Review

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Article

Combining Fractional Derivatives and Machine Learning: A Review

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Abstract: Fractional calculus gained a lot of attention in the last couple of years. Researchers discovered that processes in various fields follow rather fractional dynamics than ordinary integer-ordered dynamics, meaning the corresponding differential equations feature non-integer valued derivatives. There are several arguments for why this is the case, one of them being that fractional derivatives' inherit spatiotemporal memory and/or the ability to express complex naturally occurring phenomena. Another popular topic nowadays is machine learning, i.e., learning behavior and patterns from historical data. In our ever-changing world with ever-increasing amounts of data, machine learning is a powerful tool for data analysis, problem-solving, modeling, and prediction. It further provides many insights and discoveries in various scientific d isciplines. As these two modern-day topics provide a lot of potential for combined approaches to describe complex dynamics, this article reviews combined approaches of fractional derivatives and machine learning from the past, puts them into context, and thus provides a list of possible combined approaches and the corresponding techniques. Note, however, that this article does not deal with neural networks, as there already is profound literature on neural networks and fractional calculus. We sorted past combined approaches from the literature into three categories, i.e., preprocessing, machine learning & fractional dynamics, and optimization. The contributions of fractional derivatives to machine learning are manifold as they provide powerful preprocessing and feature augmentation techniques, can improve physically informed machine learning, and are capable of improving hyperparameter optimization. Thus, this article serves to motivate researchers dealing with data-based problems, to be specific machine learning practitioners, to adopt new tools and enhance their existing approaches.

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1. Introduction

Machine learning and fractional calculus are tools capable of dealing with and describing complex real-life phenomena and its relation to its inherent non-linear properties.

Whereas machine learning dynamically learns complex behavior from data in most cases, the framework of fractional calculus was in the past used to describe complex phenomena by statically modeling them. The issue of a fractional derivative came up first in a letter from de l'Hopital to Leibniz in 1695, i.e., what one would obtain from a derivative of non-integer order n, e.g., $n = \frac{1}{2}$. Leibniz's famous response to this day was: "It will lead to a paradox from which one day useful consequences will be drawn," [1].

Nowadays, both frameworks feature a multitude of applications in various disciplines.

The applications of fractional calculus range from physics, where one can describe anomalous diffusion in complex liquids or frequency-dependent acoustic wave propagation, to, e.g., image processing, where one can use fractional derivatives as filters or enhance images, [2]. Another area of research in which fractional calculus is used to describe various phenomena are the environmental sciences. Here fractional calculus is applied to interpret

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hydrological cycles or analyze underground water chemical reactions, [2,3]. As discussed by Du et al. [4], fractional calculus introduces memory into modeling processes, which in turn is required for specific approaches in, e.g., biology or psychology. Thus the researchers suggest interpreting the fractional order of a process as a degree of memory of this process, [4].

This article aims to find overlaps between machine learning and fractional calculus, specifically fractional derivatives. Further, to discuss these overlaps or combined applications and put them into a bigger context and finally, give some recommendations on using fractional derivatives to enhance machine learning approaches. At this point, we need to clarify that, when employing the popular term machine learning and narrowing down the scope of this article, we only talk about supervised non-neural-network machine learning applications. The reason for not including neural networks is that there already is an excellent article on the combined approaches of neural networks and fractional calculus. Thus, we're not going to discuss this topic again but instead, refer to the article by Viera et al. [5]. We also want to clarify that the scope of this article is to provide additional tools from fractional calculus to the experienced machine learning practitioner rather than introducing people familiar with fractional calculus to the realm of machine learning. Thus the outcome of this article is a focused list of possible combined applications of machine learning and fractional derivatives and, based on the discussed literature, a discussion on the context of the combination of these two tools as well as recommendations for future research.

This article is structured as follows:

Section 2 describes how this review was performed, the employed search, and the exclusion criteria. Section 3 briefly introduces fractional calculus and describes the different types of fractional derivatives. Section 4 introduces supervised machine learning. Next, we present the results of our literature review in Section 5. We discuss our findings and put them into context in Section 6 and conclude this article in Section 7. We further added Appendix A, which provides a table containing all reviewed articles.

