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Strict Sense Minimizers Which Are Not Extended Minimizers and Abnormality

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Abstract: We consider a constrained optimal control problem and an extension of it, in which the set of strict-sense trajectories is enlarged. Extension is a common procedure in optimal control, used to derive necessary and sufficient optimality conditions for the original problem from the extended one, which usually admits a minimizer and has a more regular structure. However, this procedure fails if the two problems have different infima. It is therefore relevant to identify such situations. Following on from earlier work by Warga but adopting perturbation techniques developed in non-smooth analysis, we investigate the relation between the occurrence of an infimum gap and abnormality of necessary conditions. For a notion of local minimizer based on control distance and an extension including the impulsive one, we prove that (i) a local extended minimizer which is not a local minimizer of the original problem and (ii) a local strict-sense minimizer which is not a local minimizer of the extended problem, both satisfy the extended maximum principle in abnormal form. The main novelty is result (ii), as until now it had only been shown that a strict-sense minimizer which is not an extended minimizer is abnormal for an 'averaged version' of the maximum principle.

Keywords: optimal control problems; maximum principle; state constraints; gap phenomena; impulsive optimal control

MSC: 49K15; 34K45; 49N25

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1. Introduction

It is common practice in the fields of Calculus of Variations and Optimal Control to extend the space of solutions for problems that cannot be solved in a, say, ordinary space, or if the solution is difficult to find, even with numerical approximation. This process, known as *extension*, involves compactifying and regularizing the problem, resulting in a more manageable structure and the possibility of obtaining necessary and sufficient conditions for optimality. However, for it to be considered a well-posed extension, it is crucial that there is no gap between the infimum of the original problem and the infimum of the extended problem. Otherwise, the extended problem will not provide any useful information about the original problem. This gap also causes issues for the method of dynamic programming, as the solution to the corresponding Hamilton-Jacobi equation typically coincides with the value function of the extended problem. However, even if the set of strict-sense solutions is L^∞-dense in the set of extended paths, the presence of constraints often leads to the occurrence of an infimum gap. In particular, this problem arises when all strict-sense solutions close to a feasible extended trajectory, for instance a local minimizer, fail to meet the constraints.

The main purpose of this paper is to establish new necessary conditions for an infimum gap to occur between problem (P) and its extension (P_e) below. The fulfillment of these conditions requires the use of necessary optimality conditions, expressed through the maximum principle, in abnormal form (i.e., with cost multiplier zero for some set of multipliers).

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Fixed $T > 0$ and $\check{x}_0 \in \mathbb{R}^n$, we consider the minimization of

$$\Psi(y(T))$$

subject to the dynamic constraint

$$\dot{y}(t) = \mathcal{F}(t, y(t), \omega(t), \alpha(t)) \quad \text{for a.e. } t \in [0, T], \qquad y(0) = \check{x}_0, \tag{1}$$

and to the state and endpoint constraints

$$h(t, y(t)) \leq 0 \quad \forall t \in [0, T], \qquad y(T) \in \mathcal{T}. \tag{2}$$

The data comprise the bounded, but not necessarily closed set of control values $V \subset \mathbb{R}^m$, the compact set $A \subset \mathbb{R}^q$, the closed target set $\mathcal{T} \subset \mathbb{R}^n$, and the functions $\Psi : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{F} : \mathbb{R} \times \mathbb{R}^n \times \overline{V} \times A \to \mathbb{R}^n$, and $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ (see Section 2). We set $\mathcal{A} := L^1([0, T]; A)$, and then introduce two sets of admissible controls:

$$\mathcal{V} := L^1([0, T], V) \quad \text{and} \quad \mathcal{W} := L^1([0, T]; \overline{V})$$

($\overline{V}$ denotes the closure of V). We refer to any triple (ω, α, y) as an *extended process*, or simply a *process*, when $(\omega, \alpha) \in \mathcal{W} \times \mathcal{A}$ are the control functions and $y \in W^{1,1}([0, T]; \mathbb{R}^n)$ is the solution of (1) associated with (ω, α). A process (ω, α, y) is a *strict sense process* if $\omega \in \mathcal{V}$. A strict sense or extended process (ω, α, y) is *feasible* when it satisfies the constraints (2). Let Γ_s and Γ_e denote the subsets of feasible strict sense processes and feasible extended processes, respectively. Hence, we consider the following optimal control problems:

$$(P_s) \qquad \inf_{(\omega,\alpha,y)\in\Gamma_s} \Psi(y(T)) \qquad (\textit{strict sense problem})$$

and

$$(P_e) \qquad \inf_{(\omega,\alpha,y)\in\Gamma_e} \Psi(y(T)) \qquad (\textit{extended problem}).$$

Clearly, $\Gamma_s \subseteq \Gamma_e$, so that $\inf_{\Gamma_e} \Psi(y(T)) \leq \inf_{\Gamma_s} \Psi(y(T))$. When this inequality is strict, one usually says that *there is an infimum gap* or that *the Lavrentiev phenomenon occurs*. In particular, introducing a notion of local minimizer based on control distance (see Section 2), we distinguish:

- a *type-E local infimum gap*, when the cost of a local minimizer of the extended problem is strictly smaller than the local infimum of the original, strict sense problem,
- *type-S local infimum gap*, whether a local strict sense minimizer is not a local minimizer of the extended problem.

Under the assumptions specified in Section 2, the main results of this paper can be summarized as follows:

(i) *if at $(\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_e$ there is a type-E local infimum gap, then $(\bar{\omega}, \bar{\alpha}, \bar{y})$ satisfies the maximum principle in abnormal form;*

(ii) *if $(\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_s$ is a local minimizer of (P), then it satisfies the same maximum principle as the extended problem. If in addition at $(\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_s$ there is a type-S local Ψ-infimum gap, then $(\bar{\omega}, \bar{\alpha}, \bar{y})$ is an abnormal minimizer.*

As we will illustrate in Section 5, problem (P) and its extension (P_e) include the impulsive extension of a class of non-smooth, constrained optimal control problems. These problems involve unbounded original dynamics and the customary assumptions of coercivity, which prevent minimizing sequences to converge to discontinuous paths, are not invoked. We point out that consideration of impulsive systems is crucial in many applications (see, e.g. [8,12,22]). For example, instances in mechanics are situations where some state parameters $(u_1, \ldots, u_m)$ are treated as controls [9,11].

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Warga was the first to study the correlation between the presence of an infimum gap and the validity of the maximum principle in abnormal form for a classical extension by relaxation. He announced the result for a type-S local infimum gap in his early paper [42], which focused on state constraint-free optimal control problems with smooth data. In the monograph [43], Warga proved the relationship between gap and abnormality for a type-E local infimum gap in optimal control problems with state constraints. His subsequent work [45] extended this result to include nonsmooth data, utilizing the concept of 'derivative containers' introduced in [44]. Vinter and Palladino [37] proved the above mentioned correlation both in case of type-E and type-S local infimum gap for the classical extension by convex relaxation of a class of non-smooth state-constrained optimal control problems which subsume those considered by Warga, and under less restrictive hypotheses on data. Their techniques differ significantly from Warga's, as Warga used approximating cones to reachable sets, while Vinter and Palladino utilize the non-smooth maximum principle, expressed in terms of subdifferentials, originally formulated by Clarke [13]. More recently, following the latter approach, results of this kind were established in [34], [16] for the impulsive extension of optimal control problems with unbounded dynamics (without and with state constraints, respectively), and in [17–19] for an abstract extension, including both relaxation and impulsive extension as special cases. In particular, in [16–18] we also provide, for the first time, sufficient conditions for the nondegeneracy of the abnormality condition related with a type-E infimum gap.

However, all these works focus primarily on type-E local infimum gap and consider L^∞-local minimizers (i.e. local minimizers with respect to the L^∞-distance of trajectories). Specifically, apart from Warga's initial work, type-S local infimum gap is only studied in [37], for the extension by convexification of the dynamics, and in proceeding [19], for a more general extension. In both papers, the result is not entirely satisfactory, however, because it is shown that a strict sense L^∞-local minimizer that is not also an extended minimizer, satisfies in abnormal form an 'averaged version' of the maximum principle, which is much less informative than the actual maximum principle.

In this paper, for the extension under consideration, on the one hand, we fill the gap that was left in the previous literature between the results obtained for type-E infimum gap and type-S infimum gap, respectively, by showing that in both cases the local minimizer is abnormal for the maximum principle associated with the extended problem. On the other hand, we extend the previous results for type-E local infimum gap to the case of a notion of local minimizer based on control distance rather than trajectory distance. Note that, by the continuity property of the input-output map associated with the control system, this implies that the present results imply the previous ones.

The paper is organized as follows. In Section 2 we collect notation, useful definitions, and the precise assumptions. In Section 3 we rigorously introduce the concepts of type-E and type-S local infimum gap and state our main results, proved in Section 5. Section 4 is devoted to apply these results to the impulsive extension of a control-affine system with unbounded controls. We also give an example. Section 6 contains some concluding remarks.

