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Article

Accelerating Expanding Universe by Newton's Equations of Celestial Mechanics

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Abstract: This article, by using Newton's equations for celestial mechanics, provides asymptotic formulas for the rate of acceleration of an expanding universe. This formula, based on the distance between two celestial bodies raised to a certain power, is independent of dark energy.

Keywords: Newton's equations for celestial mechanics; N-body problem; expanding universe; accelerated expansion; dark energy; asymptotic methods; asymptotic formulas

MSC: 85A40; 70F15

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1. Introduction

The expansion of the universe has interested physicists, mathematicians, and philosophers for centuries. As a case in point, consider the questions Richard Bentley posed to Newton in 1687 [37].

i.) In a finite universe if all the stars are drawn to each other by gravitation, should not they collapse into a single point?
ii.) In an infinite universe with infinitely many stars, would not every star be pulled apart by infinite forces acting in all directions?

Newton agreed with i.) and favored an infinite universe with infinitely many stars, so that each star would be drawn in all directions equally, the forces would cancel, and no collapse would occur. Newton acknowledged the problem with this later scenario is that the stars would have to be precisely placed to maintain such an unstable equilibrium without collapse. Later Newton claimed that God prevented this collapse by making constant minute correction. In other words, he conceded that continual miracle is needed to prevent the Sun and the fixed stars from rushing together through gravity [8,15]. Neither Newton nor Bentley lived long enough to witness the rigorous mathematical work which showed that N-point masses obeying Newton's celestial mechanics equations (NCME) could escape to infinity. This happens when proper initial conditions are chosen. Moreover, the distance between any two masses also grows without bound by merely obeying NCME. See Gingold and Solomon [11] and references therein for the latter. Note that unlike Einstein's field equations of general relativity, NCME does not require a cosmological parameter that prevents collapse of the universe.

The proceeding discussion underscores the necessity of producing a "real world" explanation for the experimental data which shows that the universe is expanding rather than collapsing. Moreover, there is a need to explain that not only is the universe expanding but also that its expansion is accelerating. In this paper we propose to explain this accelerated expansion via the closed form solutions of escaping trajectories of NCME described within [11] since that work provides the detailed asymptotic approximations necessary to facilitate the results obtained herein. The main results of [11] differ from the mathematical works of [17,30,31] in the following respects.

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i.) By using successive approximations of integral equations, [11] actually computes the position vectors of the trajectories or solutions of NCME as a function of time rather than rely on integrals of motion.
ii.) Consequently, [11] does not need to assume the existence of solutions that are free of singularities on a semi-infinite interval. Unlike [11,17,30,31] proves the existence of solutions that are singularities free on a semi-infinite interval.
iii.) [11] has approximations of solutions to any level of accuracy desired.
iv.) Unlike [11,17,30,31] does not need to assume an extra condition on the ratio of the total energy to the potential energy of solutions that escape to infinity as $t \to \infty$.

We could not find elsewhere in the literature the methodology and applications that are presented in this article. The articles [2,12–14,18–22,24,25,33] relate to the expansion of the universe. However, they are fundamentally different than our approach even though they use titles or key words like Newtonian cosmology. A case in point are the two articles by Milne [21,22] and Tipler [33]. The derivations in [21] are premised on the Lorentz transformation and postulates of the theory of special relativity. The derivations in [33] are based on the equations of the theory of general relativity. The postulates of general relativity allow for modifications of NCME which yields the new equation (NCMEMN)

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{K}{R^2}, \tag{1.1}$$

where R is the distance from the center of a homogeneous and isotropic sphere of matter with density $\rho(t)$, G is Newton's gravitational constant, and K is a certain constant. We observe that for $K = 0, 1, -1$, Equation (1.1) coincides with Friedman's equation of cosmology which itself was derived from Einstein's theory of general relativity. It is noteworthy that the derivation arguments of (1.1) replace derivations arguments of McCrea and Milne [18] that were found to be defective. We observe that the nature of the equation (1.1) could be very different than the nature of NCME. Thus the name "Newtonian cosmology" attached to (1.1) is not representative of its nature.

Traditional cosmology derived from Einstein's field equations uses the Friedman equation that predicts the rate of change of the radius $R(t)$ of an isotropic and homogeneous expanding universe. Our approach is different in the following sense. We use NCME to calculate the the trajectories of individual point masses of an expanding universe that is neither isotropic nor homogeneous. After which we use mathematical methods like averaging in order to obtain the rate of expansion of a isotropic and homogeneous universe. Our models offer a substantial amount of parameters that can be adjusted to make our rates of change fit experimental data.

The main goals of this article are

1. To demonstrate that NCME are capable of explaining the accelerated expansion.
2. To derive rates of accelerated expansion based on NCME and thus provide an explanation to an accelerated expansion that is independent of dark energy.
3. To introduce parameters in measuring the rate of acceleration in order to facilitate the fitting of experimental data with our proposed rates of acceleration.

We distinguish between two types of measured accelerated expansion that are parametrized with a parameter α. The first type is a vectorial acceleration that is based on asymptotic approximations of $\|r_j''(t) - r_k''(t)\|^\alpha$. The second type is the scalar measure $[\|r_j(t) - r_k(t)\|^\alpha]''$. These two are related but are not the same. In each of the two categories we utilize pointwise measures to be contrasted with averaging and statistical measures. As such we obtain three types of NCME based measures for accelerated expansion.

1. The first type measures the rate of accelerated expansion of the distance between any two point masses raised to a power that is a parameter.
2. The second type utilizes a statistical measure.

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3. The third type conforms to the big bang theory. It shows that asymptotically, the N point masses lie on a sphere with center at the origin of the coordinate system. All rates of acceleration are shown to be positive quantities as the time is large.

A main result of our work is the formula

$$\frac{d^2\|r_j(t)-r_k(t)\|^\alpha}{dt^2} \sim \alpha(\alpha-1)\|a_j-a_k\|^\alpha t^{\alpha-2} > 0,\ t\to\infty,\ \alpha>1, \tag{1.2}$$

where $r_j(t), r_k(t)$, with $j \neq k$, are two position vectors of the two point masses N-body problem, and where $\alpha > 0$ is a parameter and a_j, a_k are the asymptotic velocities of the two bodies as time $t \to \infty$. This formula shows that the acceleration of the scalar distance between raised to a power α is a positive quantity. This is the content of Theorem 3.2. Noteworthy is the case of $\alpha > 2$. Then the acceleration of $\|r_j(t) - r_k(t)\|^\alpha$ is not just a positive quantity, but it is growing to infinity at a rate proportional to $t^{\alpha-2} \to \infty$ as $t \to \infty$. The latter holds even though our asymptotic analysis shows that

$$\left\|\frac{d^2[r_j(t)-r_k(t)]}{dt^2}\right\| \to 0, \qquad t\to\infty; \tag{1.3}$$

see Proposition 5.2.

We also use Theorem 3.2 to calculate two classes of ratios which are generalizations of the Hubble scale factor $\frac{dR}{dt}/R$, namely

$$\frac{\frac{d}{dt}\|r_j-r_k\|^\alpha}{\|r_j-r_k\|^\beta}, \qquad \alpha,\beta>0, \tag{1.4}$$

and

$$\frac{\frac{d^2}{dt^2}\|r_j-r_k\|^\alpha}{\|r_j-r_k\|^\beta}, \qquad \alpha>1 \text{ and } \beta>0. \tag{1.5}$$

We then compare and contrast these ratios to the kinematic vector ratios

$$\frac{\|r'_j-r'_k\|^\alpha}{\|r_j-r_k\|^\beta}, \qquad \frac{\|r''_j-r''_{k'}\|^\alpha}{\|r_j-r_k\|^\beta}, \qquad \alpha,\beta\geq 1. \tag{1.6}$$

Our derivations are based on robust asymptotic approximations [11]. In Section 2 we cite the main theorem of [11] in order to provide a self contained study. In Section 3 we derive Equation (1.2) via Theorem 3.2. In Section 4 we use Theorem 3.2 to compute the alpha Hubble approximations/ratios of (1.4) and (1.5). In Section 5 we derive (1.3) as a result of Proposition 5.2, and in Section 6 we use Proposition 5.2 to calculate the vectorial alpha Hubble approximations/ratios of (1.6). Finally, in Section 7 we use Theorem 3.2 and Proposition 5.2 as tools for deriving statistical measures of expansion of the universe.