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2. Methodology

For this review, we performed an online search on google scholar. We searched for the keywords listed in the following list:

- machine learning fractional calculus
- machine learning fractional derivatives
- support vector fractional calculus
- support vector fractional derivatives
- decision tree fractional calculus
- decision tree fractional derivative
- random forest fractional calculus
- random forest fractional derivative
- XGBoost fractional calculus
- XGBoost fractional derivative
- regression fractional calculus
- regression fractional derivative
- ridge regression fractional calculus
- ridge regression fractional derivative
- lasso regression fractional calculus
- lasso regression fractional derivative
- logistic regression fractional calculus
- logistic regression fractional derivative
- naive bayes fractional calculus
- naive bayes fractional derivative
- KNN fractional calculus
- KNN fractional derivative
- nearest neighbor fractional calculus
- nearest neighbor fractional derivative
- LightGBM fractional calculus
- LightGBM fractional derivative
- extreme learning fractional calculus
- extreme learning fractional derivative
- extreme boosting fractional calculus
- extreme boosting fractional derivative

We acknowledge that this list is incomplete; further, it can never be a complete list. This is because of the nowadays somewhat blurry usage of the machine learning terminology and lingo and the ever-changing landscape of new algorithms and ideas in the field. Here we want to refer to two homepages [6,7], which are both glossaries that keep up with the fast-changing machine learning terminology. Thus the previous list contains popular machine learning algorithms, prevalent machine learning, and fractional calculus keywords, and occasionally some more exotic keywords, which are included because of the authors' domain knowledge.

Further, the authors added some articles that did not appear in the mentioned search queries but are deemed to be a relevant addition to this article based on the authors domain knowledge.

As we usually found articles fitting our exclusion criteria within the first ten results, we prioritized our search and analysis to the first 20 results for each search query.

As already mentioned, we employed a range of exclusion criteria to keep this review focused and to reduce the number of featured articles. Thus, the following exclusion criteria were used:

- We excluded articles primarily targeting and/or only targeting neural networks.
  This was done because there already exists an excellent review on neural networks and fractional calculus, [5]. Still, we will peripherally mention neural network applications or feature articles that include neural networks along with other machine learning algorithms.
- We excluded articles dealing with control problems.
  This criterion was employed to keep this review focused, as including control problems would have unnecessarily blown up the discussed methodology and made it harder to identify key takeaways for hybrid applications of fractional calculus and machine learning.
- We discarded everything related to grey models.
  Grey models do not coincide with the classic supervised learning approaches we're looking for in this review.

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- We excluded publications that dealt with fractional integrals rather than with fractional derivatives.
- We excluded publications that featured fractional complexity measures, e.g. fractional entropy measures.
  Still we're briefly mentioning them in our discussion if they provide valuable insights into hybrid applications.
- We excluded unsupervised, and reinforcement learning approaches.
  Thus, we're only dealing with supervised learning.

3. Fractional Derivatives

Fractional derivatives are a generalization of integer-ordered derivatives such that they allow for derivatives of order $n = \frac{1}{2}$ instead of only integer-valued derivatives, i.e., $n = 1, 2, 3, \ldots$.

Contrary to integer order derivatives, they do not behave locally, i.e., the fractional derivative of a function $f(t)$ depends on all values of the same function, thus inducing memory about every point of a signal into the derivative.

Also, there are many ways to define a fractional derivative; thus, many researchers proposed their version of the fractional derivative, sometimes explicitly tailored to match a particular problem/application. To further illustrate this idea and to give the reader a grasp on the multitude of possible fractional derivatives, we give the list from Wikipedia [8]: The listed fractional derivatives are the Grünwald–Letnikov derivative, the Sonin–Letnikov derivative, the Liouville derivative, the Caputo derivative, the Hadamard derivative, the Marchaud derivative, the Riesz derivative, the Miller–Ross derivative, the Weyl derivative, the Erdélyi–Kober derivative, the Coimbra derivative, the Katugampola derivative, the Hilfer derivative, the Davidson derivative, the Chen derivative, the Caputo-Fabrizio derivative, and the Atangana–Baleanu derivative.