2. Notation ands basic assumptions

2.1. Notation and preliminaries

Given $T > 0$ and a set $X \subseteq \mathbb{R}^k$, we write $W^{1,1}([0,T],X)$ $L^1([0,T],X)$, $L^\infty([0,T],X)$, for the space of absolutely continuous functions, Lebesgue integrable functions, essentially bounded functions defined on $[0,T]$ and with values in X, respectively. For all the classes of functions introduced so far, we will not specify domain and codomain when the meaning is clear and we will use $\|\cdot\|_{L^1(0,T)}$, $\|\cdot\|_{L^\infty(0,T)}$, or also $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^\infty}$ to denote the L^1 and the ess-sup norm, respectively. Furthermore, we denote by $\ell(X)$, $\text{co}(X)$, $\overline{X}$, ∂X the Lebesgue measure, the convex hull, the closure, and the boundary of X, respectively. As customary, χ_X is the characteristic function of X, namely $\chi_X(x) = 1$ if $x \in X$ and $\chi_X(x) = 0$ if $x \in \mathbb{R}^k \setminus X$. Given a closed set $\mathcal{O} \subseteq \mathbb{R}^k$ and a point $z \in \mathbb{R}^k$, we define the distance of z from $\mathcal{O}$ as $d_{\mathcal{O}}(z) := \min_{y \in \mathcal{O}} |z - y|$. For any $a, b \in \mathbb{R}$, we write $a \vee b := \max\{a, b\}$. We use $NBV^+([0,T],\mathbb{R})$

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to denote the space of monotone non decreasing, real valued functions μ on $[0,T]$ of bounded variation, vanishing at the point 0 and right continuous on $]0,T[$. Each $\mu \in NBV^+([0,T],\mathbb{R})$ defines a Borel measure on $[0,T]$, still denoted by μ, its total variation function is indicated by $\|\mu\|_{TV}$ or by $\mu([0,T])$, and its support is $\text{spt}(\mu)$. If $(\mu_i) \subset NBV^+([0,T],\mathbb{R})$, we say that $\mu_i \rightharpoonup^* \mu \in NBV^+([0,T],\mathbb{R})$ if $\int_{[0,T]} \phi\mu_i(dt) \to \int_{[0,T]} \phi\mu(dt)$ for any continuous function $\phi : [0,T] \to \mathbb{R}$.

Some standard constructs from non-smooth analysis are employed in this paper. For background material we refer the reader for instance to [13,40]. A set $K \subseteq \mathbb{R}^k$ is a *cone* if $ck \in K$ for any $c > 0$, whenever $k \in K$. Take a closed set $D \subseteq \mathbb{R}^k$ and a point $\bar{x} \in D$, the *limiting normal cone* $N_D(\bar{x})$ of D at $\bar{x}$ is given by

$$N_D(\bar{x}) := \left\{ \eta \in \mathbb{R}^k : \exists x_i \overset{D}{\to} \bar{x},\, \eta_i \to \eta \;\text{ such that }\; \limsup_{x \to x_i} \frac{\eta_i \cdot (x - x_i)}{|x - x_i|} \le 0 \;\; \forall i \right\},$$

in which the notation $x_i \xrightarrow{D} \bar{x}$ is used to indicate that all points in the converging sequence $(x_i)_i$ lay in D. Take a lower semicontinuous function $G : \mathbb{R}^k \to \mathbb{R}$ and a point $\bar{x} \in \mathbb{R}^k$, the *limiting subdifferential* of G at $\bar{x}$ is

$$\partial G(\bar{x}) := \left\{ \xi \colon \exists \xi_i \to \xi,\, x_i \to \bar{x} \text{ s.t. } \limsup_{x \to x_i} \frac{\xi_i \cdot (x - x_i) - G(x) + G(x_i)}{|x - x_i|} \le 0 \;\; \forall i \right\}.$$

If $G : \mathbb{R}^k \times \mathbb{R}^h \to \mathbb{R}$ is a lower semicontinuous function and $(\bar{x},\bar{y}) \in \mathbb{R}^k \times \mathbb{R}^h$, we write $\partial_x G(\bar{x},\bar{y})$, $\partial_y G(\bar{x},\bar{y})$ to denote the *partial limiting subdifferential of G at $(\bar{x},\bar{y})$ w.r.t. x, y*, respectively. When G is differentiable, ∇G is the usual gradient operator and $\nabla_x G$, $\nabla_y G$ denote the partial derivatives of G. Given a locally Lipschitz continuous function $G : \mathbb{R}^k \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^k$, the *hybrid subdifferential* of G at $\bar{x}$ is

$$\partial^> G(\bar{x}) := \text{co} \left\{ \xi \colon \exists (x_i)_i \subset \text{diff}(G) \setminus \{\bar{x}\} \text{ s.t. } x_i \to \bar{x},\, G(x_i) > 0 \;\forall i,\; \nabla G(x_i) \to \xi \right\},$$

where $\text{diff}(G)$ is the set of differentiability points of G. Finally, given a locally Lipschitz continuous function $G : \mathbb{R}^k \to \mathbb{R}^l$ and $\bar{x} \in \mathbb{R}^k$, we write $DG(\bar{x})$ to denote the *Clarke generalized Jacobian*, defined as

$$DG(\bar{x}) := \text{ co } \left\{ \xi \colon \exists (x_i)_i \subset \text{diff}(G) \setminus \{\bar{x}\} \text{ s.t. } x_i \to \bar{x} \text{ and } \nabla G(x_i) \to \xi \right\},$$

where now ∇G denotes the classical Jacobian matrix of G. If $G : \mathbb{R}^k \times \mathbb{R}^h \to \mathbb{R}^l$ and $(\bar{x},\bar{y}) \in \mathbb{R}^k \times \mathbb{R}^h$, $D_x G(\bar{x},\bar{y})$, $D_y G(\bar{x},\bar{y})$ denote the *Clarke generalized Jacobian of G at $(\bar{x},\bar{y})$ w.r.t. x, y*, respectively. We recall that it holds

$$p \cdot DG(x) = \text{co}\, \partial (p \cdot G)(x) \qquad \forall (x,p) \in \mathbb{R}^{n+n}. \tag{3}$$

2.2. Basic assumptions

We shall consider the following hypotheses, in which $(\bar{\omega},\bar{\alpha},\bar{y})$ is a feasible extended process, which we call the *reference process* and, for some $\theta > 0$, we set

$$\Sigma_\theta := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : \;\; t \in [0,T],\; x \in \bar{y}(t) + \theta\,\mathbb{B}\}.$$

(H1) *The Borel set $A \subset \mathbb{R}^q$ is compact and the Borel set $V \subset \mathbb{R}^m$ is bounded. Moreover, there exists a sequence $(V_i)_i$ of closed subsets of V such that*

$$V_i \subseteq V_{i+1} \quad \textit{for every } i, \qquad \bigcup_{i=1}^{+\infty} V_i = V.$$

(H2) *The cost function Ψ is Lipschitz continuous on a neighborhood of $\bar{y}(T)$. The target $\mathcal{T} \subseteq \mathbb{R}^n$ is closed. The constraint function h is upper semicontinuous and there exists $K_h > 0$ such that*

$$|h(t,x) - h(t,x')| \le K_h|x - x'| \quad \text{for any } (t,x),\ (t,x') \in \Sigma_\theta.$$

(H3) *For all $(x,w,a) \in \{x \in \mathbb{R}^n : (t,x) \in \Sigma_\theta \text{ for some } t \in [0,T]\} \times \overline{V} \times A$, $\mathcal{F}(\cdot,x,w,a)$ is Lebesgue measurable on $[0,T]$. Moreover, there exists $k \in L^1([0,T],[0,+\infty[)$ such that*

$$|\mathcal{F}(t,x,w,a)| \le k(t), \quad |\mathcal{F}(t,x',w,a) - \mathcal{F}(t,x,w,a)| \le k(t)|x' - x|, \tag{4}$$

for all (t,x,w,a), $(t,x',w,a) \in \Sigma_\theta \times \overline{V} \times A$. Furthermore, there exists some continuous increasing function $\varphi : [0,+\infty[\to [0,+\infty[$ with $\varphi(0) = 0$ such that for any $(t,x,a) \in \Sigma_\theta \times A$, we have

$$|\mathcal{F}(t,x,w',a) - \mathcal{F}(t,x,w,a)| \le k(t)\varphi(|w' - w|) \qquad \forall w',\ w \in \overline{V},$$
$$D_x\mathcal{F}(t,x,w',a) \subseteq D_x\mathcal{F}(t,x,w,a) + k(t)\varphi(|w' - w|)\,\mathbb{B} \qquad \forall w',\ w \in \overline{V}.$$

Remark 1. Condition **(H1)**, which is always satisfied when the set V is relatively open, implies (and in general is stronger than) the density of $\mathcal{V}$ in $\mathcal{W}$ in the L^1-norm. In particular, for any $\bar{\omega} \in \mathcal{W}$ and any $\varepsilon > 0$ there exists an integer i_ε such that the Hausdorff distance $d_H(V_i, \overline{V}) < \varepsilon/T$ for every $i \ge i_\varepsilon$. Hence, by the selection theorem [7, Theorem 2, p. 91] there is a measurable function $\omega_\varepsilon(t) \in \mathrm{proj}_{V_{i_\varepsilon}}(\omega(t))$ for a.e. t, such that

$$\|\omega_\varepsilon - \bar{\omega}\|_{L^1} \le T\|\omega_i - \omega\|_{L^\infty} \le T d_H(V_i, \overline{V}) \le \varepsilon.$$

Remark 2. Condition **(H3)** is satisfied, for instance, when

$$\mathcal{F}(t,x,w,a) = \mathcal{F}_1(t,x,a) + \mathcal{F}_2(t,x,w,a),$$

where $\mathcal{F}_1$, $\mathcal{F}_2$ verify hypothesis (4) and, in addition, the function $\mathcal{F}_2(t,\cdot,w,a)$ is C^1 and $\nabla_x\mathcal{F}_2$ is continuous on the compact set $\Sigma_\theta \times \overline{V} \times A$. Another situation where condition **(H3)** is verified, is when the dynamics function has a polynomial dependence on the control variable w, with locally Lipschitz continuous coefficients in the state variable.