2. Notations and Reference Theorem

Recall that Newton's equations of celestial mechanics, NCEM, imply that the position vectors $r_i(t) \in \mathbb{R}^3$ of N point masses m_i, $i = 1, \ldots, N$, satisfy

$$\frac{d^2 r_i}{dt^2} = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(r_j-r_i)}{\|r_i-r_j\|^3}, = -\nabla_{r_i}\Phi, \tag{2.1}$$

where

$$\Phi := \sum_{1\leq i<j\leq N} \frac{m_i m_j}{\|r_i-r_j\|}.$$

The units have been chosen so that Newton's gravitational constant $G = 1$.

In order to state and prove the main results of this paper we remake the main result of Gingold and Solomon [11] using the following Landau asymptotic notation.

Given two scalar functions $g(t), h(t)$ such that $g(t)/h(t)$ is well defined on $[t_0, \infty)$, we say that

$$g(t) \sim h(t) \text{ as } t \to \infty \iff \lim_{t\to\infty} \frac{g(t)}{h(t)} = 1,$$

Given two scalar functions $g(t), h(t)$ such that $g(t)/h(t)$ is well defined on $[t_0, \infty)$, we say that

$$g(t) = o(h(t)) \text{ as } t \to \infty \iff \lim_{t\to\infty} \frac{g(t)}{h(t)} = 0,$$

Given two scalar functions $g(t), h(t)$ such that $g(t)/h(t)$ is well defined on $[t_0, \infty)$, we say that

$$g(t) = O(h(t)) \text{ as } t \to \infty \iff \text{ there exists } K \geq 0 \text{ such that } \left|\frac{g(t)}{h(t)}\right| \leq K.$$

In the following theorem we let $\overrightarrow{0}$ denote zero vector that is either 3-dimensional or $3N$-dimensional.

Theorem 2.1. *(Theorem 1 of [11]) Given any N-tuple of masses $(m_i)_{1\leq i\leq N}$, given any $3N$-tuple $(c_i)_{1\leq i\leq N}$ of constant vectors $c_i \in \mathbb{R}^3$, and given any $3N$-tuple $(a_i)_{1\leq i\leq N}$ of constant vectors $a_i \in \mathbb{R}^3$ which satisfies the constraint*

$$\|a_j - a_i\| \neq 0, \qquad \text{whenever } i \neq j,$$

there exists an interval $[t_0, \infty)$, with $t_0 > 1$ such that

i. *The differential system (2.1) possesses unique vector solutions where*

$$r_i(t) = f_i(t) + \delta_i(t), \qquad f_i(t) := a_i t + b_i \ln t + c_i, \qquad \delta_i(t) \in \mathbb{R}^3. \tag{2.2}$$

Furthermore, there exists a constant $\omega > 0$ with the property that for $i \neq j$, (where $i, j = 1...N$),

$$\|f_i(t) - f_j(t)\| \geq \omega > 0, \qquad t \in [t_0, \infty). \tag{2.3}$$

ii. *The $3N$ coefficients b_i are uniquely determined by the $3N$ coefficients a_i as follows:*

$$b_i := -\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j(a_j - a_i)}{\|a_j - a_i\|^3}, \quad i = 1, \ldots, N. \tag{2.4}$$

iii. *The vectors $\delta_i(t)$ satisfy on $[t_0, \infty)$ the following asymptotic relations.*

$$\delta_j(t) = O\left(\frac{\ln t}{t}\right), \; \delta_j'(t) = O\left(\frac{\ln t}{t^2}\right), \; \delta_j''(t) = O\left(\frac{\ln t}{t^3}\right), \quad t \to \infty. \tag{2.5}$$

Remark 2.1. *The significance of the above theorem to cosmology is fourfold. First, NCME guarantee that there exist a continuum (as many as the number of points on a nonzero interval) of initial positions $r_i(t_0)$ and initial velocities $r_i'(t_0)$ of N point masses from which the universe (vidalicet N trajectories $r_i(t)$ of the N point masses) has evolved after the big bang. These initial conditions correspond to the parameters $((a_i, c_i))_{i=1}^N$, where $a_j - a_k \neq \overrightarrow{0}$ whenever $j \neq k$. Secondly, in this accelerated expansion of the universe, not only do at least $N-1$ point masses escape to infinity but the distance between any two point masses becomes unbounded as the time $t \to \infty$. Thirdly, the accelerated expansion manifested in the asymptotic approximations to the position vectors $r_i(t)$ (resp. $r_j(t) - r_k(t)$), the velocity vectors $r_i'(t)$ (resp. $r_j'(t) - r_k'(t)$), and acceleration vectors $r_i''(t)$ (resp. $r_j''(t) - r_k''(t)$) is due to the laws of nature as manifested by the NCME and is not due to dark energy.*

Fourthly, the rate of the accelerated expansion can be shown to be as large as desired depending on the parameters $((a_i, c_i))_{i=1}^N$.

Remark 2.2. *The cosmological constant in Einstein field equations of general relativity may be set to zero and yet NCME will guarantee a universe of accelerated expansion.*

3. Numerator of the Alpha Hubble Approximation

Let $\alpha > 1$. The goal of this section is to provide the asymptotic evaluation for $[\|r_j(t) - r_k(t)\|^\alpha]'$ and $[\|r_j(t) - r_k(t)\|^\alpha]''$ as $t \to \infty$. In order to calculate these asymptotic evaluations, we observe that

$$\|r_j(t) - r_k(t)\|^\alpha = \left[\|r_j(t) - r_k(t)\|^2\right]^{\frac{\alpha}{2}},$$

and use the results of the following theorem whose proof is provided in Appendix A.

Theorem 3.1. *Under assumptions of Theorem 2.1, for $j \neq k$ we have*

$$\|r_j(t) - r_k(t)\|^2 := V_{jk} + W_{jk} + Z_{jk}$$

where

$$V_{jk} := (f_j - f_k)^T (f_j - f_k), \quad W_{jk} := 2(f_j - f_k)^T (\delta_j - \delta_k), \tag{3.1}$$

and

$$Z_{jk} := (\delta_j - \delta_k)^T (\delta_j - \delta_k). \tag{3.2}$$

Then as $t \to \infty$,

$$\|r_j(t) - r_k(t)\|^2 = V_{jk}\left[1 + O\left(\frac{\ln t}{t}\right)\right] = \|f_j - f_k\|^2 \left[1 + O\left(\frac{\ln t}{t}\right)\right], \tag{3.3}$$

$$\frac{d\|r_j(t) - r_k(t)\|^2}{dt} = V'_{jk}\left[1 + O\left(\frac{\ln t}{t^2}\right)\right] = 2(f'_j - f'_k)^T (f_j - f_k)\left[1 + O\left(\frac{\ln t}{t^2}\right)\right] \tag{3.4}$$

and

$$\frac{d^2\|r_j(t) - r_k(t)\|^2}{dt^2} = V''_{jk}\left[1 + O\left(\frac{\ln t}{t}\right)\right] = 2\|a_j - a_k\|^2 \left[1 + O\left(\frac{\ln t}{t}\right)\right]. \tag{3.5}$$

Now we generalize the parameter and assume $\alpha > 1$.

Theorem 3.2. *Let $\alpha > 1$. Under assumptions of Theorem 2.1 and Theorem 3.1, for $j \neq k$ we have*

$$\frac{d}{dt}\|r_j(t) - r_k(t)\|^\alpha = \alpha \|a_j - a_k\|^\alpha t^{\alpha-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right], \tag{3.6}$$

and

$$\frac{d^2}{dt^2}\|r_j(t) - r_k(t)\|^\alpha = \alpha(\alpha - 1)\|a_j - a_k\|^\alpha t^{\alpha-2}\left[1 + O\left(\frac{\ln t}{t}\right)\right]. \tag{3.7}$$

Proof. As $t \to \infty$, we use the chain rule, the definition of Big O, along with (3.3) and (3.4) to calculate

$$
\begin{aligned}
\frac{d}{dt}\|r_j(t) - r_k(t)\|^\alpha &= \frac{d}{dt}\left[\|r_j(t) - r_k(t)\|^2\right]^{\frac{\alpha}{2}} \\
&= \frac{\alpha}{2}\left[\|r_j(t) - r_k(t)\|^2\right]^{\frac{\alpha}{2}-1} \frac{d}{dt}\|r_j(t) - r_k(t)\|^2 \\
&= \frac{\alpha}{2}\left[\|f_j(t) - f_k(t)\|^2\left[1 + O\left(\frac{\ln t}{t}\right)\right]\right]^{\frac{\alpha}{2}-1} \frac{d}{dt}\|r_j(t) - r_k(t)\|^2 \\
&= \frac{\alpha}{2}\left[\|f_j(t) - f_k(t)\|^2\right]^{\frac{\alpha}{2}-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right] \frac{d}{dt}\|r_j(t) - r_k(t)\|^2 \\
&= \alpha\left[\|f_j(t) - f_k(t)\|^2\right]^{\frac{\alpha}{2}-1}(f_j' - f_k')^T(f_j - f_k)\left[1 + O\left(\frac{\ln t}{t}\right)\right] \qquad (3.8)
\end{aligned}
$$