Because of this arbitrariness of choosing a fractional derivative, we stick to those fractional derivatives relevant in our literature review, briefly explain them, and reference sources for a discussion on them. Thus the discussed fractional derivatives are the Grünwald-Letnikov, the Reisz, the Caputo, and the Riemnann-Liouville fractional derivative. Further, we need to differentiate between left and right-sided derivatives for many of these fractional derivatives. The following list is based on the reviews by de Oliveira et al. and Aslan [9,10], which we refer the reader to for an in-depth discussion of the topic. Further, for a discussion on the discrete version of the fractional derivatives, we refer to [11,12]. The featured applications combining fractional derivatives and machine learning use the following list of fractional derivatives:

- **The Grünwald-Letnikov fractional derivative**

$$\begin{aligned}
\text{left-sided: } \mathrm{D}[f(x)]^{\alpha}_{a^+} &= \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^k\frac{\Gamma(\alpha+1)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \quad nh = x-a \\
\text{right-sided: } \mathrm{D}[f(x)]^{\alpha}_{b^-} &= \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^k\frac{\Gamma(\alpha+1)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \quad nh = b-x
\end{aligned} \tag{1}$$

- **The Caputo Fractional Derivative**

$$\begin{aligned}
\text{left-sided: } \mathrm{D}^{\alpha}_{a^+}[f(x)] &= \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-\zeta)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}[f(\zeta)]\mathrm{d}\zeta, \quad x \geq a \\
\text{right-sided: } \mathrm{D}^{\alpha}_{b^-}[f(x)] &= \frac{1}{\Gamma(n-\alpha)}\int_x^b (x-\zeta)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}[f(\zeta)]\mathrm{d}\zeta, \quad x \leq b
\end{aligned} \tag{2}$$

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- **The Riemann-Liouville fractional derivative**

$$\begin{aligned}\text{left-sided: } \mathrm{D}^{\alpha}_{a+}[f(x)] &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x (x-\zeta)^{n-\alpha-1}f(\zeta)\mathrm{d}\zeta, \quad x \geq a \\ \text{right-sided: } \mathrm{D}^{\alpha}_{b-}[f(x)] &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^b (x-\zeta)^{n-\alpha-1}f(\zeta)\mathrm{d}\zeta, \quad x \leq b\end{aligned} \tag{3}$$

- **The Riesz Fractional Derivative**

$$\mathrm{D}^{\alpha}_{x}[f(x)] = -\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)}\frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\int_{-\infty}^x (x-\zeta)^{n-\alpha-1}f(\zeta)\mathrm{d}\zeta + \int_x^{\infty} (\zeta-x)^{n-\alpha-1}f(\zeta)\mathrm{d}\zeta\right) \tag{4}$$

Remarks:

- α denotes the order of the fractional derivative. Whereas $\alpha \in \mathbb{C} : \mathbb{R} \in (n-1, n]$, $n \in \mathbb{N}$. Thus n defines the integer proximity of α. $\mathbb{R}(\cdot)$ denotes the real part of a complex number. Further, as defined here, alpha can be complex. However, every single reviewed article (see Appendix A for the summary table) featuring a combined application of fractional derivatives and machine learning uses a real or at least rational value for α.
- $n-1$ and n as $n-1 \leq \mathbb{R}(\alpha) < n$.
- $[a,b]$ is a finite interval in $\mathbb{R}$ and $k \in \mathbb{N}$. Further, $f(0) \equiv f(0^+) - f(0^-)$.
- ζ is an auxiliary variable used for integration.
- $\Gamma(\cdot)$ is the gamma function, [13].
- $\lfloor \cdot \rfloor$ denotes the floor function, i.e. $\lfloor x \rfloor \max\{z \in \mathbb{Z} : z \leq x\}$
- The previously mentioned "memory" is induced by the summation over k in the case of the Grünwald-Letnikov derivative and by the integration over ζ for the other mentioned derivatives.

4. Supervised Machine Learning

[14] Supervised machine learning is the practice of using a set of input variables, such as blood metabolite, gene expression, or environmental conditions, to predict quantitative output, such as diseases in humans or crop yields. Thus, aiming to categorize data from prior information via an algorithm trained from labeled data [15,16].