3. Type-E or type-S local infimum gap and abnormality

3.1. Type-E and type-S local infimum gap

We recall that Γ_s and Γ_e denote the sets of feasible strict sense and feasible extended processes, respectively. Given $z = (\omega,\alpha,y)$, $\hat{z} = (\hat{\omega},\hat{\alpha},\hat{y}) \in \Gamma_e$, we define the following distance:

$$\mathbf{d}(z,\hat{z}) := \|\omega - \hat{\omega}\|_{L^1} + \ell\{t \in [0,T] :\ \alpha(t) \ne \hat{\alpha}(t)\}. \tag{5}$$

Definition 1 (Local minimizer)**.** *Let $\tilde{\Gamma}$ and $(\tilde{P})$ denote Γ_e and (P_e) or Γ_s and (P_s), respectively. A process $\bar{z} := (\bar{\omega},\bar{\alpha},\bar{y}) \in \tilde{\Gamma}$ is called a* local Ψ-minimizer *for problem $(\tilde{P})$ if, for some $\delta > 0$, one has*

$$\Psi(\bar{y}(T)) = \inf\left\{\Psi(y(T)) :\ z = (\omega,\alpha,y) \in \tilde{\Gamma},\ \mathbf{d}(z,\bar{z}) < \delta\right\}.$$

The process $\bar{z}$ is a Ψ-minimizer *for problem $(\tilde{P})$ if $\Psi(\bar{y}(T)) = \inf_{\tilde{\Gamma}} \Psi(y(T))$.*

Remark 3. Under hypothesis **(H3)**, for each extended control $(\omega,\alpha) \in \mathcal{W} \times \mathcal{A}$ in a suitable $\mathbf{d}$-neighborhood of the reference control $(\bar{\omega},\bar{\alpha})$, there is one and only one solution $y := y[\omega,\alpha]$ of (1). Furthermore, the input-output map $(\omega,\alpha) \mapsto y[\omega,\alpha]$ from $\mathcal{W} \times \mathcal{A}$ to C^0 is continuous in this neighborhood, provided $\mathcal{W} \times \mathcal{A}$ is equipped with the distance $\mathbf{d}$ and C^0 with the distance induced

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by the sup-norm. Consequently, if the process $\bar{z}$ is an L^∞-local minimizer, meaning that $\bar{z}$ reaches the minimum over processes $z = (\omega, \alpha, y)$ with $\|y - \bar{y}\|_{L^\infty} < \delta$ for some $\delta > 0$, then it is also a local minimizer according to Def. 1. In general, the contrary is not true.

It is now natural to introduce two notions of local infimum gap, depending on whether the reference process is extended or strict sense.

Definition 2 (Infimum gaps)**.** *Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function.* (i) *If $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_e$ and there is some $\delta > 0$ such that*[1]

$$\Psi(\bar{y}(T)) < \inf \left\{ \Psi(y(T)) : \;\; z = (\omega, \alpha, y) \in \Gamma_s, \;\; \mathbf{d}(z, \bar{z}) < \delta \right\},$$

we say that at $\bar{z}$ there is a type-E local Ψ-infimum gap. (ii) *Let $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_s$ be a local Ψ-minimizer for problem (P_s) which is not a local Ψ-minimizer for problem (P_e), i.e. for any $\varepsilon > 0$ there exists some $(\omega, \alpha, y) \in \Gamma_e$ such that*

$$\Psi(y(T)) < \Psi(\bar{y}(T)) \quad \textit{and} \quad \mathbf{d}(z, \bar{z}) < \varepsilon.$$

Then, we say that at $\bar{z}$ there is a type-S local Ψ-infimum gap.
(iii) *We say that there is a* Ψ-infimum gap *if* $\inf_{\Gamma_e} \Psi(y(T)) < \inf_{\Gamma_s} \Psi(y(T))$.

When Ψ is clear from the context, we will often simply write *infimum gap* instead of Ψ-infimum gap.

Remark 4. As it is easy to see, thanks to the continuity of the input-output map $(\omega, \alpha) \mapsto y[\omega, \alpha]$ the notion of type-E local Ψ-infimum gap at $\bar{z}$ is actually independent of the cost function Ψ, as it is equivalent to the fact that

$$\{z = (\omega, \alpha, y) \in \Gamma_s : \;\; \mathbf{d}(z, \bar{z}) < \delta\} = \emptyset \quad \text{for some } \delta > 0 \tag{6}$$

(see [17, Proposition 2.1]). If $\bar{z}$ satisfies (6), we say that it is an *isolated process*.

3.2. Main results

Now we introduce a Pontryagin maximum principle and a notion of normal and abnormal extremal for the extended optimization problem. Then we establish a link between abnormality and occurrence of a gap phenomenon.

Definition 3 (Pontryagin maximum principle)**.** *Let $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y})$ be a feasible extended process for problem (P_e) and let hypotheses* **(H1)-(H2)-(H3)** *be satisfied. We say that $\bar{z}$ is a* Ψ-extremal, *or* satisfies the Pontryagin

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[1] As customary, when the set is empty we define the infimum equal to $+\infty$.

maximum principle, *if there exist a path* $p \in W^{1,1}([0,T],\mathbb{R}^n)$, $\gamma \geq 0$, $\mu \in NBV^+([0,T],\mathbb{R})$ *and a Borel measurable and* μ*-integrable function* $m : [0,T] \to \mathbb{R}^n$ *satisfying the following conditions:*

$$\|p\|_{L^\infty} + \|\mu\|_{TV} + \gamma \neq 0, \tag{7}$$

$$-\dot{p}(t) \in \operatorname{co}\partial_x \{q(t)\cdot\mathcal{F}(t,\bar{y}(t),\bar{\omega}(t),\bar{\alpha}(t))\} \ \ a.e.\ t \in [0,T]; \tag{8}$$

$$-q(T) \in \gamma\partial\Psi(\bar{y}(T)) + N_{\mathcal{T}}(\bar{y}(T)); \tag{9}$$

for a.e. $t \in [0,T]$, *one has*

$$q(t)\cdot\mathcal{F}(t,\bar{y}(t),\bar{\omega}(t),\bar{\alpha}(t)) = \max_{(w,a)\in\overline{V}\times A} q(t)\cdot\mathcal{F}(t,\bar{y}(t),w,a); \tag{10}$$

$$m(t) \in \partial_x^{>} h(t,\bar{y}(t)) \qquad \mu\text{-}a.e.\ t \in [0,T]; \tag{11}$$

$$spt(\mu) \subseteq \{t \in [0,T] : h(t,\bar{y}(t)) = 0\}, \tag{12}$$

where

$$q(t) := \begin{cases} p(t) + \int_{[0,t[} m(t')\mu(dt') & t \in [0,T[, \\ p(T) + \int_{[0,T]} m(t')\mu(dt') & t = T. \end{cases}$$

We will call a Ψ*-extremal* normal *if all possible choices of* (p,γ,μ,m) *as above have* $\gamma > 0$, *and* abnormal *when it is not normal. Since the notion of abnormal* Ψ*-extremal is actually independent of* Ψ, *in the following abnormal* Ψ*-extremals will be simply called* abnormal extremals.

Theorem 1. *Let* $\bar{z} := (\bar{\omega},\bar{\alpha},\bar{y}) \in \Gamma_e$ *and let hypotheses* **(H1)-(H2)-(H3)** *be satisfied. Then,*
(i) *if* $\bar{z}$ *is a local* Ψ*-minimizer for* (P_e), *then* $\bar{z}$ *is a* Ψ*-extremal. If at* $\bar{z}$ *there is a type-E local* Ψ*-infimum gap, then* $\bar{z}$ *is an abnormal extremal;*(ii) *if* $\bar{z} \in \Gamma_s$ *is a local* Ψ*-minimizer for* (P_s), *then* $\bar{z}$ *is a* Ψ*-extremal. If at* $\bar{z}$ *there is a type-S local* Ψ*-infimum gap, then* $\bar{z}$ *is an abnormal extremal.*

The proof of Theorem 1, in which the notion of local minimizer adopted in this work, based on the control distance **d**, plays a crucial role, is given in Section 5. The main novelty of Theorem 1 is statement (ii), concerning the case where $\bar{z}$ is a local minimizer of the original problem which is not a local minimizer of the extended one. Indeed, in the previous literature (see [19,37]) it was proved in this case that $\bar{z}$ is an abnormal extremal for an 'averaged version' of the maximum principle only, meaning that the adjoint equation (8) was replaced by the following weaker differential inclusion

$$-\dot{p}(t) \in \operatorname{co}\left\{ \bigcup_{(w,a)\in\overline{V}\times A} \partial_x\Big(q(t)\cdot\mathcal{F}(t,\bar{y}(t),w,a)\Big)\right\} \qquad \text{a.e. } t \in [0,T],$$

in which all information on optimal control is lost.

Remark 5. It is worth mentioning that, despite hypothesis **(H1)** implies the density of $\mathcal{V}$ in $\mathcal{W}$ in the L^1-norm, it is well-known since the earliest work by Warga [45] and Kaskovz [24] that, in general, if the set of strict sense controls is merely an L^1-dense subset of the set of extended controls, the link between gap and abnormality established in Theorem 1 may fail (see for instance the example in [36, Sec. 9]).

As a direct consequence of Theorem 1, we obtain that normality is a sufficient condition for the absence of any type of local infimum gap.

Theorem 2. *Let* $\bar{z} := (\bar{\omega},\bar{\alpha},\bar{y}) \in \Gamma_e$ *and let hypotheses* **(H1)-(H2)-(H3)** *be satisfied. Then,* (i) *if* $\bar{z}$ *is a local* Ψ*-minimizer for* (P_e) *which is a normal* Ψ*-extremal, at* $\bar{z}$ *there is no type-E local* Ψ*-infimum gap. If, in addition,* $\bar{z}$ *is a* Ψ*-minimizer for* (P_e), *then there is no* Ψ*-infimum gap;* (ii) *if* $\bar{z} \in \Gamma_s$ *is a local* Ψ*-minimizer for* (P_s) *which*

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is a normal Ψ*-extremal, at* $\bar{z}$ *there is no type-S local* Ψ*-infimum gap, namely,* $\bar{z}$ *is a local* Ψ*-minimizer for* (P_e) *as well.*

4. An application: the impulsive extension

4.1. An impulsive optimal control problem

Consider the following free end-time optimal control problem with *unbounded, control-affine* dynamics:

$$(\mathcal{P}) \begin{cases} \text{minimize } \Psi(S, x(S), v(S)) \\ \text{over } S > 0,\ u \in L^1([0,S];U),\ (x,v) \in W^{1,1}([0,S];\mathbb{R}^{n+1}), \text{ s.t.} \\ (\dot{x}(s), \dot{v}(s)) = \Big(f(s,x(s)) + \sum_{j=1}^m g_j(s,x(s))u^j(s)),\ |u(s)|\Big) \quad \text{a.e. } s \in [0,S], \\ (x(0), v(0)) = (\check{x}_0, 0), \\ h(s,x(s)) \le 0 \quad \text{for all } s \in [0,S], \qquad (S, x(S)) \in \mathcal{T}^*, \qquad v(S) \le K, \end{cases}$$

in which $U \subseteq \mathbb{R}^m$, $\mathcal{T}^* \subset \mathbb{R}^{1+n}$, $f : \mathbb{R}^{1+n} \to \mathbb{R}^n$, $g_j : \mathbb{R}^{1+n} \to \mathbb{R}^n$ for any $j = 1, \dots, m$, $\Psi : \mathbb{R}^{1+n+1} \to \mathbb{R}$, and $h : \mathbb{R}^{1+n} \to \mathbb{R}$. We make the following assumptions on data:

