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Random Triangle Theory: a Computational Approach

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Abstract

In this work we study the following problem, from a computational point of view: *If three points are selected in the unit square at random, what is the probability that the triangle obtained is obtuse, acute or right?* We provide two convergent strategies: the first derived from the ideas introduced in [2] and the second built on the combinatorics theory. The combined use of these two methods allows us to address the random triangle theory from a new perspective and, we hope, to work out a general method of dealing with some classes of computational problems.

Keywords: Random Triangle; Quasiorthogonal Dimension; Combinatorics; Computational Problems

1 Introduction

In [1] we read: " *We hope to encourage others to look again (and differently) at triangles*". Following this encouragement we study random triangle generation from a merely computational point of view. The square triangle problem is defined in [5] as " *The selection of triples of points (corresponding to the endpoints of a triangle) randomly placed inside a square (for simplicity here we consider the unit square)*". Given three points chosen inside a unit square at random, one may evaluate the average area of the triangle determined by these points [5] or study the probability that the related random triangle is acute, both from a historical and from a modern point of view [1]; or obtuse [3, 4, 5]. Using computational lenses, we study a more general problem: *if three points are selected in a unit square at random, how can we compute the probability that the triangle obtained is obtuse, acute or right? And what error may occur in this calculation?* To the purpose we introduce two convergent computational methods. The first is based

on a simple idea: in $\mathbb{R}^2$ we allow a tolerance in angle measurement, to obtain a new way of classifying angles and then geometrical figures. This approach

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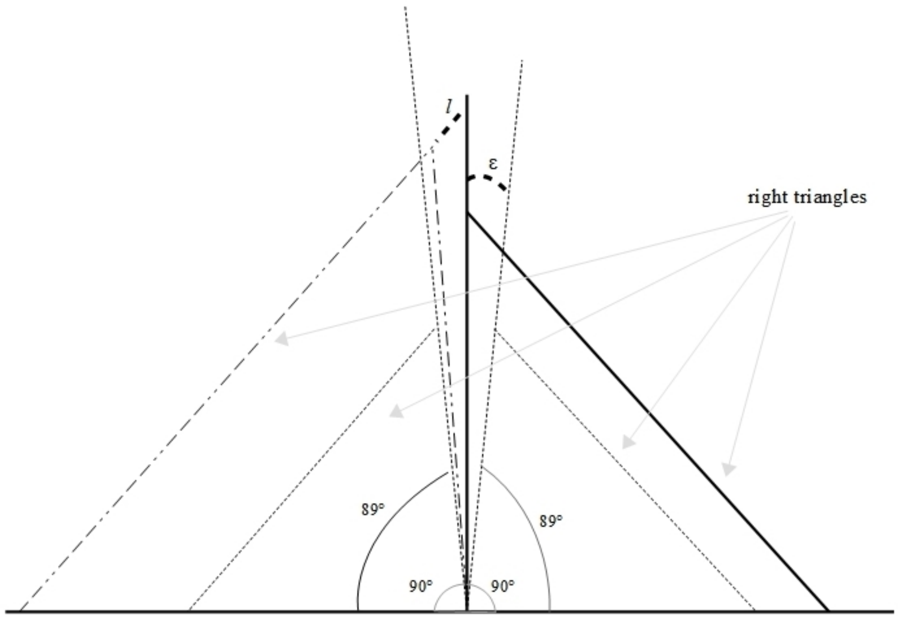

Figure 1: Angles, segments and triangles with measurement tolerance

also finds its elegant mathematical foundation in [2]. The second method is built on combinatorics, that allow us to enumerate (count) specific structures like triangles. The paper is organized as follows: Sections 2 and 3 define the first and second computational environment and provide experimental results; Section 4 highlights some links between the two approaches. Section 5 focuses on discussion and open problems.