We use (9.1) and (9.9) to further estimate the factors of $\|f_j(t) - f_k(t)\|^2$ and $(f_j' - f_k')^T(f_j - f_k)$ within (3.8) as follows.

$$
\begin{aligned}
\frac{d}{dt}\|r_j(t) - r_k(t)\|^\alpha &= \alpha\left[\|f_j(t) - f_k(t)\|^2\right]^{\frac{\alpha}{2}-1}\|a_j - a_k\|^2 t\left[1 + O\left(\frac{\ln t}{t}\right)\right]\left[1 + O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha\left[\|f_j(t) - f_k(t)\|^2\right]^{\frac{\alpha}{2}-1}\|a_j - a_k\|^2 t\left[1 + O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha\|a_j - a_k\|^2 t\left[\|a_j - a_k\|^2 t^2\left[1 + O\left(\frac{\ln t}{t}\right)\right]\right]^{\frac{\alpha}{2}-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha\|a_j - a_k\|^2 t\left[\|a_j - a_k\|^2 t^2\right]^{\frac{\alpha}{2}-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right]\left[1 + O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha\|a_j - a_k\|^\alpha t^{\alpha-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right]
\end{aligned}
$$

Next we calculate the second derivative as follows.

$$
\frac{d^2}{dt^2}\|r_j(t) - r_k(t)\|^\alpha = \frac{d}{dt}\left(\frac{\alpha}{2}\left[\|r_j(t) - r_k(t)\|^2\right]^{\frac{\alpha}{2}-1}\frac{d}{dt}\|r_j(t) - r_k(t)\|^2\right) = S_{jk} + Q_{jk}, \qquad (3.9)
$$

where

$$
S_{jk} := \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right)\left[\|r_j(t) - r_k(t)\|^2\right]^{\frac{\alpha}{2}-2}\left(\frac{d}{dt}\|r_j(t) - r_k(t)\|^2\right)^2. \qquad (3.10)
$$

and

$$
Q_{jk} := \frac{\alpha}{2}\left[\|r_j(t) - r_k(t)\|^2\right]^{\frac{\alpha}{2}-1}\frac{d^2}{dt^2}\|r_j(t) - r_k(t)\|^2. \qquad (3.11)
$$

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We now calculate the asymptotic approximations to S_{jk} and Q_{jk}. Equations (3.3) and (3.4), along with (9.1) and (9.9), imply that

$$
\begin{aligned}
S_{jk} &= \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\left[\|f_j(t)-f_k(t)\|^2\left[1+O\left(\frac{\ln t}{t}\right)\right]\right]^{\frac{\alpha}{2}-2}\left(\frac{d}{dt}\|r_j(t)-r_k(t)\|^2\right)^2 \\
&= \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\left[\|f_j(t)-f_k(t)\|^2\right]^{\frac{\alpha}{2}-2}\left[1+O\left(\frac{\ln t}{t}\right)\right]\left[2(f_j'-f_k')^T(f_j-f_k)\left[1+O\left(\frac{\ln t}{t^2}\right)\right]\right]^2 \\
&= \alpha(\alpha-2)\left[\|f_j(t)-f_k(t)\|^2\right]^{\frac{\alpha}{2}-2}\left[(f_j'-f_k')^T(f_j-f_k)\right]^2\left[1+O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha(\alpha-2)\left[\|a_j-a_k\|^2t^2\left[1+O\left(\frac{\ln t}{t}\right)\right]\right]^{\frac{\alpha}{2}-2}\left[(f_j'-f_k')^T(f_j-f_k)\right]^2\left[1+O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha(\alpha-2)\left[\|a_j-a_k\|^2t^2\right]^{\frac{\alpha}{2}-2}\left[\|a_j-a_k\|^2t\left[1+O\left(\frac{\ln t}{t}\right)\right]\right]^2\left[1+O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha(\alpha-2)\left[\|a_j-a_k\|^2t^2\right]^{\frac{\alpha}{2}-2}\|a_j-a_k\|^4t^2\left[1+O\left(\frac{\ln t}{t}\right)\right]\left[1+O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha(\alpha-2)\|a_j-a_k\|^\alpha t^{\alpha-2}\left[1+O_1\left(\frac{\ln t}{t}\right)\right].
\end{aligned}
\tag{3.12}
$$

A similar calculation using (3.5) instead of (3.4) shows that

$$
\begin{aligned}
Q_{jk} &= \frac{\alpha}{2}\left[\|f_j(t)-f_k(t)\|^2\left[1+O\left(\frac{\ln t}{t}\right)\right]\right]^{\frac{\alpha}{2}-1}\frac{d^2}{dt^2}\|r_j(t)-r_k(t)\|^2 \\
&= \frac{\alpha}{2}\left[\|f_j(t)-f_k(t)\|^2\right]^{\frac{\alpha}{2}-1}\left[1+O\left(\frac{\ln t}{t}\right)\right]\left[2\|a_j-a_k\|^2\left[1+O\left(\frac{\ln t}{t}\right)\right]\right] \\
&= \alpha\left[\|a_j-a_k\|^2t^2\left[1+O\left(\frac{\ln t}{t}\right)\right]\right]^{\frac{\alpha}{2}-1}\|a_j-a_k\|^2\left[1+O\left(\frac{\ln t}{t}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha\|a_j-a_k\|^\alpha t^{\alpha-2}\left[1+O\left(\frac{\ln t}{t}\right)\right]\left[1+O\left(\frac{\ln t}{t}\right)\right] \\
&= \alpha\|a_j-a_k\|^\alpha t^{\alpha-2}\left[1+O_2\left(\frac{\ln t}{t}\right)\right].
\end{aligned}
\tag{3.13}
$$

If we add together (3.12) and (3.13), as long as $\alpha > 1$, we obtain (3.7). $\square$

Remark 3.1. *A careful reading of the proof shows that Equation (3.6) is valid for $\alpha > 0$.*

4. Alpha Hubble Approximations

The results of Theorem 3.2 will allow us to calculate the following two ratios for large t, namely

$$
\frac{\frac{d}{dt}\|r_j-r_k\|^\alpha}{\|r_j-r_k\|^\beta}, \qquad \alpha,\beta>0,
\tag{4.1}
$$

and

$$
\frac{\frac{d^2}{dt^2}\|r_j-r_k\|^\alpha}{\|r_j-r_k\|^\beta}, \qquad \alpha>1 \text{ and } \beta>0.
\tag{4.2}
$$

Since both of the ratios in (4.1) are generalizations of the Hubble scale factor $\frac{dR}{dt}/R$, we refer to them as *alpha Hubble approximations/ratios*. For large t, we calculate the numerator of the (4.1) via (3.6). It just remains to estimate the denominator. By combining (3.3) with (9.1), we have already shown that for large t

$$
\|r_j-r_k\|^2 = \|a_j-a_k\|^2t^2\left[1+O\left(\frac{\ln t}{t}\right)\right]
$$

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Hence for $\beta > 0$, we find that as $t \to \infty$

$$\begin{aligned}
\|r_j - r_k\|^\beta &= \left[\|r_j - r_k\|^2\right]^{\frac{\beta}{2}} \left[\|a_j - a_k\|^2 t^2 \left[1 + O\left(\frac{\ln t}{t}\right)\right]\right]^{\frac{\beta}{2}} \\
&= \|a_j - a_k\|^\beta t^\beta \left[1 + O\left(\frac{\ln t}{t}\right)\right]^{\frac{\beta}{2}} \\
&= \|a_j - a_k\|^\beta t^\beta \left[1 + \frac{\beta}{2} O\left(t^{-1}\ln t\right) + \sum_{k=2}^{\infty} \binom{\frac{\beta}{2}}{k} \left(O\left(\frac{\ln t}{t}\right)\right)^k\right] \\
&= \|a_j - a_k\|^\beta t^\beta \left[1 + O\left(\frac{\ln t}{t}\right)\right]. \qquad (4.3)
\end{aligned}$$