(H4) $K > 0$ *is a fixed constant possibly equal to* $+\infty$*, the (unbounded) set of control values* U *is a closed cone, the target* $\mathcal{T}^*$ *is a closed set, the dynamics functions* f, g_j*, the constraint function* h*, and the cost function* Ψ *are locally Lipschitz continuous.*

Notice that $v(s)$ (sometimes called *fuel* or *energy*) coincides with the L^1-norm of the control function u on $[0,s]$. Assuming, as usual, the function $v \mapsto \Psi(s,x,v)$ merely monotone nondecreasing (see e.g. [32]), this problem is non-coercive, i.e. there are no conditions that prevent a minimizing sequence of trajectories from having increasing velocities and converging to a discontinuous path. Hence, adopting a by now standard extension, we embed the original problem into the *space-time* or *extended* problem (P_e) below, where the extended state variable is $(y^0, y, \nu) := (s, x, v)$, and extended trajectories are (s, x, v)-paths which are (reparameterized) L^∞-limits of graphs of the original trajectories [10,27,30,38,41]: [2]

$$(P_e) \begin{cases} \text{minimize } \Psi(y^0(T), y(T), \nu(T)) \\ \text{over } T > 0,\ (\omega^0, \omega) \in \mathcal{W}(T),\ (y^0, y, \nu) \in W^{1,1}([0,T];\mathbb{R}^{1+n+1}), \text{ s.t.} \\ \dot{y}^0(t) = \omega^0(t) \quad \text{a.e. } t \in [0,T], \\ \dot{y}(t) = f(y^0(t), y(t))\omega^0(t) + \sum_{j=1}^m g_j(y^0(t), y(t))\omega^j(t) \quad \text{a.e. } t \in [0,T], \\ \dot{\nu}(t) = |\omega(t)| \quad \text{a.e. } t \in [0,T], \\ (y^0, y, \nu)(0) = (0, \check{x}_0, 0), \qquad \big(y^0(T), y(T), \nu(T)\big) \in \mathcal{T}^* \times]-\infty, K], \\ h(y^0(t), y(t)) \le 0 \ \text{ for all } t \in [0,T], \end{cases}$$

where $\mathcal{W}(T) := L^1([0,T];W)$, being W the control set given by

$$W := \Big\{ (w^0, w) \in [0, +\infty[\times U : \ w^0 + |w| = 1 \Big\}.$$

[2] As it is well-known, a distributional approach, where u is replaced by a Radon measure, does not work unless $g_i = g_i(x)$ and the Lie brackets $[g_i, g_j](x) \equiv 0$ for every $i, j = 1, \dots, m$ (see e.g. [10,23]).

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Notice that, with any process (S,u,x,v) of the original problem $(\mathcal{P})$, by setting

$$\sigma(s) := s + v(s) \quad \text{for any } s \in [0,S], \qquad T := \sigma(S),$$

through the time-change $y^0 := \sigma^{-1}$ we can associate a process $(T,\omega^0,\omega,y^0,y,\nu)$ for (P_e) with $\omega^0 = \dot{y}^0 > 0$ a.e.. In particular, problem $(\mathcal{P})$ can be identified with the restriction of problem (P_e) to the set of processes with $\omega^0 > 0$ a.e.. In the following, we will refer to such restriction as *strict sense problem* (P_s) and to such processes as *strict sense processes*.

Therefore, the extension consists in considering extended processes $(T,\omega^0,\omega,y^0,y,\nu)$ where ω^0 may be zero on nondegenerate subintervals of $[0,T]$. On these intervals, the time variable $s = y^0$ is constant, while the state variable y evolves according to the 'fast' dynamics $\dot{y}(t) = \sum_{j=1}^m g_j(y^0(t),y(t))\omega^j(t)$. This explains why (P_e) is also called the *impulsive extension* of problem $(\mathcal{P})$, although it is a conventional optimization problem with bounded controls. In fact, one could give an equivalent s-based description of this extension using bounded variation trajectories and controls [2,5,26,29,31,33,39,46].

Adopting terminology of the present paper, we say that an extended or strict sense process $(T,\omega^0,\omega,y^0,y,\nu)$ is *feasible* [resp., an original process (S,u,x,v) is *feasible*] if it satisfies all constraints of problem (P_e) [resp., $(\mathcal{P})$]. The sets of feasible original, feasible extended and feasible strict sense processes are denoted by Γ^*, Γ_e and Γ_s, respectively. Given $z = (T,\omega^0,\omega,y^0,y,\nu)$ and $\hat{z} = (\hat{T},\hat{\omega}^0,\hat{\omega},\hat{y}^0,\hat{y},\hat{\nu}) \in \Gamma_e$, we define the distance:[3]

$$\mathbf{d}_{\text{imp}}(z,\hat{z}) := |T-\hat{T}| + \|(\omega^0,\omega)-(\hat{\omega}^0,\hat{\omega})\|_{L^1(0,T\wedge\hat{T})}. \tag{13}$$

At this point, the definitions of local minimizer and of type-E and type-S local Ψ-infimum gap (see Def. 1 and Def. 2) can be easily adapted to the impulsive extension by replacing the distance $\mathbf{d}$ defined in (5) with the distance $\mathbf{d}_{\text{imp}}$ given by (13). The unmaximized Hamiltonian associated with problem (P_e) above is given by

$$H(s,x,p_0,p,\pi,w^0,w) := p_0 w^0 + p \cdot \Big(f(s,x)w^0 + \sum_{j=1}^m g_j(s,x)w^j\Big) + \pi|\omega|$$

for all $(s,x,p_0,p,\pi,w^0,w) \in \mathbb{R}^{1+n+1+n+1} \times W$.

[3] Notice that $\mathbf{d}_{\text{imp}}$ is equivalent to the distance obtained replacing $T\wedge\hat{T}$ with $T\vee\hat{T}$ in the L^1-norm (possibly extending the controls to $\mathbb{R}$ constantly equal to 0), as $\|(\omega^0,\omega)-(\hat{\omega}^0,\hat{\omega})\|_{L^1(0,T\vee\hat{T})} - \|(\omega^0,\omega)-(\hat{\omega}^0,\hat{\omega})\|_{L^1(0,T\wedge\hat{T})} \le M|T-\hat{T}|$ for some constant $M > 0$.

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Definition 4. *We say that* $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{v}) \in \Gamma_e$ *is a* Ψ-extremal *if there exist a path* $(p_0, p) \in W^{1,1}([0,\bar{T}], \mathbb{R}^{1+n})$, $\gamma \geq 0$, $\pi \leq 0$, $\mu \in NBV^+([0,\bar{T}], \mathbb{R})$ *and Borel-measurable and* μ*-integrable functions* $(m_0, m) : [0,\bar{T}] \to \mathbb{R}^{1+n}$ *satisfying the following conditions:*

$$\|p_0\|_{L^\infty} + \|p\|_{L^\infty} + \mu([0,\bar{T}]) + \gamma \neq 0 \tag{14}$$

$$-(\dot{p}_0, \dot{p})(t) \in \operatorname{co} \partial_{s,x} H(\bar{y}^0(t), \bar{y}(t), q_0(t), q(t), \pi, \bar{\omega}^0(t), \bar{\omega}(t)) \qquad \text{a.e. } t \tag{15}$$

$$-(q_0(\bar{T}), q(\bar{T}), \pi) \in \gamma \partial \Psi(\bar{y}^0(\bar{T}), \bar{y}(\bar{T}), \bar{v}(\bar{T})) + N_{\mathcal{T}^* \times]-\infty, K]}(\bar{y}^0(\bar{T}), \bar{y}(\bar{T}), \bar{v}(\bar{T})) \tag{16}$$

$$\begin{aligned} &H(\bar{y}^0(t), \bar{y}(t), q_0(t), q(t), \pi, \bar{\omega}^0(t), \bar{\omega}(t)) \\ &\quad = \max_{(w^0,w)\in W} H(\bar{y}^0(t), \bar{y}(t), q_0(t), q(t), \pi, w^0, w) = 0 \quad \text{a.e. } t \end{aligned} \tag{17}$$

$$(m_0, m)(t) \in \partial^{>}_{s,x} h(\bar{y}^0(t), \bar{y}(t)) \qquad \mu\text{-a.e. } t \tag{18}$$

$$\operatorname{spt}(\mu) \subseteq \{t \in [0,\bar{T}] : h(\bar{y}^0(t), \bar{y}(t)) = 0\}, \tag{19}$$

where $(q_0, q) : [0,\bar{T}] \to \mathbb{R}^{1+n}$ *is given by*

$$(q_0, q)(t) := \begin{cases} (p_0, p)(t) + \int_{[0,t[} (m_0, m)(t')\mu(dt') & t \in [0,\bar{T}[, \\ (p_0, p)(\bar{T}) + \int_{[0,\bar{T}]} (m_0, m)(t')\mu(dt') & t = \bar{T}. \end{cases}$$

Moreover, if $\gamma \partial_v \Psi(\bar{y}^0(\bar{T}), \bar{y}(\bar{T}), \bar{v}(\bar{T})) = 0$ *and* $\bar{v}(\bar{T}) < K$*, then* $\pi = 0$*. Furthermore, if* $\bar{y}^0(0) < \bar{y}^0(\bar{T})$*, then* (14) *can be strengthened with*

$$\|p\|_{L^\infty} + \mu([0,\bar{T}]) + \gamma \neq 0. \tag{20}$$

We say that a Ψ*-extremal is* normal *if all sets of multipliers* $(p_0, p, \gamma, \pi, \mu, m_0, m)$ *as above have* $\gamma > 0$*, and* abnormal *when it is not normal.*

From Theorem 1 we can deduce the following results.