2 Inexact angle "computations"

In [2] Kainen and Kůrková, introduce the notion of dimension when an angle is subject to a measurement tolerance. We use the idea of inexact angle measurement[1] in Euclidean space and its generalization for the inexact measurement of a segment to classify a random triangle on a plane. To explain how the concept of "inexact angle measurement" is used here, consider for example a right angle; in geometry and trigonometry it is an angle of exactly 90°: now, suppose we are not able to measure this angle exactly.

We can set, for example, a tolerance of 1° in right angle measurement: angles whose measure is included between 89° and 90° (symmetrically 90° and 91°) are right (Figure 1). From this assumption we obtain a new way of classifying angles:

[1] Here "measure" and "computation" can be considered synonyms.

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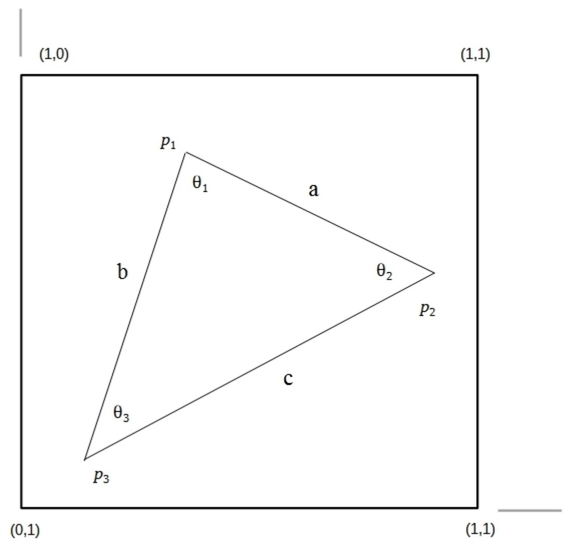

Figure 2: A random triangle in a unit square with its relative notation.

angles with degree between [89° 90°] and [90° 91°] are right angles; with degree between [1° 89°] and [91° 179°] are acute; if they have a value $> 91°$ ($< 89°$) they are obtuse. We observe that angle tolerance also introduces a tolerance in the length of the related segments[2]. Since triangles can be classified according to their angles (with the Dot product or Cosine Law) or to their sides (with the Pythagorean Theorem), we have a new schema to classify them: for example all triangles shown in Figure 1 are rectangle. We can generalize this idea. For any $\varepsilon, l \geq 0$ and for any three random points $(p_1, p_2, p_3)_n$, n=1,2, .., N, in a unit square, corresponding to the endpoints of triangles $\triangle_n$, we classify the resulting triangles (Figure 2) in three ways:

Method 1 (*Pythagorean Theorem*). Given a triangle in which c denotes the length of the hypotenuse and a and b denote the length of the other two sides:

$$if\left(a^2+b^2-l\leq c^2\right)\wedge\left(c^2\leq a^2+b^2+l\right)\implies\triangle \text{ is a Right Triangle},$$

$$if\left(c^2<a+b^2-l\right)\implies\triangle \text{ is an Acute Triangle},$$

$$if\left(c^2>a+b^2+l\right)\implies\triangle \text{ is an Obtuse Triangle}.$$

Method 2 (*Dot product-scalar product*). We compute:

$$\theta_1=\arccos\frac{\mathrm{a}\cdot\mathrm{b}}{\|\mathrm{a}\|\,\|\mathrm{b}\|},\theta_2=\arccos\frac{\mathrm{a}\cdot\mathrm{c}}{\|\mathrm{a}\|\,\|\mathrm{c}\|},\theta_3=\arccos\frac{\mathrm{b}\cdot\mathrm{c}}{\|\mathrm{b}\|\,\|\mathrm{c}\|}$$

[2]We use ε to indicate a tolerance in angle measurement and l to indicate the segment length.