There are numerous supervised learning algorithms, such as linear regression, support vector machines, decision trees, and random forests, to name a few.

One can further partition supervised machine learning into classification and regression tasks. In classification tasks, the algorithm predicts categorical variables such as types of diseases. Contrary to that, in regression, the algorithm predicts continuous numerical outputs such as wheat yields in, e.g., Hg/Ha.

We will no further discuss supervised learning for several reasons. First, this review aims to educate machine learning practitioners about possible applications of fractional derivatives, thus assuming that the reader is somehow familiar with supervised learning. Second, supervised learning and machine learning, in general, is a popular multidisciplinary area of research nowadays. Therefore, it suffers from a fast-changing and ever-increasing landscape of algorithms, terminology, and approaches. Third, there is a vast amount of literature and tutorials available that we cannot cover here. Still, we want to recommend the books by Jason Brownlee for a tutorial-heavy introduction to the topic, [17–19].

5. Results: Combined Approaches of Fractional Derivatives and Machine Learning

Given the previously defined methodology (Section 2), we found 60 relevant publications. We then separated this list of publications in this section into three main categories on how to combine machine learning and fractional derivatives, i.e., preprocessing, machine learning & fractional modeling, and optimization. In the following, we will elaborate on these categories separately and further divide them into subcategories wherever necessary.

5.1. *Preprocessing*

In this section, we discuss the applications of fractional derivatives for data preprocessing in machine learning applications. Part of this discussion are data pre-treatment in general, additional features based on fractional derivatives, and feature extraction based on fractional derivatives. These ideas, though distinct, often overlap, especially when discussing additional features and feature extraction. All these techniques have in common that they are applied before the actual machine learning, e.g., the training process. Thus we named this section *preprocessing*.

We sort relevant publications by their type of application, e.g., fractional derivatives used for preprocessing in spectroscopy. In this process we focus on three categories of applications, i.e., spectroscopy, biomedical and engineering applications.

5.1.1. Spectroscopy

An important area of research for combined approaches of fractional derivatives and machine learning is spectroscopy. Here fractional derivatives are used as a preprocessing step to enhance the spectral data and, thus, improving the accuracy of the machine learning algorithm. We can again differ between two main categories, i.e., visual and near-infrared (Vis-NIR) spectroscopy and hyperspectral analysis.

Regarding visual and near-infrared spectroscopy, we find a range of agricultural applications. In [20], the researchers estimate organic matter content in arid soil from Vis-NIR spectral data. Here, the Grünwlad Letnikov fractional derivative is used as a pretreatment for the spectral data, i.e., the fractional derivative is applied to the obtained spectral data to improve the accuracy of the employed machine learning algorithms, random forest, and partial least squares regression. Similar to this approach is the approach in [21], where again fractional derivatives are used as a pretreatment to augment the Vis-NIR data to estimate soil organic content. Here additionally to the previously used algorithms, the researchers also used memory-based learning.

The same idea, i.e., to augment Vis-NIR data using the G-L derivative to improve machine learning estimates, is used in both [22] and [23] to estimate the nitrogen content of rubber tree cultivations and in [24] for cotton farming. The employed machine learning algorithms range from partial least squares regression, extreme learning machines, convolutional neural networks, and support vector machines.

For another application in agriculture, Bhadra et al. employed the Grünwlad Letnikov fractional derivative to augment Vis-NIR data to quantify the leaf chlorophyll in sorghum using various machine learning algorithms such as again, partial least squares regression, random forest, support vector machines, and extreme learning machines, [25].

Urbanization and industrialization are putting a lot of stress on the environment and further pollute agricultural-used land with, e.g., various metals. Here [26,27], and [28] provide ideas to identify Zinc Lead and other heavy metals from Vis-NIR spectra by, again, employing data augmentation based on the Grünwlad Letnikov fractional derivative to improve the accuracy of machine learning algorithms such as random forest, XGBoost, extreme learning machines, support vector machines, and partial least squares regression.