Theorem 3. *Let* $\bar{z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{v}) \in \Gamma_e$ *and assume hypothesis* **(H4)**. *Then,*
(i) *if* $\bar{z}$ *is a local* Ψ*-minimizer for* (P_e)*, then* $\bar{z}$ *is a* Ψ*-extremal. If at* $\bar{z}$ *there is a type-E local* Ψ*-infimum gap, then* $\bar{z}$ *is an abnormal extremal;*(ii) *if* $\bar{z} \in \Gamma_s$ *is a local* Ψ*-minimizer for* (P_s)*, then* $\bar{z}$ *is a* Ψ*-extremal. If at* $\bar{z}$ *there is a type-S local* Ψ*-infimum gap, then* $\bar{z}$ *is an abnormal extremal.*

Proof. The impulsive extended problem (P_e) has a free end-time, so the theory developed in the previous sections for fixed end-time problems does not apply straightforwardly. However, through a standard time rescaling procedure that applies to free end-time problems with Lipschitz continuous time dependence, we can embed problem (P_e) into a fixed end-time optimization problem, satisfying all the assumptions of Theorem 1 and for which, for example, $\bar{z}$ is still a local minimizer if it was so for (P_e). Precisely, let $\mathcal{W} := \mathcal{W}(\bar{T})$, $\mathcal{D} := L^1([0,\bar{T}]; [-1/2, 1/2])$ and consider the *rescaled problem*:

$$(P_e^r) \begin{cases} \qquad\qquad \text{minimize } \Psi(y^0(\bar{T}), y(\bar{T}), v(\bar{T})) \\ \text{over } (\omega^0, \omega) \in \mathcal{W},\ d \in \mathcal{D},\ (y^0, y, v) \in W^{1,1}([0,\bar{T}]; \mathbb{R}^{1+n+1}), \text{ s.t.} \\ \dot{y}^0(t) = (1 + d(t))\, \omega^0(t) \quad \text{a.e. } t \in [0,\bar{T}], \\ \dot{y}(t) = (1 + d(t))\, \mathcal{F}(y^0(t), y(t), \omega^0(t), \omega(t)) \quad \text{a.e. } t \in [0,\bar{T}], \\ \dot{v}(t) = (1 + d(t))\ |\omega(t)| \quad \text{a.e. } t \in [0,\bar{T}], \\ (y^0, y, v)(0) = (t_1, \check{x}_0, 0), \\ h(y^0(t), y(t)) \leq 0 \ \text{ for all } t \in [0,\bar{T}], \quad \big(y^0(\bar{T}), y(\bar{T}), v(\bar{T})\big) \in \mathcal{T}^* \times]-\infty, K], \end{cases}$$

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where, for any $(t, x, w^0, w) \in \mathbb{R}^{1+n} \times W$, we have set

$$\mathcal{F}(t, x, w^0, w) := f(t, x)\, w^0 + \sum_{j=1}^{m} g_j(t, x)\; w^j.$$

We refer to any element $(\omega^0, \omega, d, y^0, y, \nu)$ satisfying all constraints in (P_e^r) as a feasible rescaled extended process. If $\omega^0 > 0$ a.e., then $(\omega^0, \omega, d, y^0, y, \nu)$ is called a feasible rescaled strict sense process. For any pair of feasible rescaled extended processes $\zeta := (\omega^0, \omega, d, y^0, y, \nu)$, $\hat{\zeta} := (\hat{\omega}^0, \hat{\omega}, \hat{d}, \hat{y}^0, \hat{y}, \hat{\nu})$ we define the distance

$$\mathbf{d}^{\mathbf{r}}(\zeta, \hat{\zeta}) := \|(\omega^0, \omega, d) - (\hat{\omega}^0, \hat{\omega}, \hat{d})\|_{L^1(0,\bar{T})}.$$

Let us associate with the given reference process $\bar{z} = (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{\nu})$, the (feasible) rescaled process $\bar{\zeta} := (\bar{\omega}^0, \bar{\omega}, \bar{d} = 0, \bar{y}^0, \bar{y}, \bar{\nu})$. From a straightforward application of the chain rule and standard calculations it follows that for any $\delta > 0$ there exists some $\varepsilon \in]0, \delta[$ such that with each feasible rescaled extended process $\zeta := (\tilde{\omega}^0, \tilde{\omega}, \tilde{d}, \tilde{y}^0, \tilde{y}, \tilde{\nu})$ with $\mathbf{d}^{\mathbf{r}}(\zeta, \bar{\zeta}) < \varepsilon$, using the time-change

$$\tau(s) = \int_0^s \frac{ds'}{1 + \tilde{d}(s')}, \qquad s \in [0, \bar{T}],$$

we can associate the following feasible extended process

$$z = (T, \omega^0, \omega, y^0, y, \nu) := (\tau(\bar{T}), (\tilde{\omega}^0, \tilde{\omega}, \tilde{y}^0, \tilde{y}, \tilde{\nu}) \circ \tau).$$

satisfying $\mathbf{d}_{\text{imp}}(z, \bar{z}) < \delta$. Moreover, $\Psi((\tilde{y}^0, \tilde{y}, \tilde{\nu})(\bar{T})) = \Psi((y^0, y, \nu)(T))$.

As a consequence, if $\bar{z}$ is a local Ψ-minimizer for (P_e) for some $\delta > 0$, then $\bar{\zeta}$ is a local Ψ-minimizer for (P_e^r), at which there is a type-E local infimum gap as soon as at $\bar{z}$ there is a type-E local infimum gap. At this point, the proof of Theorem 3 can be derived applying Theorem 1 to the rescaled problem. We omit the details, which follow the same line as the proofs of [40, Theorem 8.7.1] and [17, Theorem 4.1]. □

Remark 6. With similar arguments as in [17], what we have done in this section can be easily generalized to control-polynomial impulsive problems, by which we mean that the dynamics of the original problem $(\mathcal{P})$ can be replaced by

$$(\dot{x}, \dot{v})(t) = \left(f(t, x) + \sum_{k=1}^{d} \Big(\sum_{1 \le j_1 \le \cdots \le j_k \le m} g^k_{j_1, \ldots, j_k}(t, x)\; u^{j_1} \cdots u^{j_k} \Big),\; |u|^d \right) \text{ a.e. } t,$$

where d is an integer ≥ 1. This generalization may be relevant for some applications to Lagrangian Mechanics, where dynamics are usually control-polynomial with degree $d = 2$ (see [11]).

4.2. An example

The following example tells us that both a type-S local infimum gap and a type-E local infimum gap may occur. Moreover, we exhibit sets of abnormal multipliers, which exists in accordance with Theorem 3. Consider the optimization problem with scalar, unbounded controls:

$$(\mathcal{P}) \begin{cases} \quad \text{minimize } |x^1(1) - 1| \\ \text{over } u \in L^1([0,1]; [0, +\infty[), \;\; (x^1, x^2) \in W^{1,1}([0,1]; \mathbb{R}^2) \text{ s.t.} \\ (\dot{x}^1(s), \dot{x}^2(s)) = (u(s), 2) \qquad \text{a.e. } s \in [0,1], \\ (x^1, x^2)(0) = (-1, -1), \quad x^2(1) = 1, \quad \int_0^1 u(s)\, ds \le 3, \\ h(x^1(s), x^2(s)) := 1 - |x^1(s)| \vee |x^2(s)| \le 0 \text{ for all } s \in [0,1]. \end{cases}$$

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Let $W := \{(w^0, w) \in [0,+\infty[\times[0,+\infty[: \ w^0 + w = 1\}$, then the space-time extension of the above problem is given by

$$(P_e) \begin{cases} \quad \text{minimize } |y^1(T) - 1| \\ \text{over } T > 0,\ (\omega^0, \omega) \in L^1([0,T];W),\ (y^0, y^1, y^2, \nu) \in W^{1,1}([0,T];\mathbb{R}^4) \text{ s.t.} \\ (\dot y^0, \dot y^1, \dot y^2, \dot \nu)(t) = (\omega^0, \omega, 2\omega^0, \omega)(t) \qquad \text{a.e. } t \in [0,T], \\ (y^0, y^1, y^2)(0) = (0,-1,-1), \quad y^0(T) = 1, \quad y^2(T) = 1, \quad \nu(T) \le 3, \\ h(y^1(t), y^2(t)) = 1 - |y^1(t)| \vee |y^2(t)| \le 0 \text{ for all } t \in [0,T]. \end{cases}$$

Type-S local infimum gap. Let $\bar z := (\bar T, \bar\omega^0, \bar\omega, \bar y^0, \bar y^1, \bar y^2, \bar \nu)$ be the following strict sense process, where $\bar T = 1$, the control $(\bar\omega^0, \bar\omega)$ is given by the constant pair

$$(\bar\omega^0, \bar\omega)(t) = (1,0) \quad \forall t \in [0,1],$$

and

$$(\bar y^0, \bar y^1, \bar y^2, \bar\nu)(t) = (t, -1, -1+2t, 0) \quad \forall t \in [0,1].$$

It is easy to see that $\bar z$, which corresponds to the process of $(\mathcal{P})$ associated with the control $\bar u \equiv 0$, is trivially a strict sense minimizer, as $(\bar y^0, \bar y^1, \bar y^2, \bar\nu)$ is the unique feasible strict sense trajectory. However, $\bar z$ is not a local minimizer for the extend problem (P_e). Indeed, let us fix $\varepsilon > 0$ sufficiently small and let us consider the extended process $z_\varepsilon = (T_\varepsilon, \omega^0_\varepsilon, \omega_\varepsilon, y^0_\varepsilon, y^1_\varepsilon, y^2_\varepsilon, \nu_\varepsilon)$ where $T_\varepsilon = 1 + \varepsilon$ and $(\omega^0_\varepsilon, \omega_\varepsilon)$ is given by

$$(\omega^0_\varepsilon, \omega_\varepsilon)(t) := \begin{cases} (1,0) & \text{if } t \in [0,1] \\ (0,1) & \text{if } t \in]1, 1+\varepsilon], \end{cases}$$

so that one has

$$(y^0_\varepsilon, y^1_\varepsilon, y^2_\varepsilon, \nu_\varepsilon)(t) = \begin{cases} (t, -1, -1+2t, 0) & \text{if } t \in [0,1] \\ (1, -2+t, 1, t-1) & \text{if } t \in]1, 1+\varepsilon]. \end{cases}$$