and use the following classification:

$$\begin{aligned} if\ &\left[\left(\frac{\pi}{2}-\varepsilon\leq\theta_1\right)\wedge\left(\theta_1\leq\frac{\pi}{2}+\varepsilon\right)\right]\vee\\ &\left[\left(\frac{\pi}{2}-\varepsilon\leq\theta_2\right)\wedge\left(\theta_2\leq\frac{\pi}{2}+\varepsilon\right)\right]\vee\\ &\left[\left(\frac{\pi}{2}-\varepsilon\leq\theta_3\right)\wedge\left(\theta_3\leq\frac{\pi}{2}+\varepsilon\right)\right]\\ &\Longrightarrow\triangle\text{ is a Right Triangle,}\end{aligned}\tag{1}$$

$$if\ \left(\theta_1<\frac{\pi}{2}-\varepsilon\right)\vee\left(\theta_2<\frac{\pi}{2}-\varepsilon\right)\vee\left(\theta_3<\frac{\pi}{2}-\varepsilon\right)\Longrightarrow\triangle\text{ is an Acute Triangle,}\tag{2}$$

$$if\ \left(\theta_1>\frac{\pi}{2}+\varepsilon\right)\vee\left(\theta_2>\frac{\pi}{2}+\varepsilon\right)\vee\left(\theta_3>\frac{\pi}{2}+\varepsilon\right)\Longrightarrow\triangle\text{ is an Obtuse Triangle.}\tag{3}$$

Method 3 (*Cosine Law*). We compute:

$$\theta_1=\arccos\left(\frac{\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2}{2\cdot\mathrm{a}\cdot\mathrm{b}}\right),$$

$$\theta_2=\arccos\left(\frac{\mathrm{a}^2+\mathrm{c}^2-\mathrm{b}^2}{2\cdot\mathrm{a}\cdot\mathrm{c}}\right),$$

$$\theta_3=\arccos\left(\frac{\mathrm{b}^2+\mathrm{c}^2-\mathrm{a}^2}{2\cdot\mathrm{b}\cdot\mathrm{c}}\right)$$

and then apply the same classification rules (Equation 1,2 and 3) used in Method 2.

The right triangles in $\mathbb{R}^2$ are "impossible"; however, the concept of "*Quasiorthogonal Triangle*" introduced above allows inaccuracy in angle and segment measurement, making the right triangle in $\mathbb{R}^2$ "possible"!

Method 1 classifies triangles using segment measures; it uses l to introduce a tolerance in segment measurement. On the other side, **Method 2** and **3** classify triangles using angle measures and ε to allow inexact measure for angles.

In this scenario we generate 1,2 ... N triplets of random points $(p_1, p_2, p_3)_n$; then we compute the average area and classify these triangles using the three above formulas as functions of ε and l. The main objectives are:

- replicate the theoretical results known until now;
- verify the numerical convergence of the three methods;
- empirically find a relation between ε, l and the probability to have an obtuse, acute or right triangle.

In more detail, we compute the probability of having an obtuse, acute or right triangle in this way: given a fixed ε (or l) and N, we generate 1,2 ... N triplets of random points $(p_1, p_2, p_3)_n$: for each triplet we use one of the above methods

Probability	**Method 1**	**Method 2**	**Method 3**
Right Triangle	0	0	0
Acute Triangle	0.27471	0.27471	0.27471
Obtuse Triangle	0.72529	0.72529	0.72529
Sum	1	1	1

Table 1: Experimental results for $\varepsilon = l = 0$ and N=1.000.000

Probability	**Method 2**	**Method 3**
Right Triangle	0.000427	0.000427
Acute Triangle	0.27496	0.27496
Obtuse Triangle	0.72461	0.72461
Sum	1	1

Table 2: Experimental results for Method 2 with $\varepsilon =$ 0.0175 and N=1.000.000

to classify the related triangle on the plane. At the end of the classification procedure we will have n_r right triangles, n_o obtuse triangles and n_a acute triangles, with $n_r + n_o + n_a = N$. Then we can compute

$$P(\triangle = right) = \frac{n_r}{N},$$
$$P(\triangle = obtuse) = \frac{n_o}{N},$$
$$P(\triangle = acute = \frac{n_a}{N} \quad (4)$$

Implementing our code with MATLAB (or Octave), we obtain the following results for $\varepsilon = l = 0$ (no tolerance in measure is allowed) and N=1.000.000:

Average Triangle Area= 0.076405 ~ $\frac{11}{144}$,

Probability: (Table 1).