Thus we find that as $t \to \infty$,

$$\begin{aligned}
\frac{\frac{d}{dt}\|r_j - r_k\|^\alpha}{\|r_j - r_k\|^\beta} &= \frac{\alpha\|a_j - a_k\|^\alpha t^{\alpha-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right]}{\|a_j - a_k\|^\beta t^\beta \left[1 + O\left(\frac{\ln t}{t}\right)\right]} \\
&= \alpha\|a_j - a_k\|^{\alpha-\beta} t^{\alpha-\beta-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right]\left[1 + O\left(\frac{\ln t}{t}\right)\right]^{-1} \\
&= \alpha\|a_j - a_k\|^{\alpha-\beta} t^{\alpha-\beta-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right]\left[1 - O\left(\frac{\ln t}{t}\right) + \sum_{k=2}^{\infty}(-1)^k\left(O\left(\frac{\ln t}{t}\right)\right)^k\right] \\
&= \alpha\|a_j - a_k\|^{\alpha-\beta} t^{\alpha-\beta-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right]. \qquad (4.4)
\end{aligned}$$

If $\alpha = \beta = 1$, Equation (4.4) becomes

$$\frac{\|r_j - r_k\|'}{\|r_j - r_k\|} = t^{-1}\left[1 + O\left(\frac{\ln t}{t}\right)\right] \text{ as } t \to \infty. \qquad (4.5)$$

To compute (4.2) we use (3.7), along with (4.3), to find that

$$\begin{aligned}
\frac{\frac{d^2}{dt^2}\|r_j - r_k\|^\alpha}{\|r_j - r_k\|^\beta} &= \frac{\alpha(\alpha-1)\|a_j - a_k\|^\alpha t^{\alpha-2}\left[1 + O\left(\frac{\ln t}{t}\right)\right]}{\|a_j - a_k\|^\beta t^\beta\left[1 + O\left(\frac{\ln t}{t}\right)\right]} \\
&= \alpha(\alpha-1)\|a_j - a_k\|^{\alpha-\beta} t^{\alpha-\beta-2}\left[1 + O\left(\frac{\ln t}{t}\right)\right]. \qquad (4.6)
\end{aligned}$$

If $\alpha = 2$ and $\beta = 1$, Equation (4.6) becomes

$$\frac{[\|r_j - r_k\|^2]''}{\|r_j - r_k\|} = 2t^{-1}\|a_j - a_k\|\left[1 + O\left(\frac{\ln t}{t}\right)\right] \text{ as } t \to \infty. \qquad (4.7)$$

5. The Asymptotic Calculation of Kinematic Vector Quantities

We begin this section with a variation of Proposition 9.2 which calculates the asymptotic evaluation of the vector difference of the velocity of two point masses.

Proposition 5.1. *Assume that for all $j, k = 1, \ldots, N$, $a_j - a_k \neq \overrightarrow{0}$. Then*

$$\left\|r_j'(t) - r_k'(t)\right\|^2 = \|a_j - a_k\|^2\left[1 + O\left(\frac{1}{t}\right)\right] \quad \text{as } t \to \infty. \qquad (5.1)$$

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Proof: Since $r_j(t) = a_j t + b_j \ln t + c_j + \delta_j(t)$, we find that

$$\begin{aligned}
\left\| r_j'(t) - r_k'(t) \right\|^2 &= [r_j'(t) - r_k'(t)]^T [r_j'(t) - r_k'(t)] \\
&= \left[a_j - a_k + \frac{b_j - b_k}{t} + \delta_j' - \delta_k' \right]^T \left[a_j - a_k + \frac{b_j - b_k}{t} + \delta_j' - \delta_k' \right] \\
= \|a_j - a_k\|^2 &\left[1 + \frac{2(a_j - a_k)^T (b_j - b_k)}{t \|a_j - a_k\|^2} + \frac{2(a_j - a_k)^T (\delta_j' - \delta_k')}{\|a_j - a_k\|^2} \right] \\
+ \|a_j - a_k\|^2 &\left[\frac{\|b_j - b_k\|^2}{t^2 \|a_j - a_k\|^2} + \frac{2(b_j - b_k)^T (\delta_j' - \delta_k')}{t \|a_j - a_k\|^2} + \frac{(\delta_j' - \delta_k')^T (\delta_j' - \delta_k')}{\|a_j - a_k\|^2} \right]. \qquad (5.2)
\end{aligned}$$

Since Equation (2.5) implies that $\delta_j'(t) = O\left(\frac{\ln t}{t^2}\right)$ as $t \to \infty$, for large t, Equation (5.2) becomes

$$\begin{aligned}
\left\| r_j'(t) - r_k'(t) \right\|^2 &= \|a_j - a_k\|^2 \left[1 + O\left(\frac{1}{t}\right) + O\left(\frac{\ln t}{t^2}\right) + O\left(\frac{1}{t^2}\right) \right] \\
&+ \|a_j - a_k\|^2 \left[\frac{O\left(\frac{\ln t}{t^2}\right)}{t} + O\left(\frac{\ln t}{t^2}\right) O\left(\frac{\ln t}{t^2}\right) \right] \\
= \|a_j - a_k\|^2 &\left[1 + O\left(\frac{1}{t}\right) + O\left(\frac{\ln t}{t^2}\right) + O\left(\frac{1}{t^2}\right) + O\left(\frac{\ln t}{t^3}\right) + O\left(\frac{\ln^2 t}{t^4}\right) \right]. \qquad (5.3)
\end{aligned}$$

Since

$$\frac{\frac{\ln t}{t^2}}{\frac{1}{t}} = \frac{\ln t}{t}, \quad \frac{\frac{\ln t}{t^3}}{\frac{1}{t}} = \frac{\ln t}{t^2}, \quad \frac{\frac{\ln^2 t}{t^4}}{\frac{1}{t}} = \frac{\ln t}{t} \frac{\ln t}{t^2},$$

and since

$$\lim_{t \to \infty} \frac{\ln t}{t} = 0, \qquad \lim_{t \to \infty} \frac{\ln t}{t^2} = 0,$$

we may use the definition of little o to rewrite (5.3) as

$$\left\| r_j'(t) - r_k'(t) \right\|^2 = \|a_j - a_k\|^2 \left[1 + O\left(\frac{1}{t}\right) + o\left(\frac{1}{t}\right) \right] = \|a_j - a_k\|^2 \left[1 + O\left(\frac{1}{t}\right) \right]. \qquad \square$$

The asymptotic evaluation of the magnitudes of the vector difference of the accelerations of two point masses is obtained from the following proposition.

Proposition 5.2. *Assume that for all $j, k = 1, \ldots, N$, $a_j - a_k \neq \vec{0}$. If for a fixed pair (j, k), with $j \neq k$, we have*

$$b_j - b_k \neq \vec{0}, \quad \text{where } b_i := -\sum_{j \neq i} \frac{m_j (a_j - a_i)}{\|a_j - a_i\|^3}, \quad 1 \leq i \leq N, \qquad (5.4)$$

then

$$\left\| r_j''(t) - r_k''(t) \right\|^2 = t^{-4} \|b_j - b_k\|^2 \left[1 + O\left(\frac{\ln t}{t}\right) \right] \quad \text{as } t \to \infty. \qquad (5.5)$$

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Proof: Since $r_j(t) = a_j t + b_j \ln t + c_j + \delta_j(t)$, we find that

$$\begin{aligned}
\left\|r_j''(t) - r_k''(t)\right\|^2 &= [r_j''(t) - r_k''(t)]^T [r_j''(t) - r_k''(t)] \\
&= \left[-t^{-2}[b_j - b_k - t^2(\delta_j'' - \delta_k'')]\right]^T \left[-t^{-2}[b_j - b_k - t^2(\delta_j'' - \delta_k'')]\right] \\
= t^{-4}\Big[(b_j - b_k)^T(b_j - b_k) &- 2t^2(b_j - b_k)^T(\delta_j'' - \delta_k'') + t^4(\delta_j'' - \delta_k'')^T(\delta_j'' - \delta_k'')\Big] \\
= t^{-4}\|b_j - b_k\|^2 &\left[1 - \frac{2t^2(b_j - b_k)^T(\delta_j'' - \delta_k'') - t^4\left\|\delta_j'' - \delta_k''\right\|^2}{\|b_j - b_k\|^2}\right].
\end{aligned} \tag{5.6}$$

Since Equation (2.5) of Theorem 2.1 implies that $\delta_j''(t) = O\left(\frac{\ln t}{t^3}\right)$ as $t \to \infty$, for large t we may rewrite Equation (5.6) as

$$\begin{aligned}
\left\|r_j''(t) - r_k''(t)\right\|^2 &= t^{-4}\|b_j - b_k\|^2 \left[1 - \frac{2t^2(b_j - b_k)^T O\left(\frac{\ln t}{t^3}\right) - t^4\left[O\left(\frac{\ln t}{t^3}\right)\right]^2}{\|b_j - b_k\|^2}\right] \\
&= t^{-4}\|b_j - b_k\|^2 \left[1 - \frac{2(b_j - b_k)^T O\left(\frac{\ln t}{t}\right) - O\left(\frac{\ln^2 t}{t^2}\right)}{\|b_j - b_k\|^2}\right].
\end{aligned} \tag{5.7}$$

Observe that

$$\frac{\frac{\ln^2 t}{t^2}}{\frac{\ln t}{t}} \equiv \frac{\ln t}{t} \to 0 \text{ as } t \to \infty \Longrightarrow O\left(\frac{\ln^2 t}{t^2}\right) = o\left(\frac{\ln t}{t}\right) \text{ as } t \to \infty. \tag{5.8}$$

If we substitute (5.8) into (5.7) we obtain (5.5) as desired □.