For any $\varepsilon > 0$, this is the description in the state-space of a discontinuous state trajectory $(x^1_\varepsilon, x^2_\varepsilon)$ for problem $(\mathcal{P})$ which first reaches the point $(-1,1)$ using the control $u = 0$ and then jumps to the position $(-1+\varepsilon, 1)$ with an impulse. Notice that $\bar z_\varepsilon$ is a feasible extended process that satisfies

$$\mathbf{d}_{\text{imp}}(z_\varepsilon, \bar z) = |T_\varepsilon - \bar T| + \|(\omega^0_\varepsilon, \omega_\varepsilon) - (\bar\omega^0, \bar\omega)\|_{L^1(0,1\wedge(1+\varepsilon))} = \varepsilon$$

and whose cost is strictly less that the cost corresponding to $\bar z$, because it holds

$$|y^1_\varepsilon(1+\varepsilon) - 1| = 2 - \varepsilon < 2 = |\bar y^1(1) - 1|.$$

Thus, by the arbitrariness of $\varepsilon > 0$, at $\bar z$ there is a type-S local infimum gap. Indeed, a set of abnormal multipliers corresponding to $\bar z$ is given by $(p_0, p, \gamma, \pi, \mu, m_0, m)$, where $\gamma = \pi = 0$, $p_0 \equiv 0$, $\mu \equiv 0$, $p = (p_1, p_2) \equiv (0,1)$, $m_0 \equiv 0$ and $m(t) = (m_1, m_2)(t) \in \partial^> h(\bar y^1(t), \bar y^2(t))$ for any $t \in [0,1]$.

Type-E local infimum gap. Consider now the following extended process $\hat z := (\hat\omega^0, \hat\omega, \hat y^0, \hat y_1, \hat y_2, \hat\nu)$, where $\hat T = 3$ and $(\hat\omega^0, \hat\omega)$ is given by

$$(\hat\omega^0, \hat\omega)(t) := \begin{cases} (1,0) & t \in [0,1] \\ (0,1) & t \in]1,3], \end{cases}$$

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so that one has

$$(\hat{y}^0, \hat{y}^1, \hat{y}^2, \hat{v})(t) = \begin{cases} (t, -1, -1+2t, 0) & t \in [0,1] \\ (1, -2+t, 1, t-1) & t \in]1,3]. \end{cases}$$

It is easy to see that $\hat{z}$ is a minimizer for (P_e), as it is feasible and its corresponding cost is equal to zero. Moreover, at $\hat{z}$ there is type-E local infimum gap, since $\bar{z}$ defined in the previous step is the unique feasible strict sense process. Indeed, a set of abnormal multipliers corresponding to $\hat{z}$ is given by $(p_0, p, \gamma, \pi, \mu, m_0, m)$, where $\gamma = \pi = 0$, $p_0 \equiv 0$, $\mu(\{0\}) = 2$, $\mu(]0,1]) = 0$, $p = (p_1, p_2) \equiv (-2, 0)$, $m_0 \equiv 0$, $m(0) = (m_1, m_2)(0) = (1, 0)$ and $m(t) = (m_1, m_2)(t) \in \partial^{>} h(\bar{y}^1(t), \bar{y}^2(t))$ for any $t \in]0,1]$.

5. Proof of Theorem 1

Preliminarily, let us observe that, since the proofs of statements (i)-(ii) involve only extended processes with trajectories close to the reference trajectory $\bar{y}$ and the controls take values in compact sets, using standard cut-off techniques we can assume that hypotheses **(H2)** and **(H3)** are satisfied in the whole space $\mathbb{R}^{1+n}$. Therefore, the input-output map $(\omega, \alpha) \mapsto y[\omega, \alpha]$ associated with (1) is well-defined and continuous (actually, uniformly continuous).

5.1. Proof of statement (i)

If $\bar{z}$ is a local Ψ-minimizer for (P_e), the fact that it satisfies the Pontryagin maximum principle in Def. 3 can be easily derived by [40, Theorem 9.3.1]. The proof that whenever at $\bar{z}$ there is a type-E local infimum gap, then it is an abnormal extremal, requires instead a careful adaptation of the arguments used in the proof of [17, Theorem 2.1], where the same result is obtained for a notion of type-E local infimum gap in which the distance **d** between controls is replaced by the L^∞ distance of the trajectories. The proof is divided into several steps in which successive sequences of optimization problems are introduced that have as admissible controls the strict sense ones, and costs that measure how much a process violates the constraints. Thanks to the Ekeland principle, for these problems it is possible to find a sequence of minimizers which converge to the reference process $\bar{z} = (\bar{\omega}, \bar{\alpha}, \bar{y})$. Furthermore, applying a maximum principle to these approximating problems with reference to the above mentioned minimizers, we obtain in the limit a set of multipliers with $\gamma = 0$ for the extended problem with reference to $\bar{z}$.

Step 1. Define the function $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$, given by

$$\Phi(x, c) := d_{\mathcal{T}}(x) \vee c$$

and for any $y \in W^{1,1}([0,T]; \mathbb{R}^n)$, introduce the payoff

$$\mathcal{J}(y) := \Phi\Big(y(T), \max_{t \in [0,T]} h(t, y(t))\Big).$$

Fix a sequence $(\varepsilon_i)_i$ satisfying $\varepsilon_i \downarrow 0$ and let $(\rho_i)_i$ be such that

$$\rho_i^2 = \sup\{\mathcal{J}(y) : \; z = (\omega, \alpha, y) \in \Gamma_s, \; \mathbf{d}(z, \bar{z}) \le \varepsilon_i\}.$$

By the uniform continuity of the input-output map and the Lipschitz continuity of Φ, it follows that $\lim_{i \to +\infty} \rho_i^2 = 0$. Moreover, $\rho_i > 0$ for every i large enough, since $\bar{z}$ is an isolated process in view of Remark 4.

According to hypothesis **(H1)** and Remark 1, for any i there exist an element of the sequence $(V_j)_j$, which we denote by V_{ε_i}, and some $\hat{\omega}_i \in \mathcal{V}_{\varepsilon_i} := L^1([0,T]; V_{\varepsilon_i})$ such that $\|\hat{\omega}_i - \bar{\omega}\|_{L^1} \le \varepsilon_i$. Hence, let

$\hat{z}_i = (\hat{\omega}_i, \hat{\alpha}_i, \hat{y}_i)$ be such that $\hat{\alpha}_i \equiv \bar{\alpha}$ and $\hat{y}_i = y[\hat{\omega}_i, \hat{\alpha}_i]$. As a consequence, $\hat{z}_i$ is a ρ_i^2-minimizer for the optimization problem $(\hat{P}_i)$ given by

$$(\hat{P}_i) \begin{cases} \text{Minimize} \quad \mathcal{J}(y) \\ \text{over } z = (\omega, \alpha, y) \in \Gamma^i \end{cases}$$

where

$$\Gamma^i := \{ (\omega, \alpha, y) \in \mathcal{V}_{\varepsilon_i} \times \mathcal{A} \times W^{1,1}([0,T], \mathbb{R}^n) \text{ satisfying } (1) \}.$$

It is an easy task to show that, if we equip Γ^i with the distance $\mathbf{d}$, then it turns out to be a complete metric space. Accordingly, in view of the Ekeland's variational principle, there exists $z_i = (\omega_i, \alpha_i, y_i) \in \Gamma^i$ which is a minimizer for problem (P_i) given by

$$(P_i) \begin{cases} \text{Minimize} \quad \mathcal{J}(y) + \rho_i \int_0^T [|\omega(t) - \omega_i(t)| + \vartheta_i(t, \alpha(t))]\, dt \\ \text{over } z = (\omega, \alpha, y) \in \Gamma^i, \end{cases}$$

where $\vartheta_i : [0,T] \times A$ is defined as

$$\vartheta_i(t,a) := \begin{cases} 0 & \text{if } a = \alpha_i(t) \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, one has $\mathbf{d}(z_i, \hat{z}_i) \le \rho_i$, so that $\mathbf{d}(z_i, \bar{z}) \le \rho_i + \varepsilon_i \to 0$. In particular, it holds

$$\omega_i \to \bar{\omega} \text{ in } L^1, \qquad \ell(\{t \in [0,T] : \alpha_i(t) \ne \bar{\alpha}(t)\}) \to 0. \tag{21}$$

Furthermore, in view of the continuity of the input-output map associated with control system (1), one has

$$y_i \to \bar{y} \text{ in } L^\infty, \qquad \dot{y}_i \rightharpoonup \dot{\bar{y}} \text{ weakly in } L^1. \tag{22}$$

By the previous convergence analysis and since $\bar{z}$ is isolated, it follows that $\mathcal{J}(y_i) > 0$ for any i. Therefore, possibly passing to a subsequence, for any i we have

$$\text{either} \quad d_{\mathcal{T}}(y_i(T)) > 0 \quad \text{or} \quad c_i := \max_{t\in[0,T]} h(t, y_i(t)) > 0. \tag{23}$$

Step 2. From the above reasonings it follows that

$$(z_i, c_i) = (\omega_i, \alpha_i, y_i, \max_{t\in[0,T]} h(t, y_i(t)))$$

is a minimizer for the optimal control problem (Q_i), given by

$$(Q_i) \begin{cases} \text{Minimize } \Big(d_{\mathcal{T}}(y(T)) \ \vee \ c(T)\Big) + \rho_i \int_0^T [|\omega(t) - \omega_i(t)| + \vartheta_i(t, \alpha(t))]\, dt \\ \qquad \text{over } (\omega, \alpha, y, c) \in \mathcal{V}_{\varepsilon_i} \times \mathcal{A} \times W^{1,1}([0,T], \mathbb{R}^{n+1}) \text{ satisfying} \\ (\dot{y}(t),\ \dot{c}(t)) = (\mathcal{F}(t, y(t), \omega(t), \alpha(t)),\ 0) \qquad \text{a.e. } t \in [0,T], \\ y(0) = \check{x}_0, \\ \tilde{h}(t, y(t), c(t)) := h(t, y(t)) - c(t) \le 0 \qquad \forall t \in [0,T]. \end{cases}$$

Possibly passing to a subsequence, only one of the following two cases occurs:

Case $(\mathbf{a})$: $c_i > 0$ for any i.

