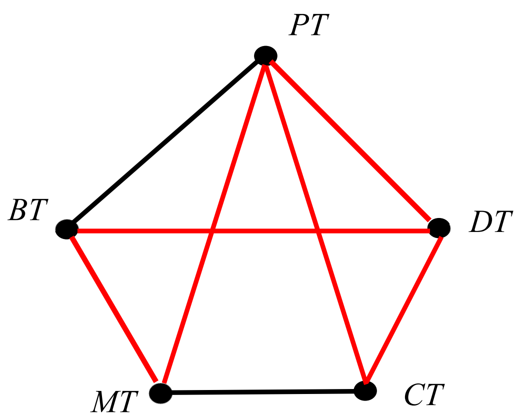

Figure 5A. Bi-colored, complete graph, formed by the five vertices/theorems $T = \{PT, BT, MT, CT, DT\}$. Triangle PT, DT, CT is a mono-colored red one.

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The bi-colored, complete graphs, shown in Figure 5-B, formed by the five vertices/theorems $T = \{PT, BT,\ MT, CT,\ DT\}$ contain the mono-colored, red triangle PT, DT, CT. Moreover, any graph built of five theorems of projective geometry will necessarily contain at least one mono-colored red triangle. Let us demonstrate this theorem. First of all, consider that such a graph may include maximum two black links/edges. Indeed, we adopted that any theorem has only one dual one. Thus, adjacent black links/edges possessing the common vertex, are forbidden. Hence, if the location of the black link is prescribed, there are possible only two possible locations of the black link/edge (as shown in Figures 5A-B), and the third black link could not be introduced into the graph. In the terms of the graphs theory we say: the black clique does not exist (Li and Lin, 2020).

It should be emphasized, that the suggested coloring procedure is applicable to any system of theorems/propositions related each to other by the mutually exclusive relations of "duality" and "non duality".

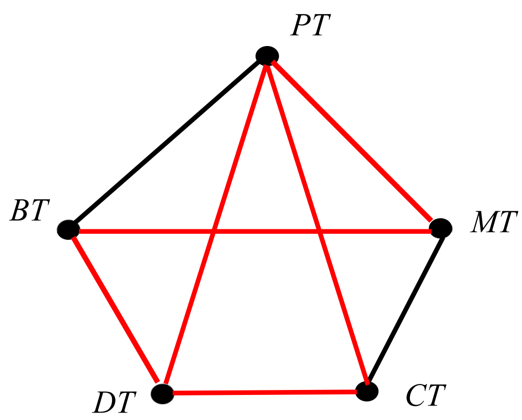

Figure 5B. Bi-colored, complete graph, formed by the five vertices/theorems $T = \{PT, BT,\ MT, CT,\ DT\}$. Triangle PT, DT, CT is a mono-colored red one.

It also should be stressed, that the relation "not being dual" is not transitive. Consider the following example: The Brianchon theorem is not dual to the Desargues theorem, the Desargues theorem is not dual to the Pascal one; however, the Brianchon theorem is dual to the Pascal one. This is important in a view of the fact, that the Ramsey numbers calculated for the transitive graphs are different from those obtained for non-transitive ones (Choudum and Ponnusamy, 1999, Gilevich *et al*. 2024).

3. Discussion

The introduced scheme, enabling conversion of the set of theorems/propositions into the bi-colored, complete graph, is applicable for any (finite or infinite) set of theorems for which the relations of duality are pre-established. The emerging graph is different from the classical Ramsey graph. Indeed, for the classical Ramsey graph $R(3,3) = 6$ is true. This means that in the complete, bi-colored graph, built of six vertices, at least one mono-colored triangle is necessarily present. In the graph introduced in the paper the mono-colored triangle is already present in the graph built of five vertices/theorems. This observation follows from the inner logical structure of the graph, addressed in detail in the previous Section.

Our approach develops the idea, introduced by Demey and Smessaert, to present the logical relations between propositions as a graph (Smessaert and Demey 2014, Demey, 2018). We re-shaped this idea in the form of complete, bi-colored graph, similar to, but different from the classical Ramsey graphs. Within our approach "black cliques" (cliques of "friends") are impossible within the suggested bi-colored complete graph.

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Extension to the infinite set of theorems/propositions related each to other with the mutually exclusive relation of duality/non duality is straightforward. Consider the infinite set of theorems/propositions. Now, the infinite Ramsey theorem is applicable for the emerging bi-colored, complete, infinite graph (Soifer, 2009, pp. 263-265, Bormashenko, 2025). The infinite Ramsey theorem, in turn, is formulated as follows: let K_ω denote the complete graph on the vertex set N. For every $c > 0$, if we color the edges of K_ω with c colors, then there must be present an infinite monochromatic clique. The infinite Ramsey theorem re-formulates the seminal Dirichlet pigeonhole principle, which states that if there exists n pigeonholes containing $n+1$ pigeons, one of the pigeonholes necessarily must contain at least two pigeons (Soifer, 2009, pp. 263-265, Bormashenko, 2025). In our case we have two colors (black/dual and red/not dual), and black cliques are forbidden. Thus, the presence of the red infinite clique in the infinite graph built of theorems/propositions is inevitable.

4. Conclusions

Synthesis of the concepts of mathematical duality and mathematical logic becomes possible within the graph theory. Theorems/propositions of the mathematics are seen as the vertices of the graph connected each to other with the mutually exclusive relations of duality/non duality. Theorems may be “dual” or “not dual” each to other. We also adopt that a single dual theorem corresponds to the given one. Thus, the bi-colored, complete graph emerges. Dual theorems are considered as the “friends” in the terms of the Ramsey theory, and not dual theorems are treated as “strangers”. This approach gives rise to the meta-mathematical envelope to any of mathematical theories, in which relations of duality between propositions are established.

The emerging complete, bi-colored graph is different from the classical Ramsey graph. For the classical Ramsey graph the Ramsey number $R(3,3) = 6$ is true. This means, that in the complete, bi-colored graph, built of six vertices, at least one mono-colored triangle is necessarily present. In the graph introduced in the paper the mono-colored triangle is already present in the graph built of five vertices/theorems. This observation follows from the inner logical structure of the graph; we assume that a single dual theorem exists for a given theorem. Thus, the clique built of dual theorems (“friends” in the terms of the Ramsey theory) is impossible in the graph.

The extension of the suggested approach to the infinite, however countable set of theorems/propositions is straightforward. Again, cliques built of dual theorems are forbidden. An infinite clique built of non-dual theorems will be necessarily present in the infinite graph. We conclude that the graph interpretation of mathematical duality yields fruitful meta-mathematical approach to the finite and infinite sets of theorems/propositions. Multi-colored logical graphs should addressed in the future investigations (Smessaert and Demey 2014, Demey, 2018).

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