Finally, the Grünwlad Letnikov fractional derivative is also be used to improve the discussed Vis-NIR data to improve the estimation of soil salinity, soil salt, and other water-soluble ions in [29] and [30]. Here, again, the researchers use the Grünwlad Letnikov fractional derivative to augment the spectral data and/or obtain spectral indices. Again, various ML algorithms are employed, such as partial least squares regression, random forests, and extreme learning machines.

When it comes to Hyperspectral data, we find similar approaches to the discussed Vis-NIR spectral applications, i.e., the spectral data is preprocessed using the Grünwlad Letnikov fractional derivative and afterward a measured observable such as, e.g., soil organic matter is predicted using a machine learning algorithm such as, e.g., partial least squares regression, [31]. In [32], estimates on top soil organic carbon are given using a Grünwlad Letnikov fractional derivative and Random random forest.

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Also, (soil) salt content can be estimated using hyperspectral data and a range of ML algorithms, as done in [33–35]. Another agricultural application is to assess the nitrogen content in apple tree canopies via support vector machines, random forest algorithms, and fractional-derivative-augmented hyperspectral data, [36].

Cheng et al. predict the photosynthetic pigments in apple leaves using fractional derivatives to augment the data and range of machine learning algorithms, i.e., support vector machines, k nearest neighbor, random forest, and neural networks, [37]. In [38], it is shown that a similar approach though using a different algorithm, XGBoost, can be used to estimate agricultural soil moisture content. And in [39], XGBoost, LightGBM, Random Forests, and other algorithms are used to predict the soil electrical conductivity from hyperspectral data using fractional-derivative-augmented data. Though related, but not necessarily an agricultural application, is discussed in [40], where augmented hyperspectral data is used to estimate the clay content in desert soils using partial least squares regression.

Further, these ideas of using fractional derivatives to preprocess hyperspectral data to improve ML approaches are not restricted to being used for remotely sensed data of soils. These ideas are applied to water in both [41] and [42]. In [41], fractional derivatives are used to improve the prediction of total nitrogen in water using XGBoost and Random Forest, whereas in [42], a similar approach is used for estimating total suspended matter in water using random forest.

5.1.2. Biomedical Applications

We also find various applications when it comes to biomedical applications of fractional derivatives combined with ML.

Starting with the classification of EEG signals, we find three publications that combine machine learning and fractional derivatives. Ref. [43] proposes to classify Electroencephalography (EEG) signals into ictal and seizure-free signals by employing a fractional linear prediction from a fractional order system to model the signals, and/or parts of the signal and then to use the so obtained parameters as features for a support vector machine classification approach. Similar to this is the approach from AAruni et al. [44], where EEG signals are modeled using the transfer function of a fractional order system. Contrary to the previous publication, the researchers use the error and signal energy as features for support vector machines to differentiate between ictal and seizure-free signals. The third publication dealing with this subject, i.e., modeling EEG signals using fractional transfer functions and classifying between different categories of EEG signals, e.g., seizure and seizure-free, is [45]. Here, similar to the two previously referenced publications, the fractional modeling transfer function is used to obtain the error and signal energy to be fed into a k nearest neighbor classifier.

Further, similar approaches can be used to analyze Electrocardiography (ECG) signals. Again in [46], a fractional order transfer function is applied to the QRS complex signal (from ECG measurements) to obtain six parameters which are then fed into a k nearest neighbor classifier for person identification. Ref. [47] describes an approach to do the same to obtain five parameters from a QRS complex signal to differentiate between healthy and three types of arrhythmic signals using a k nearest neighbor classifier.

Given the results shown in [48], one can use the same approach, e.g., fractional transfer functions in combination with a k nearest neighbor classifier, to analyze the respiratory properties of humans. In this process, three model parameters are used as features to classify respiratory diseases.

In [49], the researchers are using the Grünwald-Letnikov fractional derivative to preprocess online handwriting data. Afterward, this preprocessed data is fed into support vector machine and random forest algorithms to detect/classify Parkinson's disease from a person's handwriting.

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Contrary to all previously listed publications in this section, in [50], a discrete fractional mask is used to preprocess CT images and to improve the identification of tumors using support vector machines, random forest, J48, and Simple Cart machine learning models.