Case $(\mathbf{b})$: $c_i \le 0$ for any i.

Let us first analyze **Case (a)**. Since in this case $h(t, y_i(t)) - c_i > 0$ implies $h(t, y_i(t)) > 0$, one has $\partial^{>}_{x,c}\tilde{h}(t, x, c) = \partial^{>}_{x} h(t, x) \times \{-1\}$. Moreover, in view of the max rule for subdifferentials (see e.g. [40, Sec. 5]), if $(\beta_i^1, \beta_i^2) \in \partial\Phi(y_i(T), c_i)$, then there exist σ_i^1, $\sigma_i^2 \geq 0$ such that $\sigma_i^1 + \sigma_i^2 = 1$, $\beta_i^1 \in \sigma_i^1(\partial d_{\mathcal{T}}(y_i(T)) \cap \partial\mathbb{B})$ and $\beta_i^2 = \sigma_i^2$. Furthermore, $\sigma_i^k = 0$ for $k = 1, 2$ whenever $d_{\mathcal{T}}(y_i(T)) \vee c_i$ is strictly greater than the k-th term in the maximization. Thanks to the above reasonings and applying the maximum principle to problem (Q_i) with reference to its minimizer (z_i, c_i) we deduce that there exist $(p_i, \pi_i) \in W^{1,1}([0, T], \mathbb{R}^{n+1})$, $\lambda_i \geq 0$, $\mu_i \in NBV^+([0, T], \mathbb{R})$, σ_i^1, $\sigma_i^2 \geq 0$ such that $\sigma_i^1 + \sigma_i^2 = 1$, and a Borel-measurable and μ_i-integrable map $m_i : [0, T] \to \mathbb{R}^n$ satisfying conditions (i)′–(vi)′ below:

(i)′ $\|p_i\|_{L^\infty} + \lambda_i + \mu_i([0, T]) + \|\pi_i\|_{L^\infty} = 1$;
(ii)′ $-\dot{p}_i(t) \in \mathrm{co}\,\partial_x\{q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t))$ and $\dot{\pi}_i(t) = 0$ for a.e. $t \in [0, T]$;
(iii)′ $-q_i(T) \in \lambda_i\sigma_i^1\Big(\partial\Phi(y_i(T)) \cap \partial\mathbb{B}\Big)$, $\pi(0) = 0$, $-\pi(T) + \mu_i([0, T]) = \lambda_i\sigma_i^2$;
(iv)′ $m_i(t) \in \partial^{>}_x h(t, y_i(t))$ μ_i-a.e. $t \in [0, T]$;
(v)′ $\mathrm{spt}(\mu_i) \subset \{t \in [0, T] : h(t, y_i(t)) - c_i = 0\}$;
(vi)′ $\int_0^T q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t))\, dt$
$\geq \int_0^T [q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega(t), \alpha(t)) - \rho_i\lambda_i(|\omega_i(t) - \omega(t)|) + \vartheta_i(t, \alpha(t))]\, dt$
$\geq \int_0^T [q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega(t), \alpha(t)) - \rho_i\lambda_i(1 + \mathrm{diam}(\overline{V}))]\, dt$
for any $(\omega, \alpha) \in \mathcal{V}_{\varepsilon_i} \times \mathcal{A}$,

where $\mathrm{diam}(\overline{V})$ is the diameter of the compact set $\overline{V}$ and $q_i : [0, T] \to \mathbb{R}^n$ is defined as

$$q_i(t) := \begin{cases} p_i(t) + \int_{[0,t[} m_i(t')\mu_i(dt') & \text{if } t \in [0, T[, \\ p_i(T) + \int_{[0,T]} m_i(t')\mu_i(dt') & \text{if } t = T. \end{cases} \tag{24}$$

From (ii)′ and (iii)′ we deduce that $\pi_i \equiv 0$ and $\mu_i([0, T]) = \lambda_i\sigma_i^2$. Since $\|m_i\|_{L^\infty} \leq K_h$, from (iii)′ we also have $\lambda_i\sigma_i^1 = |q_i(T)| \leq \|p_i\|_{L^\infty} + K_h\mu_i([0, T])$. By summing up these relations and (i)′ we get

$$2\|p_i\|_{L^\infty} + (2 + K_h)\mu_i([0, T]) + \lambda_i \geq 1 + \lambda_i\sigma_i^1 + \lambda_i\sigma_i^2,$$

so that $\|p_i\|_{L^\infty} + \mu_i([0, T]) \geq \frac{1}{2+K_h}$. By rescaling the multipliers, one obtains $\|p_i\|_{L^\infty} + \mu_i([0, T]) = 1$ and $\lambda_i \geq 2 + K_h$.

If instead **Case (b)** occurs, then $d_{\mathcal{T}}(y_i(T)) > 0$ for any i in view of (23). Hence, for $\delta > 0$ small, the process $(z_i, c_i + \delta)$ still is a minimizer for (Q_i) and $h(t, y_i(t)) - (c_i + \delta) < 0$ for all $t \in [0, T]$. By applying the maximum principle to (Q_i) with reference to $(z_i, c_i + \delta)$, we deduce the existence of $p_i \in W^{1,1}([0, T], \mathbb{R}^n)$ and $\lambda_i > 0$ [4] satisfying conditions (i)′–(vi)′ above for $\mu_i \equiv 0$, $\sigma_i^2 = 0$ (hence, $\sigma_i^1 = 1$). In this case, by (iii)′ we deduce $0 < \lambda_i = |q_i(T)| \leq \|p_i\|_{L^\infty}$. By summing up this relation with (i)′ we get $2\|p_i\|_{L^\infty} + \lambda_i > 1 + \lambda_i$, so that $\|p_i\|_{L^\infty} > \frac{1}{2}$. By rescaling the multipliers, we have $\|p_i\|_{L^\infty} = 1$ and $\lambda_i \leq 2 \leq 2 + K_h$.

Step 3. For both **Case (a)** and **Case (b)** we have proved that for any i there exist $p_i \in W^{1,1}([0, T], \mathbb{R}^n)$, $\mu_i \in NBV^+([0, T], \mathbb{R})$ and a Borel-measurable and μ_i-integrable map $m_i : [0, T] \to \mathbb{R}^n$ satisfying relations (i)–(vi) below

(i) $\|p_i\|_{L^\infty} + \mu_i([0, T]) = 1$;
(ii) $-\dot{p}_i(t) \in \mathrm{co}\,\partial_x\{q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t))$ a.e. $t \in [0, T]$;
(iii) $-q_i(T) \in [0, 2 + K_h]\Big(\partial\Phi(y_i(T)) \cap \partial\mathbb{B}\Big)$;
(iv) $m_i(t) \in \partial^{>}_x h(t, y_i(t))$ μ_i-a.e. $t \in [0, T]$;
(v) $\mathrm{spt}(\mu_i) \subset \{t \in [0, T] : h(t, y_i(t)) - c_i = 0\}$;

[4] If it were $\lambda_i = 0$, then $q_i(T) = p_i(T) = 0$, so that the linearity of the adjoint equation (ii)′ implies $p_i \equiv 0$, contradicting (i)′.

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(vi) $\int_0^T q_i(t)\cdot\mathcal{F}(t,y_i(t),\omega_i(t),\alpha_i(t))\,dt$
$\geq \int_0^T [q_i(t)\cdot\mathcal{F}(t,y_i(t),\omega(t),\alpha(t)) - \rho_i(2+K_h)(1+\mathrm{diam}(\overline{V}))]\,dt$
for any $(\omega,\alpha)\in\mathcal{V}_{\varepsilon_i}\times\mathcal{A}$,

where $q_i : [0,T]\to\mathbb{R}^n$ is as in (24). Employing a standard convergence analysis (see [16] for more details) we deduce that there exist $(p,\mu)\in W^{1,1}([0,T],\mathbb{R}^n)\times NBV^+([0,T],\mathbb{R})$ and a Borel-measurable and μ-integrable map $m : [0,T]\to\mathbb{R}^n$ such that, up to a subsequence, we have

$$\mu_i \rightharpoonup^* \mu, \quad m_i(t)\mu_i(dt) \rightharpoonup^* m(t)\mu(dt),$$
$$p_i \to p \text{ in } L^\infty, \quad q_i \to q \text{ in } L^1, \quad \dot{p}_i \rightharpoonup \dot{p} \text{ weakly in } L^1. \tag{25}$$

Therefore, using (22) and passing to the limit in conditions (i), (iv) and (v) we obtain

$$\|p\|_{L^\infty} + \mu([0,T]) = 1, \quad m(t)\in\partial_x^> h(t,\bar{y}(t)) \ \mu\text{-a.e. } t\in[0,T],$$
$$\mathrm{spt}(\mu)\subset\{t\in[0,T] : h(t,\bar{y}(t)) = 0\}.$$

Moreover, using basic properties of subdifferentials and the fact that $\partial d_{\mathcal{T}}(x) = N_{\mathcal{T}}(x)\cap\mathbb{B}$ for any $x\in\mathcal{T}$ (see [40]), passing to the limit in (iii) we deduce that

$$-q(T)\in N_{\mathcal{T}}(\bar{y}(T)),$$

where $q : [0,T]\to\mathbb{R}^n$ is given by

$$q(t) := \begin{cases} p(t) + \int_{[0,t[} m(t')\mu(dt') & \text{if } t\in[0,T[\\ p(T) + \int_{[0,T]} m(t')\mu(dt') & \text{if } t = T. \end{cases}$$