For the Average Triangle Area and for Probability, the three methods provide the same results, that are in accord with the previous theoretical analysis (mean area = 0.07638 and probability that three points form an obtuse triangle = 0.72520 [3, 4, 5]). Furthermore, **Method 2** and **Method 3** use ε to introduce a tolerance in angle measurement: for example, we set ε=0.0175 (assuming a tolerance of 1 degree in angle measurement); the results can be seen in Table 2. For ε=0.025 the results are in Table 3 , and so on. As expected, **Method 2** and **Method 3** behave in the same way: for the sake of simplicity, in the next we consider only **Method 1** and **Method 2**.

Finally, to empirically find a relation between l and the probability of having an obtuse, acute or right triangle, we consider l=0, l=0.01, l=0.02, ... l=1; then we generate N=10.000 random triangles (random points $(p_1, p_2, p_3)_n$) for each value of l and classify them using **Method 1**. The results are shown in Figure 3. In the same way, using **Method 2** we consider ε=0, ε=0.0175, ..., $\varepsilon = \frac{\pi}{2}$, and N=10.000 (Figure 4). With the results in Figures 3 and 4 it is possible to find an approximation for the probability of obtaining an obtuse triangle. There

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Probability	**Method 2**	**Method 3**
Right Triangle	0.000646	0.000646
Acute Triangle	0.27427	0.27427
Obtuse Triangle	0.72508	0.72508
Sum	1	1

Table 3: Experimental results for Method 2 with $\varepsilon =$ 0.025 and N=1.000.000

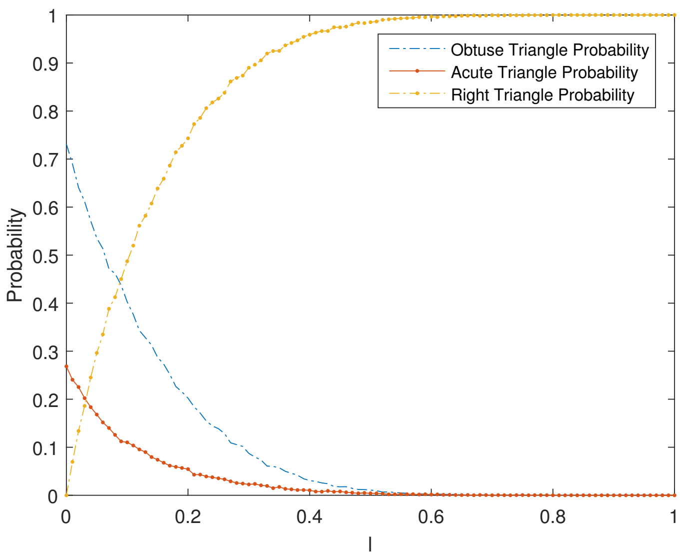

Figure 3: Probability of having an obtuse, acute or right triangle, as a function of l, using **Method 1**

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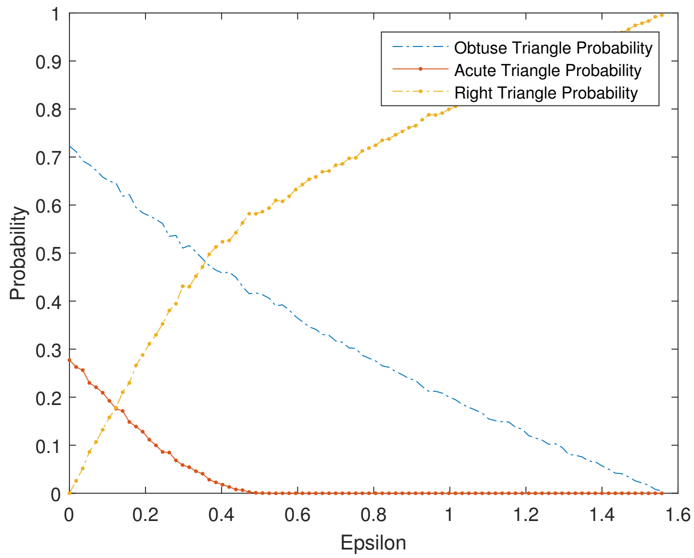