5.1.3. Engineering

Apart from the two previously discussed categories, we found three publications that didn't fit into spectroscopical or biomedical applications. Instead, we consider these publications to be engineering approaches.

In [51], the researchers use histogram peak distributions and the Grünwald Letnikov fractional derivative to preprocess images of solar panels to then identify defective solder joints via extreme learning machines.

Ref. [52] uses the Reisz fractional derivative to preprocess spectral data obtained from a fiber Bragg grating sensor. Afterward, this data is fed into three machine learning algorithms, i.e., random forest, linear regression, decision trees, and a multi-layer perceptron, to detect the temperature peaks on solar panels from the FBG spectrum.

In [53], the researchers identify construction equipment activity from image/video data using a combination of a fractional derivative and a range of machine learning algorithms. The employed machine learning algorithms are random forest, neural networks (a multi-layer perceptron), and Support Vector Machines. The fractional derivative employed is the Riemann-Liouville fractional derivative, and it is used to preprocess the images, which are fed into the algorithms in a later step.

5.2. Machine Learning & Fractional Dynamics

In this section, we discuss publications where fractional dynamics and the corresponding modeling via fractional differential equations has been used in combination with machine learning. There are several ways to do this, e.g., fractional modeling and afterward employing machine learning for a regression or classification using features obtained from the modeling approach or using machine learning to improve the identification of fractional models/equations from data.

Although the publications by [43–48] have already been categorized as preprocessing, they also serve as a hybrid machine learning and fractional dynamics approach. Specifically, as authors employed the transfer function from a fractional order model to first, model a signal and then to obtain the functions parameters or the corresponding error and signal energy of the model as features for classification.

When identifying the correct transfer function and the underlying dynamics, i.e., with the corresponding fractional differential operators, we find an excellent exemplary application in [54]. Here, the employed Machine Learning technique is Gaussian process regression, whereas the unknown equation parameters are treated as hyperparameters of a physics-informed Gaussian process kernel. This framework is used to identify linear space-fractional differential equations from data, and its applicability is shown for stock market data, e.g., S&P 500 stock data.

In [55], similar ideas are employed to model general damping systems. Here k-means is employed to first differ between viscous and hysteretic damping. Next, the viscous damping dynamics are recovered using a linear regression approach, and afterward, the hysteretic damping is identified using a sparse regression approach. This framework also allows for fractional-order models, and its applicability is shown for, e.g., a fractional-order viscoelastic damper.

We further find an application for machine learning to identify fractional order population growth models from data, as done in [56]. The researchers developed a least squares support vector machine with an orthogonal kernel for simulating this equation. This orthogonal kernel consists of fractional rational Legendre functions.

In [57], the researchers employ ridge regression as a numerical boundary treatment for the fractional KdV-Burgers equation. Another application for identifying fractional models is given by [58], where the researchers use a Gaussian process regression to identify a

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fractional chaotic model in finance from data. Further, as the extrapolation of the so-identified model does not perform well, the researchers suggest employing a neural network to improve prediction accuracy.

In [59], both a fractional order model and a first order resistor-capacitor model are employed to describe the electrical behavior of battery cells with external short circuit faults. Afterward, a random forest classifier is fed with features from the so-obtained fractional order model to classify if a battery cell incurs leakage.

5.3. Optimization

In this section, we differ between derivative-free and gradient-based methods within the optimization context. It is also to say that the derivative-free algorithms, e.g., particle swarm algorithms, still employ fractional derivatives in the form of, e.g., fractional velocities. The featured derivative-free algorithms include bio-inspired algorithms such as, e.g., the fractional bee colony algorithm and the fractional particle swarm algorithm.

5.3.1. Fractional Gradient Based Optimization

When discussing optimization, the first big topic is gradient-based algorithms. Here the basic idea is to alter the gradient-based approach by switching to a fractional gradient. One benefit of doing so is that these algorithms do not fall trap to the local minimum problem as much as regular gradient-based algorithms (find a good citation here).