Let us now derive the adjoint equation (8). Let $\Omega_i := \{t\in[0,T] : \alpha_i(t) = \bar{\alpha}(t)\}$, so that $\ell(\Omega_i)\to 0$ in view of (21). Using (3) and hypothesis **(H3)**, for a.e. $t\in\Omega_i$, we get

$$\begin{aligned}(-\dot{p}_i(t),\dot{y}_i(t)) &\in \Big(\mathrm{co}\,\partial_x\{q_i(t)\cdot\mathcal{F}(t,y_i(t),\omega_i(t),\bar{\alpha}(t))\},\ \mathcal{F}(t,y_i(t),\omega_i(t),\bar{\alpha}(t))\Big)\\ &\subseteq \Big(q_i(t)\cdot D_x\mathcal{F}(t,y_i(t),\bar{\omega}(t),\bar{\alpha}(t)) + |q_i(t)|k(t)\varphi(|\omega_i(t)-\bar{\omega}(t)|)\mathbb{B},\\ &\qquad \mathcal{F}(t,y_i(t),\bar{\omega}(t),\bar{\alpha}(t)) + k(t)\varphi(|\omega_i(t)-\bar{\omega}(t)|)\mathbb{B}\Big)\\ &\subseteq \Big(\mathrm{co}\,\partial_x\{q(t)\cdot\mathcal{F}(t,y_i(t),\bar{\omega}(t),\bar{\alpha}(t)),\ \mathcal{F}(t,y_i(t),\bar{\omega}(t),\bar{\alpha}(t))\}\Big) + r_i(t)\mathbb{B}\end{aligned}$$

where, since $\|q_i\|_{L^\infty}\leq\|p_i\|_{L^\infty} + K_h\mu_i([0,T])\leq 1+K_h$, the map $r_i : [0,T]\to\mathbb{R}$ is given by

$$r_i(t) = |q_i(t)-q(t)|k(t) + 2(1+K_h)k(t)\varphi(|\omega_i(t)-\bar{\omega}(t)|).$$

By the continuity of φ, (21) and (25) we deduce that, up to a subsequence, $r_i(t)\to 0$ for a.e. $t\in[0,T]$. Moreoveor, it holds

$$|r_i(t)|\leq 2(1+K_h)(1+\varphi(\mathrm{diam}(\overline{V})))k(t)\in L^1.$$

Hence, by the dominated convergence theorem, $r_i\to 0$ in L^1 (in particular, $\varphi(|\omega_i-\bar{\omega}|)\to 0$ in L^1). From the compactness of trajectories theorem (see [40, Theorem 2.5.3]) it follows that for a.e. $t\in[0,T]$ it holds

$$(-\dot{p}(t),\dot{\bar{y}}(t))\in\Big(\mathrm{co}\,\partial_x\{q(t)\cdot\mathcal{F}(t,\bar{y}(t),\bar{\omega}(t),\bar{\alpha}(t))\},\mathcal{F}(t,\bar{y}(t),\bar{\omega}(t),\bar{\alpha}(t))\Big)$$

Now it remains to prove (10). Let $(\omega, \alpha) \in \mathcal{W} \times \mathcal{A}$ and, as a consequence of hypothesis **(H1)**, let $(v_i)_i \subset \mathcal{V}$ be such that $v_i \in \mathcal{V}_{\varepsilon_i}$ for any i and $\|\omega - v_i\|_{L^1} \leq \varepsilon_i \downarrow 0$. Condition (vi) implies that

$$\int_0^T q_i(t) \cdot \dot{\bar{y}}_i(t) dt \geq \int_0^T [q_i(t) \cdot \mathcal{F}(t, y_i(t), v_i(t), \alpha(t)) - \rho_i(1 + \mathrm{diam}(\overline{V}))(2 + K_h)] dt$$

Up to a subsequence, the right hand side of the above inequality converges to $\int_0^T [q(t) \cdot \mathcal{F}(t, \bar{y}(t), \omega(t), \alpha(t))]\, dt$, by the dominated convergence theorem. At the same time it holds

$$\int_0^T q_i(t) \cdot \dot{y}_i(t) dt = \int_0^T q(t) \cdot \dot{\bar{y}}(t) dt + \int_0^T (q_i(t) - q(t)) \cdot \dot{y}_i(t) dt + \int_0^T q(t) \cdot (\dot{y}_i(t) - \dot{\bar{y}}(t)) dt.$$

But now the second term in the right hand side of the equality above tends to zero in view of the dominated convergence theorem, while the third one converges to zero because of (22) and the fact that $q \in L^\infty$. Therefore, we have proved that for any $(\omega, \alpha) \in \mathcal{W} \times \mathcal{A}$ one has

$$\int_0^T q(t) \cdot \dot{\bar{y}}(t)\, dt \geq \int_0^T q(t) \mathcal{F}(t, \bar{y}(t), \omega(t), \alpha(t))\, dt.$$

From a measurable selection theorem (10) immediately follows.

5.2. *Proof of statement (ii)*

Let $\bar{z} = (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_s$ be a local Ψ-minimizer for (P_s). We can derive that it is an extremal of the Pontryagin maximum principle from ([40] Theorem 9.3.1). In particular, the maximality condition (10) still holds with the maximum taken over $\overline{V} \times A$, since we assume that the dynamics function is continuous with respect to the w-variable.

If $\bar{z}$ is a local Ψ-minimizer for (P_s) which is not a local Ψ-minimizer for (P_e), then, on the one hand, there exists $\delta > 0$ such that $\Psi(\bar{y}(T)) \leq \Psi(y(T))$ for any $z = (\omega, \alpha, y) \in \Gamma_s$ such that $\mathbf{d}(z, \bar{z}) \leq 2\delta$. On the other hand, taken $(\varepsilon_i)_i \subset]0, \delta[$ with $\varepsilon_i \downarrow 0$, for each i there exists some $z_i = (\omega_i, \alpha_i, y_i) \in \Gamma_e$ such that $\mathbf{d}(z_i, \bar{z}) \leq \varepsilon_i < \delta$ and $\Psi(y_i(T)) < \Psi(\bar{y}(T))$. Hence, for any $z = (\omega, \alpha, y) \in \Gamma_s$ such that $\mathbf{d}(z_i, z) \leq \delta$, one has $\mathbf{d}(z, \bar{z}) \leq 2\delta$, so that we have by construction

$$\Psi(y_i(T)) < \Psi(\bar{y}(T)) \leq \Psi(y(T)).$$

Since the strict sense process z is arbitrary, this proves that at z_i there is a type-E local infimum gap for any i. Hence, in view of Theorem 1, (i) for any i there exist $p_i \in W^{1,1}([0, T], \mathbb{R}^n)$, $\mu_i \in NBV^+([0, T], \mathbb{R})$ and a Borel-measurable and μ_i-integrable map $m_i : [0, T] \to \mathbb{R}^n$ satisfying conditions (i)–(vi) below:

(i) $\|p_i\|_{L^\infty} + \mu_i([0, T]) = 1$;
(ii) $-\dot{p}_i(t) \in \mathrm{co}\, \partial_x \{q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t))$ a.e. $t \in [0, T]$;
(iii) $-q_i(T) \in N_{\mathcal{T}}(y_i(T))$;
(iv) $m_i(t) \in \partial_x^> h(t, y_i(t))$ μ_i-a.e. $t \in [0, T]$;
(v) $\mathrm{spt}(\mu_i) \subset \{t \in [0, T] : h(t, y_i(t)) - c_i = 0\}$;
(vi) $q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t)) = \max_{(w,a) \in \overline{V} \times A} q_i(t) \cdot \mathcal{F}(t, y_i(t), w, a)$ a.e. t,

where $q_i : [0, T] \to \mathbb{R}^n$ is as in (24). We do observe that our construction implies $\mathbf{d}(z_i, \bar{z}) \to 0$, so that (21) and (22) hold true. We can thus conclude the proof employing a standard convergence analysis similar to that in the *Step 3* of the proof of Theorem 1, (i).

6. Concluding remarks

In this paper we investigate infimum gap phenomena that may occur when we pass from an optimal control problem with non-smooth data, endpoints, and state constraints, to an extended version of it, in a framework that includes the impulsive extension of a class of non-coercive problems

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with unbounded dynamics. In particular, we consider a type-E and a type-S local infimum gap: in the former an extended minimizer has cost which is strictly smaller than the infimum cost over close feasible strict sense processes, in the latter a local strict sense minimizer does not locally minimize the extended problem. Following on from Warga's previous research, but utilizing more recent perturbation techniques from non-smooth analysis, we prove that whenever at a process there is either a type-E or a type-S local infimum gap for a notion of local minimizer based on the control distance **d**, then it satisfies a non-smooth, constrained version of the Pontryagin maximum principle in abnormal form. Compared to previous results, in which there was an 'asymmetry' between the necessary abnormality conditions derived for type-E and type-S local infimum gap, for the extension under consideration we obtain the same condition for both.

As a corollary, we provide sufficient conditions in the form of a normality test for the absence of local infimum gap phenomena. Although a normality test for gap avoidance might seem completely theoretical and hardly verifiable, it can actually be very useful because in certain situations normality follows from easily verifiable criteria. These criteria take the form of constraint and endpoint qualification conditions for normality and have been extensively explored in the literature (see e.g. [6,14,15,25] and references therein). As shown in [16,33,34], where several explicit conditions for normality in control-affine impulsive extensions are presented, these criteria are generally weaker than those previously established to directly determine the absence of a gap, as in [1,28].

The framework introduced in this paper may have implications for future infimum gap research in several directions. On the one hand, it may be the starting point for some generalizations, such as, for instance: (i) determine a higher-order maximum principle also for local minimizers of the strict sense problem and prove that in the case of a type-S local infimum gap, there is abnormality of the higher-order conditions as well. So far, results of this kind are only known for extended minimizers and for type-E infimum gap, limited to the impulsive extension case (see [3,4,35]); (ii) explore infimum gap phenomena for the impulsive extension of optimal control problems involving control-affine systems with time delays, for which necessary optimality conditions have very recently been established by Fusco, Motta, and Vinter in [20,21].

Another interesting problem might be to consider different extension procedures for classes of control systems not considered in this paper (such as distributed parameters systems or multistage problems).

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