Remark 5.1. *If $b_j - b_k = \vec{0}$, then $r_j''(t) - r_k''(t) = \delta_j'' - \delta_k''$, and*

$$\|r_j'' - r_k''\|^2 = (r_j'' - r_k'')^T(r_j'' - r_k'') = (\delta_j'' - \delta_k'')^T(\delta_j'' - \delta_k'') = O\left(\frac{\ln^2 t}{t^6}\right), \tag{5.9}$$

since $\delta_j'' = O(t^{-3}\ln t)$.

6. Vectorial Alpha Hubble Approximations

In this section we look at a variation of the alpha Hubble ratios inspired by Propositions 5.1 and 5.2. In particular, for large t we want to analyze the behavior of

$$\frac{\|r_j' - r_k'\|^\alpha}{\|r_j - r_k\|^\beta}, \qquad \frac{\|r_j'' - r_{k'}''\|^\alpha}{\|r_j - r_k\|^\beta}, \qquad \alpha, \beta \geq 1. \tag{6.1}$$

We will refer to the above as *vectorial alpha Hubble approximations/ratios*. The denominator of each type of vectorial alpha Hubble approximation is calculated via (4.3). To calculate the numerator of the first vectorial alpha Hubble approximation we adjust (5.1) as follows.

$$\begin{aligned}
\|r_j' - r_k'\|^\alpha = \left[\|r_j' - r_k'\|^2\right]^{\frac{\alpha}{2}} &= \left[\|a_j - a_k\|^2\left[1 + O\left(\frac{1}{t}\right)\right]\right]^{\frac{\alpha}{2}} \\
&= \|a_j - a_k\|^\alpha\left[1 + \frac{\alpha}{2}O\left(t^{-1}\right) + \sum_{k=2}^{\infty}\binom{\frac{\alpha}{2}}{k}\left(O\left(\frac{1}{t}\right)\right)^k\right] \\
&= \|a_j - a_k\|^\alpha\left[1 + O\left(\frac{1}{t}\right)\right].
\end{aligned} \tag{6.2}$$

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Hence, as $t \to \infty$

$$\begin{aligned}\frac{\|r'_j - r'_k\|^\alpha}{\|r_j - r_k\|^\beta} &= \frac{\|a_j - a_k\|^\alpha \left[1 + O\left(\frac{1}{t}\right)\right]}{\|a_j - a_k\|^\beta t^\beta \left[1 + O\left(\frac{\ln t}{t}\right)\right]} \\ &= \|a_j - a_k\|^{\alpha-\beta} t^{-\beta} \left[1 + O\left(\frac{1}{t}\right)\right] \left[1 + O\left(\frac{\ln t}{t}\right)\right]^{-1} \\ &= \|a_j - a_k\|^{\alpha-\beta} t^{-\beta} \left[1 + O\left(\frac{1}{t}\right)\right] \left[1 + O\left(\frac{\ln t}{t}\right)\right] \\ &= \|a_j - a_k\|^{\alpha-\beta} t^{-\beta} \left[1 + O\left(\frac{\ln t}{t}\right)\right]. \end{aligned} \tag{6.3}$$

If $\alpha = \beta = 1$, Equation (6.3) becomes

$$\frac{\|r'_j - r'_k\|}{\|r_j - r_k\|} = t^{-1} \left[1 + O\left(\frac{\ln t}{t}\right)\right], \text{ as } t \to \infty. \tag{6.4}$$

This is the same order of growth as provided by (4.5).

Next we turn our attention to the second vectorial alpha Hubble approximation. We will assume that $b_j - b_k \neq \overrightarrow{0}$. We can adjust (5.5) for large t as follows.

$$\begin{aligned}\|r''_j - r''_k\|^\alpha = \left[\|r''_j - r''_k\|^2\right]^{\frac{\alpha}{2}} &= \left[t^{-4}\|b_j - b_k|^2 \left[1 + O\left(\frac{\ln t}{t}\right)\right]\right]^{\frac{\alpha}{2}} \\ = t^{-\frac{\alpha}{2}} \|b_j - b_k\|^\alpha \left[1 + O\left(\frac{\ln t}{t}\right)\right]^{\frac{\alpha}{2}} &= t^{-\frac{\alpha}{2}} \|b_j - b_k\|^\alpha \left[1 + O\left(\frac{\ln t}{t}\right)\right] \end{aligned} \tag{6.5}$$

We then combine the above calculation with (4.3) to discover that as $t \to \infty$,

$$\begin{aligned}\frac{\|r''_j - r''_k\|^\alpha}{\|r_j - r_k\|^\beta} &= \frac{t^{-\frac{\alpha}{2}} \|b_j - b_k\|^\alpha \left[1 + O\left(\frac{\ln t}{t}\right)\right]}{\|a_j - a_k\|^\beta t^\beta \left[1 + O\left(\frac{\ln t}{t}\right)\right]} \\ &= t^{-\frac{\alpha}{2}-\beta} \frac{\|b_j - b_k\|^\alpha}{\|a_j - a_k\|^\beta} \left[1 + O\left(\frac{\ln t}{t}\right)\right] \left[1 + O\left(\frac{\ln t}{t}\right)\right] \\ &= t^{-\frac{\alpha}{2}-\beta} \frac{\|b_j - b_k\|^\alpha}{\|a_j - a_k\|^\beta} \left[1 + O\left(\frac{\ln t}{t}\right)\right]. \end{aligned} \tag{6.6}$$

If $\alpha = \beta = 1$, Equation (6.6) becomes

$$\frac{\|r''_j - r''_k\|}{\|r_j - r_k\|} = t^{-\frac{3}{2}} \frac{\|b_j - b_k\|}{\|a_j - a_k\|} \left[1 + O\left(\frac{\ln t}{t}\right)\right], \text{ as } t \to \infty. \tag{6.7}$$

If $\alpha = 2$ and $\beta = 1$, Equation (6.6) becomes

$$\frac{\|r''_j - r''_k\|^2}{\|r_j - r_k\|} = t^{-2} \frac{\|b_j - b_k\|^2}{\|a_j - a_k\|} \left[1 + O\left(\frac{\ln t}{t}\right)\right], \text{ as } t \to \infty, \tag{6.8}$$

which is a much smaller than the order of growth provided by (4.7), namely $O(t^{-2})$ versus $O(t^{-1})$.