Figure 4: Probability of having an obtuse, acute or right triangle, as a function of ε, using **Method 2**

are different approximation methods: if we use a 4^{th} polynomial (in MATLAB tools, the Norm of residuals is 0.031396) for the curve in Figure 3, we have:

$$f(l) = 2.523 \cdot l^4 - 7.451 \cdot l^3 + 8.176 \cdot l^2 - 3.968 \cdot l + 0.7229 \tag{5}$$

If we use a cubic polynomial (in MATLAB tools, the Norm of residuals is 0.041426), for obtuse triangle probability in Figure 3, we obtain:

$$f(l) = -0.077 \cdot l^3 + 0.308 \cdot l^2 - 0.756 \cdot l + 0.7228 \tag{6}$$

Equations 4 and 5 can be seen as reformulations and generalizations of the results in [3, 4]: as a matter of fact, for l=0:

$$f(0) = 0.7228(9) \sim \frac{97}{150} + \frac{1}{40} \cdot \pi$$

Better approximations may be found and used instead of equations 5 and 6, either with a more accurate step for l and ε or with a increase in the value of N. In the same way it is very easy to derive an approximation function for the probability of having acute and right triangles.

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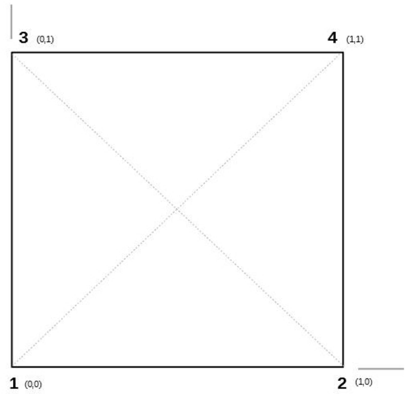

Figure 5: A grid with step g=1 on a unit square

3 Triangle Counting

In this section we try to count triangles starting from basic combinatorial concepts. The strategy is the following: we consider a unit square[3] and set a grid with a different step length (the distance between adjacent points in the grid): g=1, $g = \frac{1}{2}$, ... etc. When g=1, we have the square vertex; when $g \to 0$ we have the continuum. Given a g on a square, we select three random points, trying to answer our initial question: *What is the probability that the triangle obtained is obtuse, acute or right?* For g=1 (Figure 4) we have a grid with only four points (the square vertices):

$\mathbf{3}_{coord(0,1)}$ $\mathbf{4}_{coord(1,1)}$

$\mathbf{1}_{coord(0,0)}$ $\mathbf{2}_{coord(1,0)}$

If *three of these points are selected at random, we can generate the following triangles*[4]:

"△124" "△142" "△123" "△132" "△134" "△143"

The same happens when starting from vertices **2**, **3** and **4**: we have 6*4=24 triangles, many of which are the same (for example "△124" and "△421", "△134" and "△143" ... or "△321" and "△123"). To compute the number of distinct triangles, we need to compute the combination without an repetitions of k (=3) objects from n (=4):

$$C_{n,k} = \frac{n!}{k!(n-k)!} = \frac{4!}{3!1!} = 4$$

[3] Simply square in the next.

[4] With the string "△124" we indicate the triangle that has point vertices in 1 2 4 nodes in the grid.