If $b_j - b_k = \overrightarrow{0}$, we use (5.9) and observe that

$$\|r''_j - r''_k\|^\alpha = \left[\|r''_j - r''_k\|^2\right]^{\frac{\alpha}{2}} = \left[O\left(\frac{\ln^2 t}{t^6}\right)\right]^{\frac{\alpha}{2}} = O\left(\frac{\ln^\alpha t}{t^{3\alpha}}\right). \tag{6.9}$$

In that case, the second vectorial alpha Hubble approximation for large t becomes

$$\begin{aligned}\frac{\|r_j'' - r_k''\|^\alpha}{\|r_j - r_k\|^\beta} &= \frac{O\left(\frac{\ln^\alpha t}{t^{3\alpha}}\right)}{\|a_j - a_k\|^\beta t^\beta \left[1 + O\left(\frac{\ln t}{t}\right)\right]} \\ &= \|a_j - a_k\|^{-\beta} t^{-\beta} O\left(\frac{\ln^\alpha t}{t^{3\alpha}}\right)\left[1 + O\left(\frac{\ln t}{t}\right)\right]^{-1} \\ &= \|a_j - a_k\|^{-\beta} t^{-\beta} O\left(\frac{\ln^\alpha t}{t^{3\alpha}}\right)\left[1 + O\left(\frac{\ln t}{t}\right)\right] \\ &= \|a_j - a_k\|^{-\beta} t^{-\beta} O\left(\frac{\ln^\alpha t}{t^{3\alpha}}\right),\end{aligned} \tag{6.10}$$

where in the last equality we implicitly assumed that $\alpha \geq 0$. When $\alpha = \beta = 1$, Equation 6.10) becomes

$$\frac{\|r_j'' - r_k''\|}{\|r_j - r_k\|} = \|a_j - a_k\|^{-1} t^{-1} O\left(\frac{\ln t}{t^3}\right), \text{ as } t \to \infty, \tag{6.11}$$

while if $\alpha = 2$ and $\beta = 1$, Equation 6.10) becomes

$$\frac{\|r_j'' - r_k''\|^2}{\|r_j - r_k\|} = \|a_j - a_k\|^{-1} t^{-1} O\left(\frac{\ln^2 t}{t^6}\right), \text{ as } t \to \infty. \tag{6.12}$$

7. What is the Acceleration Rate of Our Expanding Universe? How Many Accelerated Expanding Universes Exist?

Let $t \in [t_0, \infty)$. The relation

$$H(t) := \frac{d(\ln r_i(t))}{dt} = \frac{\frac{dr_i(t)}{dt}}{w(t)} \iff \frac{dr_i(t)}{dt} = H(t) r_i(t) \tag{7.1}$$

is called a traditional Hubble law with Hubble constant $H(t)$. Equation (7.1) is a manifestation of the fact that the rate of change of the distance is proportional to the distance. In this paper, instead of using $r_i(t)$, we looked at relative differences $r_j(t) - r_k(t)$ and calculated

$$\hat{H}(t) := \frac{\frac{d}{dt}\|r_j(t) - r_k(t)\|}{\|r_j(t) - r_k(t)\|} = \frac{1}{2}\frac{d}{dt}\ln(\|r_j(t) - r_k(t)\|^2) = \frac{\frac{d}{dt}\|r_j(t) - r_k(t)\|^2}{2\|r_j(t) - r_k(t)\|^2}; \tag{7.2}$$

see (4.5). Without loss of generality, we can assume that $r_k(t) \equiv r_1(t)$ and average, for $2 \leq j \leq N$, all $\hat{H}(t)$ to compute a statistical Hubble expansion average (HAVG) as defined below.

$$HAVG := \frac{1}{N-1}\sum_{j=2}^{N} \frac{\frac{d}{dt}\|r_j(t) - r_1(t)\|}{\|r_j(t) - r_1(t)\|} \to \frac{1}{N-1}\sum_{j=2}^{N}\frac{1}{t} \quad \text{as } t \to \infty. \tag{7.3}$$

The last conclusion of (7.3) reflects the order of growth provided by (4.5).

There are other ways to obtain statistical averages of acceleration Equation (3.7) shows that any two different celestial bodies may accelerate away from each other by a different rate. We can use statistical averages in order to define one measure of acceleration by the formula

$$\text{AVA} := \frac{\sum_{1 \leq j < k \leq N} \alpha(\alpha - 1)\|a_j - a_k\|^\alpha t^{\alpha - 2}}{\binom{N}{2}}, \tag{7.4}$$

where AVA stands for Average of Acceleration.

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Remark 7.1. *Noteworthy is the case $\alpha = 2$ in (3.7) as a transitional value for the nature of the measures of acceleration. It renders the rate of acceleration to be a positive constant that depends on the velocities of two point masses. Namely, $2\|a_j - a_k\|^2$. However, for $1 < \alpha < 2$, the rate of acceleration is positive for a long time but shrinks to zero as $t \to \infty$. Namely, $\frac{\alpha(\alpha-1)\|a_j-a_k\|^\alpha}{t^{2-\alpha}} \to 0$ as $t \to \infty$. For $2 < \alpha$, the rate of acceleration is positive for large t but tends to infinity as $t \to \infty$. Namely, $\alpha(\alpha-1)\|a_j - a_k\|^\alpha t^{\alpha-2} \to \infty$ as $t \to \infty$. We also note that the value $\alpha = 1$ is also interesting transitional value. However, it requires a different asymptotic analysis.*

For a universe which endorses the big bang theory of cosmology and has one measure of accelerated expansion we can choose the parameters a_j in such a manner that asymptotically all bodies lie on an expanding sphere. For this assumption to hold we choose a positive constant ρ and put

$$r_j(t) = a_j t + b_j \ln t + c_j + \delta_j(t) = \rho U_j t + b_j \ln t + c_j + \delta_j(t), \qquad t \geq t_0. \tag{7.5}$$

Recall that $\|U_j\| = 1$ for $1 \leq j \leq N$. Evidently the arrow representing $r_j(t)$ has tail at the center of our coordinate system and head at the center of mass of the celestial body j. It satisfies

$$\|r_j(t)\| \sim \|\rho U_j t\| = \rho t, \qquad \frac{d(\|r_j(t)\|)}{dt} = \rho, \quad \text{as } t \to \infty. \tag{7.6}$$

Moreover, as $t \to \infty$,

$$\|r_j(t)\|^2 \sim \rho^2 t^2, \qquad \frac{d(\|r_j(t)\|^2)}{dt} \sim 2\rho^2 t, \qquad \frac{d^2(\|r_j(t)\|^2)}{dt^2} = 2\rho^2. \tag{7.7}$$

Formulas (7.6) and (7.7) say the following.

1. There exist spherical models of expansion of a universe. In these, all N celestial bodies lie, for large time t, on an expanding sphere with radius ρt.
2. The speed of expansion for large time t, as measured by the rate of change of the radius of the sphere, is ρ. However, the rate of expansion as measured by the rate of change of the radius squared of the sphere is $2\rho^2 t$.
3. The rate of acceleration as measured by the rate of change of the radius squared of the sphere is $2\rho^2$.

One may choose a cosmological scale $\mu > 0$ that ultimately makes the quantity $2\rho^2\mu$ fit experimental data.

We note that what humans observe and study is their particular universe. However, Newton's equations of celestial mechanics are a source of an infinite number of diverse and different universes. Each universe corresponds to and is determined by the $2N$ initial vector values $(r_j(t_0), r_j'(t_0))$. Each pair (a_j, c_j) corresponds to a unique pair of initial conditions $(r_j(t_0), r_j'(t_0))$. The different choices of $(a_j, c_j), j = 1, 2, 3, \cdots, N$, subject to $a_j - a_k \neq \overrightarrow{0}$, generates a different expanding universe.

8. Conclusion

This paper proposes asymptotic formulas for cosmological models of an accelerating expanding universe based on Newton's equations of celestial mechanics. The asymptotic formulas of Theorem 3.2 and Proposition 5.2, along with the ensuing alpha Hubble ratios of (4.1), (4.2), and (6.1), are complete with estimates of asymptotic remainders terms that measure

i.) The relative speed between any two point masses.
ii.) The relative acceleration between any two point masses.
iii.) The rate of expansion of an isotropic and homogenous universe.
iv.) Generalized types of Hubble formulas.
v.) A comparison between scalar and vectorial accelerations.

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The above derivations are absent of dark matter.

9. Appendix A: Asymptotic Analysis of $\|r_j - r_k\|^2$

This section provides the proof of Theorem 3.1 and begins with a series of auxiliary propositions. In order to make the presentation less cumbersome we will suppress, from time to time, the dependence of t in the vector functions.