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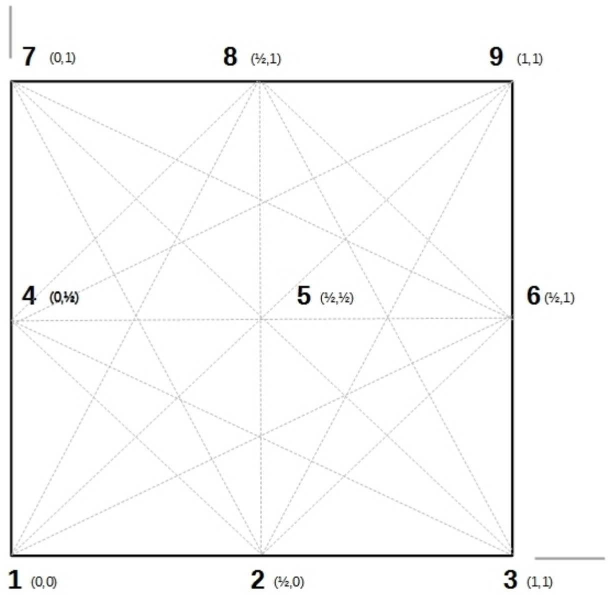

Figure 6: A grid with step $g = \frac{1}{2}$ in a unit square

For example, if we consider node 1 as the starting vertex, the triangles are: "△123" "△124" "△134" and "△234".

In this case (g=1), all possible triangles are right! In the opposite situation, when $g \to 0$, no right triangle can exist: it is the same situation as the one described in Section 2, with ε (or l)=0. Finally, we observe that for g=1 all triangles have the same area.

For $g = \frac{1}{2}$ (Figure 6), we have a grid with nine points:

$\mathbf{7}_{coord(0,1)}$ $\mathbf{8}_{coord(\frac{1}{2},1)}\mathbf{9}_{coord(1,1)}$

$\mathbf{4}_{coord(0,\frac{1}{2})}$ $\mathbf{5}_{coord(\frac{1}{2},\frac{1}{2})}\mathbf{6}_{coord(1,\frac{1}{2})}$

$\mathbf{1}_{coord(0,0)}$ $\mathbf{2}_{coord(\frac{1}{2},0)}\mathbf{3}_{coord(1,0)}$

In this case, to count the number of different triangles we can use a "recursive" approach counting the distinct rectangles (or squares) that we can have on this grid (for example the rectangle with vertices "1793", "2893", "5698") and then use the method described above for g=1. Alternatively we can compute the combination without repetitions of k (=3) objects from n (=9) and remove from this value the number of possible aligned random numbers on the grid (given three random points, from the above grid, these can be aligned, for example "159" or "654", so we have no triangles).

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Algorithm 1 Pseudo-code used for counting triangles.

Input: a grid with step g in unit square, a number N;

1. extract three random points from the grid for N times, obtaining N pseudo-random triangles;
2. from these pseudo-random N triangles remove the identical ones (if exist), obtaining the setN_1 (*detail: from n we remove: - 1 the identical extraction; for example if for g=1 the string "△412" is repeated several times, we consider it only once. - 2 identical triangles; for example. if g=1, and triangle "△421" has already been generated, we remove the other five identical triangles: "△124", "△142" etc.*);
3. remove the aligned points (if exist), from set N_1, obtaining the set of random triangles N_2;
4. compute the mean area for the triangles in set N_2, and classify them as right, obtuse or acute with the above Method 1(2), with l=0 ($\varepsilon = 0$);

Output: number of distinct triangles considered (the cardinality of N_2), mean triangle area, and probability of having obtuse, acute or right triangle (frequency of triangles in N_2).

Only 76 distinct triangles remain:

$$C_{n,k} = \frac{n!}{k!(n-k)!} - \#AlignedPointCombinations = \frac{9!}{3!6!} - 8 = 76 \qquad (7)$$

Some are not right, like triangle "△126". Using this schema, we can easily count triangles for $g = \frac{1}{4}, g = \frac{1}{5}, \ldots$ etc; an algorithm for this is the **Algorithm 1**.

We implement the pseudo-code in Alghoritm1 using MATLAB (Octave), and perform some preliminary experiments to verify the convergence of our method on the known theoretical values (Table 4).

In Table 4 , both g and N vary, so the mean area rapidly converges on its theoretical value: $\frac{11}{144}$=0.07638; the same for the probabilities (relative frequency) of having obtuse, acute or right triangles[6].

In this scenario many other experiments can be performed. We may fix g and vary N or vice versa. Given a g, one can study whether the related value of N allows generating any distinct triangles or not. As an example of the experiments that can be performed within this setting, we may fix N=10000 and vary g, starting from 1/10. From Table 4 and Equations 6, for g=1/100, corresponding to 100 points in the grid, we can have more than 10000 different triangles. This situation is similar to what happens in $\mathbb{R}^2$, but it allows us to estimate of the probability of getting right triangles. The results are shown in Table 5.

[5] Relative frequency/Probability. The same for Acute and Right triangles.
[6] To compute the probability we use the equations (4)
[7] Relative frequency/probability. The same for acute and right triangles.

g	N	N_2	Area (*mean*)	Obtuse[5]	Acute	Right
1	50000	4	0,5	0	0	1
$\frac{1}{2}$	50000	76	0,23026	0,31579	0,10526	0,57895
$\frac{1}{3}$	50000	516	0,16451	0,45736	0,15504	0,3876
$\frac{1}{4}$	50000	2156	0,13677	0,53618	0,18738	0,27644
$\frac{1}{5}$	50000	6769	0,12195	0,58133	0,20535	0,21333
$\frac{1}{10}$	100000	82598	0,09552	0,67205	0,24422	0,08373
$\frac{1}{15}$	100000	96854	0,0892	0,69298	0,2601	0,04691
$\frac{1}{20}$	150000	148004	0,08507	0,70751	0,26303	0,02945
$\frac{1}{40}$	150000	149514	0,0806	0,71969	0,27052	0,00979
$\frac{1}{50}$	200000	199606	0,07964	0,72132	0,27192	0,00676
$\frac{1}{75}$	200000	199812	0,07858	0,72489	0,27163	0,00348
$\frac{1}{100}$	250000	249871	0,07826	0,72306	0,27494	0,002
$\frac{1}{100}$	350000	349791	0,07803	0,72333	0,2746	0,00206

Table 4: Algorithm 1, preliminary experimental results.

g	N	N_2	Area (*mean*)	Obtuse[7]	Acute	Right
$\frac{1}{10}$	10000	9591	0.094381	0.67668	0.24179	0.24179
$\frac{1}{50}$	10000	9981	0.078779	0.72848	0.2644	0.007113
$\frac{1}{100}$	10000	9995	0.077257	0.72146	0.27664	0.001901
$\frac{1}{150}$	10000	9998	0.078521	0.72414	0.27495	0.0009
$\frac{1}{200}$	10000	10000	0.075715	0.7371	0.2625	0.0004
$\frac{1}{250}$	10000	9998	0.077453	0.71964	0.28016	0.0002
$\frac{1}{300}$	10000	9997	0.076193	0.73202	0.26748	0.0005

Table 5: Triangle counting with N fixed to 10000 and g variable.

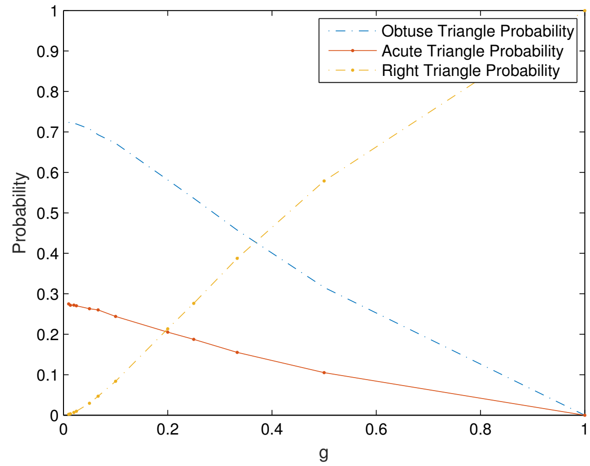

Figure 7: Probability of having an obtuse, acute or right triangle, as a function of g.