Proposition 9.1. *Let $f_i(t) = a_i t + b_i \ln t + c_i$, where a_i and c_i, for $1 \le i \le N$ are constant vector in $\mathbb{R}^3$ satisfying $\|a_j - a_i\| \ne 0$, $i \ne j$, and where b_i is defined via (2.4). Then for $j \ne k$,*

$$\|f_j - f_k\|^2 = \|a_j - a_k\|^2 t^2 \left[1 + O\left(\frac{\ln t}{t}\right)\right] \text{ as } t \to \infty, \tag{9.1}$$

and

$$\|f_j - f_k\| = \|a_j - a_k\| t \left[1 + O\left(\frac{\ln t}{t}\right)\right] \text{ as } t \to \infty. \tag{9.2}$$

Proof: By definition

$$\begin{aligned}
\|f_j - f_k\|^2 &= \left[(a_j - a_k)t + (b_j - b_k)\ln t + c_j - c_k\right]^T \left[(a_j - a_k)t + (b_j - b_k)\ln t + c_j - c_k\right] \\
&= \|a_j - a_k\|^2 t^2 \left[1 + \frac{2(a_j - a_k)^T (b_j - b_k)\ln t}{\|a_j - a_k\|^2 t} + \frac{2(a_j - a_k)^T (c_j - c_k)}{\|a_j - a_k\|^2 t}\right] \\
&+ \|a_j - a_k\|^2 t^2 \left[\frac{\|b_j - b_k\|^2 \ln^2 t}{\|a_j - a_k\|^2 t^2} + \frac{2(b_j - b_k)^T (c_j - c_k)\ln t}{\|a_j - a_k\|^2 t^2} + \frac{\|c_j - c_k\|^2}{\|a_j - a_k\|^2 t^2}\right].
\end{aligned} \tag{9.3}$$

Equation (9.1) immediately follows from (9.3) and the definition of Big O. Without loss of generality, assume that for $[t_0, \infty)$,

$$\left|O\left(\frac{\ln t}{t}\right)\right| \le \rho < 1. \tag{9.4}$$

The existence of such a t_0 is guaranteed since $\lim_{t \to \infty} \ln t / t = 0$. By working over $[t_0, \infty)$ we may take the square root of (9.1),

$$\|f_j - f_k\| = \|a_j - a_k\| t \sqrt{1 + O\left(\frac{\ln t}{t}\right)}$$

and then apply the binomial theorem to write

$$\sqrt{1 + O\left(\frac{\ln t}{t}\right)} = 1 + \frac{O\left(\frac{\ln t}{t}\right)}{2} + \sum_{k=2}^{\infty} \binom{\frac{1}{2}}{k} \left[O\left(\frac{\ln t}{t}\right)\right]^k = 1 + O\left(\frac{\ln t}{t}\right) \qquad \square.$$

Proposition 9.2. *Let $f_i(t) = a_i t + b_i \ln t + c_i$, where a_i and c_i, for $1 \le i \le N$ are constant vector in $\mathbb{R}^3$ satisfying $\|a_j - a_i\| \ne 0$, $i \ne j$, and where b_i is defined via (2.4). Then for $j \ne k$,*

$$\|f'_j - f'_k\|^2 = \|a_j - a_k\|^2 \left[1 + O\left(\frac{1}{t}\right)\right] \text{ as } t \to \infty, \tag{9.5}$$

and

$$\|f'_j - f'_k\| = \|a_j - a_k\| \left[1 + O\left(\frac{1}{t}\right)\right] \text{ as } t \to \infty. \tag{9.6}$$

Proof: The definition of $f_j(t)$ implies that

$$\begin{aligned}\|f_j' - f_k'\|^2 &= \left[a_j - a_k + \frac{b_j - b_k}{t}\right]^T \left[a_j - a_k + \frac{b_j - b_k}{t}\right] \\ &= \|a_j - a_k\|^2 \left[1 + \frac{2(a_j - a_k)^T (b_j - b_k)}{\|a_j - a_k\|^2 t} + \frac{\|b_j - b_k\|^2}{\|a_j - a_k\|^2 t^2}\right]. \end{aligned} \tag{9.7}$$

Equation (9.5) the follows from (9.7) and the definition of Big O. Without loss of generality, assume that for $[t_0, \infty)$,

$$\left|O\left(\frac{1}{t}\right)\right| \le \rho < 1. \tag{9.8}$$

The existence of such a t_0 is guaranteed since $\lim_{t\to\infty} t^{-1} = 0$. By working over $[t_0, \infty)$ we may take the square root of (9.5),

$$\|f_j - f_k\| = \|a_j - a_k\| \sqrt{1 + O\left(\frac{1}{t}\right)}$$

and then apply the binomial theorem to write

$$\sqrt{1 + O\left(\frac{1}{t}\right)} = 1 + \frac{O\left(\frac{1}{t}\right)}{2} + \sum_{k=2}^{\infty} \binom{\frac{1}{2}}{k} \left[O\left(\frac{1}{t}\right)\right]^k = 1 + O\left(\frac{1}{t}\right) \qquad \square.$$

Proposition 9.3. *Let $f_i(t) = a_i t + b_i \ln t + c_i$, where a_i and c_i, for $1 \le i \le N$ are constant vector in $\mathbb{R}^3$ satisfying $\|a_j - a_i\| \neq 0$, $i \neq j$, and where b_i is defined via (2.4). Then for $j \neq k$,*

$$(f_j' - f_k')^T (f_j - f_k) = (f_j - f_k)^T (f_j' - f_k') = \|a_j - a_k\|^2 t \left[1 + O\left(\frac{\ln t}{t}\right)\right] \text{ as } t \to \infty, \tag{9.9}$$

If $b_j - b_k \neq \vec{0}$, then

$$(f_j'' - f_k'')^T (f_j - f_k) = (f_j - f_k)^T (f_j'' - f_k'') = O\left(\frac{1}{t}\right) \text{ as } t \to \infty, \tag{9.10}$$

and

$$(f_j'' - f_k'')^T (f_j' - f_k') = (f_j' - f_k')^T (f_j'' - f_k'') = O\left(\frac{1}{t^2}\right) \text{ as } t \to \infty. \tag{9.11}$$

If $b_j - b_k = \vec{0}$, then

$$(f_j' - f_k')^T (f_j - f_k) = (f_j'' - f_k'')^T (f_j - f_k) = \vec{0}, \tag{9.12}$$

and (9.10) and (9.11) are automatically satisfied.

Proof: The definition of $f_j(t)$ implies that

$$\begin{aligned}(f_j' - f_k')^T (f_j - f_k) &= \left[a_j - a_k + \frac{b_j - b_k}{t}\right]^T [(a_j - a_k)t + (b_j - b_k)\ln t + c_j - c_k] \\ &= \|a_j - a_k\|^2 t \left[1 + \frac{(a_j - a_k)^T (b_j - b_k)\ln t}{\|a_j - a_k\|^2 t} + \frac{(a_j - a_k)^T (c_j - c_k)}{\|a_j - a_k\|^2 t}\right] \\ &+ \|a_j - a_k\|^2 t \left[\frac{(b_j - b_k)^T (a_j - a_k)}{\|a_j - a_k\|^2 t} + \frac{\|b_j - b_k\|^2 \ln t}{\|a_j - a_k\|^2 t^2} + \frac{(b_j - b_k)^T (c_j - c_k)}{\|a_j - a_k\|^2 t^2}\right]. \end{aligned} \tag{9.13}$$

Equation (9.9) immediately follows from (9.13) and the definition of Big O.

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The definition of $f_j(t)$ also implies that

$$\begin{aligned}(f_j'' - f_k'')^T(f_j - f_k) &= -\frac{1}{t^2}(b_j - b_k)^T\left[(a_j - a_k)t + (b_j - b_k)\ln t + c_j - c_k\right] \\ &= -\frac{(b_j - b_k)^T(a_j - a_k)}{t} - \frac{\|b_j - b_k\|^2 \ln t}{t^2} - \frac{(b_j - b_k)^T(c_j - c_k)}{t^2},\end{aligned} \tag{9.14}$$

and that

$$\begin{aligned}(f_j'' - f_k'')^T(f_j' - f_k') &= -\frac{1}{t^2}(b_j - b_k)^T\left[a_j - a_k + \frac{b_j - b_k}{t}\right]^T \\ &= -\frac{(b_j - b_k)^T(a_j - a_k)}{t^2} - \frac{\|b_j - b_k\|^2}{t^3}.\end{aligned} \tag{9.15}$$

Then (9.10) and (9.11) respectively follow by applying the definition of Big O to (9.14) and (9.15) respectively. $\square$.

Proof of Theorem 3.1: Since $r_j(t) = f_j(t) + \delta_j(t)$, we find that

$$\begin{aligned}\|r_j(t) - r_k(t)\|^2 &= (r_j(t) - r_k(t))^T(r_j(t) - r_k(t)) \\ &= \left[f_j(t) - f_k(t) + \delta_j(t) - \delta_k(t)\right]^T\left[f_j(t) - f_k(t) + \delta_j(t) - \delta_k(t)\right] \\ &= (f_j - f_k)^T(f_j - f_k) + 2(f_j - f_k)^T(\delta_j - \delta_k) + (\delta_j - \delta_k)^T(\delta_j - \delta_k) \\ &= \|f_j - f_k\|^2\left[1 + \frac{2(f_j - f_k)^T(\delta_j - \delta_k) + (\delta_j - \delta_k)^T(\delta_j - \delta_k)}{\|f_j - f_k\|^2}\right],\end{aligned}$$

where we implicitly made use of the fact that $\|f_j - f_k\| \neq 0$; see Condition *iv*. of Theorem 2.1. Consequently, the triangle inequality, when combined with the Cauchy-Schwartz inequality and the above observation, implies that

$$\begin{aligned}\left|\frac{2(f_j - f_k)^T(\delta_j - \delta_k) + (\delta_j - \delta_k)^T(\delta_j - \delta_k)}{\|f_j - f_k\|^2}\right| &\leq \frac{2\|f_j - f_k\|\|\delta_j - \delta_k\| + \|\delta_j - \delta_k\|^2}{\|f_j - f_k\|^2} \\ &\leq \frac{2\|\delta_j - \delta_k\|}{\|f_j - f_k\|} + \frac{\|\delta_j - \delta_k\|^2}{\|f_j - f_k\|^2} \leq \frac{2\|\delta_j - \delta_k\|}{\omega} + \frac{\|\delta_j - \delta_k\|^2}{\omega^2} \\ &= O\left(\frac{\ln t}{t}\right) + \left[O\left(\frac{\ln t}{t}\right)\right]^2 = O\left(\frac{\ln t}{t}\right),\end{aligned}$$

where the last line made use of (2.5). In summary, we have shown that

$$\|r_j(t) - r_k(t)\|^2 = \|f_j - f_k\|^2\left[1 + O\left(\frac{\ln t}{t}\right)\right] \text{ as } t \to \infty,$$

which is precisely (3.3).

Next we will verify (3.4). By definition

$$\frac{d\|r_j(t) - r_k(t)\|^2}{dt} = V_{jk}' + W_{jk}' + Z_{jk}', \tag{9.16}$$

where V_{jk}, W_{jk}, and Z_{jk} are defined via (3.1) and (3.2). We will show that V_{jk}' is the leading asymptotic term of (9.16) as $t \to \infty$. Indeed, by (9.9), we have

$$V_{jk}' = 2(f_j' - f_k')^T(f_j - f_k) = 2\|a_j - a_k\|^2 t\left[1 + O\left(\frac{\ln t}{t}\right)\right] \text{ as } t \to \infty, \tag{9.17}$$

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which implies that $V'_{jk} \neq 0$ as $t \to \infty$. Since $f'_j(t) = a_j + b_j/t$ and since (2.5) implies that

$$\delta_j - \delta_k = O\left(\frac{\ln t}{t}\right), \qquad \delta'_j - \delta'_k = O\left(\frac{\ln t}{t^2}\right),$$

we discover that as $t \to \infty$

$$\begin{aligned} W'_{jk} &= 2(f'_j - f'_k)^T(\delta_j - \delta_k) + 2(f_j - f_k)^T(\delta'_j - \delta'_k) \\ &= O\left(\frac{\ln t}{t}\right) + tO\left(\frac{\ln t}{t^2}\right) = O\left(\frac{\ln t}{t}\right), \end{aligned}$$

and

$$Z'_{jk} = 2(\delta_j - \delta_k)^T(\delta'_j - \delta'_k) = O\left(\frac{\ln t}{t}\right)O\left(\frac{\ln t}{t^2}\right) = O\left(\frac{\ln^2 t}{t^3}\right).$$

Putting all the above together we obtain

$$\begin{aligned} \frac{d\|r_j(t) - r_k(t)\|^2}{dt} &= V'_{jk} + W'_{jk} + Z'_{jk} = V'_{jk}\left[1 + \frac{W'_{jk} + Z'_{jk}}{V'_{jk}}\right] \\ &= V'_{jk}\left[1 + \frac{O(\frac{\ln t}{t}) + O(\frac{\ln^2 t}{t^3})}{2t\|a_j - a_k\|^2[1 + O(t^{-1}\ln t)]}\right] \\ &= V'_{jk}\left[1 + \frac{O(\frac{\ln t}{t^2})}{1 + O(t^{-1}\ln t)}\right] = V'_{jk}\left[1 + O\left(\frac{\ln t}{t^2}\right)\left(1 + O(t^{-1}\ln t)\right)^{-1}\right] \\ &= V'_{jk}\left[1 + O\left(\frac{\ln t}{t^2}\right)\left(1 - O\left(\frac{\ln t}{t}\right) + \sum_{p=2}^{\infty}(-1)^p\left(O\left(\frac{\ln t}{t}\right)\right)^k\right)\right] \\ &= V'_{jk}\left[1 + O\left(\frac{\ln t}{t^2}\right)\right], \end{aligned}$$

where we have assumed that we are working over an interval $[t_0, \infty)$ such that $|O(t^{-1}\ln t)| < \rho < 1$.

We turn to the second derivatives. Since

$$V''_{jk} = 2\left[(f'_j - f'_k)^T(f_j - f_k)\right]' = 2\left[(f''_j - f''_k)^T(f_j - f_k) + (f'_j - f'_k)^T(f'_j - f'_k)\right],$$

Equation (9.10) and Proposition 9.2 imply that

$$\begin{aligned} V''_{jk} &= O(t^{-1}) + 2(f'_j - f'_k)^T(f'_j - f'_k) = O(t^{-1}) + 2\|a_j - a_k\|^2\left[1 + O(t^{-1})\right] \\ &= 2\|a_j - a_k\|^2\left[1 + O(t^{-1})\right] \text{ as } t \to \infty. \end{aligned}$$

Next consider W''_{jk}. Since $f'_j(t) = a_j + b_j t^{-1}$, since $f''_j(t) = b_j t^{-2}$, and since (2.5) implies that

$$\delta_j - \delta_k = O\left(\frac{\ln t}{t}\right), \quad \delta'_j - \delta'_k = O\left(\frac{\ln t}{t^2}\right), \quad \delta''_j - \delta''_k = O\left(\frac{\ln t}{t^3}\right),$$

we find that as $t \to \infty$

$$\begin{aligned} W''_{jk} &= 2[(f'_j - f'_k)^T(\delta_j - \delta_k) + (f_j - f_k)^T(\delta'_j - \delta'_k)]' \\ &= 2[(f''_j - f''_k)^T(\delta_j - \delta_k) + 2(f'_j - f'_k)^T(\delta'_j - \delta'_k) + (f_j - f_k)^T(\delta''_j - \delta''_k)] \\ &= O(t^{-2})O\left(\frac{\ln t}{t}\right) + O\left(\frac{\ln t}{t^2}\right) + O(t)O\left(\frac{\ln t}{t^2}\right) = O\left(\frac{\ln t}{t}\right). \end{aligned}$$

Finally we turn to

$$
\begin{aligned}
Z''_{jk} &= [(\delta_j - \delta_k)^T(\delta_j - \delta_k)]'' = 2(\delta''_j - \delta''_k)^T(\delta_j - \delta_k) + 2(\delta'_j - \delta'_k)^T(\delta'_j - \delta'_k) \\
&= O\left(\frac{\ln t}{t^3}\right)O\left(\frac{\ln t}{t}\right) + O\left(\frac{\ln t}{t^2}\right)O\left(\frac{\ln t}{t^2}\right) = O\left(\frac{\ln^2 t}{t^4}\right).
\end{aligned}
$$

Putting all the above together we get

$$
\begin{aligned}
\frac{d^2\|r_j(t) - r_k(t)\|^2}{dt^2} &= V''_{jk} + W''_{jk} + Z''_{jk} = V''_{jk}\left[1 + \frac{W''_{jk} + Z''_{jk}}{V''_{jk}}\right] \\
&= V''_{jk}\left[1 + \frac{O(\frac{\ln t}{t}) + O(\frac{\ln^2 t}{t^4})}{2\|a_j - a_k\|^2[1 + O(t^{-1})]}\right] \\
&= V''_{jk}\left[1 + \left(O\left(\frac{\ln t}{t}\right) + O\left(\frac{\ln^2 t}{t^4}\right)\right)\left(1 + O(t^{-1})\right)^{-1}\right] \\
&= V''_{jk}\left[1 + \left(O\left(\frac{\ln t}{t}\right) + O\left(\frac{\ln^2 t}{t^4}\right)\right)\left(1 - O(t^{-1}) + \sum_{p=2}^{\infty}(-1)^p\left(O(t^{-1})\right)^p\right)\right] \\
&= V''_{jk}\left[1 + O\left(\frac{\ln t}{t}\right)\right]. \qquad \square
\end{aligned}
$$

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