If $g = \frac{1}{50}$ there is a good approximation for the mean Area and a probability (frequency) of obtaining an obtuse triangle. If we assume this as a plausible scenario (a good approximation for $\mathbb{R}^2$) we can say that there is a probability, though little (0.007), of having right triangles!

4 Inexact Triangle "Computations" vs Triangle "Counting"

This section sketches some links between the methods[8] described in Sections 2 and 3. Both methods converge on the theoretical values known for the mean triangle area and on the probability of having an obtuse triangle (0.07638 and 0.72520 [5]). In other words, $\varepsilon = l = 0$ corresponds to $(g, \mathrm{N}) \rightarrow \infty$ (so we are in $\mathbb{R}^2$) and vice versa; l =1 or $\varepsilon = \frac{\pi}{2}$ corresponds to $g = 1$ for N sufficiently large. In the first case no right triangle can exist; in the second case there are only right triangles. Furthermore, if the probabilities in Table 4 and g are plotted in reverse order (Figure 7), the behavior is the same as in Figures 3 and 4. The differences between the graphs are not conceptual: they are due to the different ranges of ε, l and g and to different experiment settings. In this case too we can easily derive three approximant functions for the probability (relative frequency) of getting acute, obtuse or right triangles.

[8] For the sake of simplicity we call them "Quasiorthogonal Method" and "Counting Method".

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5 Conclusion and Open Problems

In this work we have introduced a general framework, made up of two convergent methods, to study the probability that a triangle is obtuse, acute or right. This is preliminary research, so a lot of questions remain open. It may be very interesting to study the behavior of the mean triangle area within the Quasiorthogonal Method. Once obtained a good approximation for the mean area of triangles, the related probabilities may reach their correct value (see end of Section 3). The concept of "*Quasiorthogonal figure*" can be extended in many useful ways. It would be interesting to give a formal demonstration of Equations 4 and 5 and of the other four related equations, or to find a more elegant alternative. The concept could also be useful for real applications for example in artificial vision.

The curves in Figures 3 and 4 must have punctual sum 1. This constraint may be used to derive a single comprehensive approximant function. With the Counting method one may estimate the optimal relation for g and $N(N_2)$, to see what happens when, given a g, we consider fewer or more triangles for that g than the possible ones. The Counting method may also be used to exactly count the number of right, obtuse or acute triangles generated according to g.

The connections between the two methods sketched in Section 4 could be studied in a more detailed and formalized way. In principle the two methods may be collapsed into one. Furthermore, given a little or a great value for g, we may use Method 1, 2 or 3 with ε, l as variables (Algorithm 1). Our framework could be easily generalized to manage different geometrical figures (rectangles, pentagons, etc) used beyond random triangle theory, to study a generic problem from a computational perspective.

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References

[1] A. Edelman and G. Strang, "Random triangle theory with geometry and applications", Foundations of Computational Mathematics, vol. 15, no. 3, pp. 681–713, 2015.

[2] P. C. Kainen and V. Kŭrková, "Quasiorthogonal dimension of Euclidean spaces," Applied mathematics letters, vol. 6, no. 3, pp. 7–10, 1993.

[3] E. Langford, "The probability that a random triangle is obtuse," Biometrika, vol. 56, no. 3, pp. 689–690, 1969.

 doi:10.20944/preprints202205.0023.v1

[4] E. Langford, "A problem in geometrical probability," Mathematics Magazine, vol. 43, no. 5, pp. 237–244, 1970.

[5] E. W. Weisstein, "Square Triangle Picking," Delta, vol. 3, p. 4, 2000. (Weisstein, Eric W. "Square Triangle Picking." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/SquareTrianglePicking.html)