Regarding the actual applications where fractional derivatives are used for optimization in combination with machine learning algorithms, we find a range of fractional gradient descent applications. We want to point out that all the discussed combined approaches employ the Caputo-fractional derivative but use different algorithms and data sets. In [60], the researchers use ridge regression in combination with fractional gradient descent on testing environment and data sets provided by [61]. The same testing environment is used in [62]. But in this publication, the researchers use logistic regression instead of ridge regression. In both cases, the researchers report improvements to basic (non-fractional) implementations.

Two more applications of fractional gradient descent are found in [63] and [64]. In these publications, the researchers combine fractional gradient descent with support vector machines for two classic data sets, i.e., the Iris and the Rainfall data set.

Contrary to the previous modeling approaches are the ideas presented in [65]; here, a multilinear regression is modified by using a fractional derivative based on the input features to the linear regression. Thus obtaining the derivative and, consequently, the fractional derivative from the minimum of the slope, i.e., setting it to zero. This idea is used to predict the gross domestic product in Romania. It is shown that models not considering fractional derivatives can be outperformed using the presented concepts. Awadalla et al. [66] present a similar approach where the same idea is used for fractional linear and fractional squared regression, i.e., the derivatives to derive the minimum are fractionalized. This approach also shows the applicability of the proposed ideas to standard data sets.

5.3.2. Fractional Gradient-Free Optimization

Particle swarm optimization algorithms are one of the most popular gradient-free optimization techniques. In this section, we further differ between regular and Darwinian particle swarm algorithms. All particle swarm algorithms feature a velocity component, which in an attempt to make the algorithms fractional, is changed to a non-integer derivative-based velocity, [67].

Apart from gradient-based algorithms, we find a range of bio-inspired gradient-free algorithms. The following discussion includes fractional particle swarm, fractional Darwinian particle swarm, and fractional bee colony algorithms.

We're starting with the work of Chou et al. [68], which presents a fractional order particle swarm optimization to improve the classification of heart diseases via an XGBoost classifier. Further, [69] presents an improved fractional particle swarm optimization al-

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gorithm and shows its applicability to optimize support vector machine and K-Means algorithms. As used in the same domain, the researchers use a dataset where the task is to classify heart diseases.

An extension to the regular particle swarm optimization is the Darwinian particle swarm optimization algorithm, introduced in 2005 by Tillet et al. [70]. Couceiro et al. [71] introduced a fractional version of this algorithm. Ghamisi et al. later used these ideas in combination with Random Forest for classifying spectral images [72]. It is used in [73] again for spectral images, i.e., for multilevel segmentation of spectral images later to be processed by support vector machines. Of course, we could also list these tasks in preprocessing, but since particle swarm is a classical optimization technique, we decided to use them in this category, i.e., optimization. Another application is discussed by Wang et al. in [74], where the researchers use a fractional Darwinian particle swarm optimization algorithm to optimize the feature selection for extreme learning machines. Another example of using an optimization procedure for preprocessing is found in [75], where the researchers use a fractional order Darwinian particle swarm algorithm to delineate cancer lesions. In a later step, these CT scan images are classified using Decision Trees. In both [76,77], fractional-order Darwinian particle swarm is employed to segment MRI brain scans, which are then analyzed for strokes using random forest and support vector machines. Two further Biomedical applications for fractional order Darwinian particle swarm are found in [78] and [79]. In both publications, the researchers use the discussed optimization algorithm to segment medical images, i.e., for detecting brain tumors and foot ulcers, respectively. Both publications then employ a supervised ML algorithm for classification, whereas the former uses NAive Bayes and a tree classifier, and the latter a fuzzy c-means classifier.

A different approach for optimizing machine learning approaches is taken in [80], where the researchers predict water quality using a fractional order artificial bee colony algorithm and decision trees. Another example for combining fractional order artificial bee colony algorithms and ML is given in [81] for predicting nonperforming loans using a non-linear regression model.

6. Discussion

Here we summarize and discuss the performed literature analysis. We found a total of 60 publications that fit the criteria defined in Section 2. In the previous section, we separated all publications into three categories, i.e., Preprocessing (Section 5.1), Optimization (Section 5.3), and Machine Learning & Fractional Dynamics. We further subsectioned preprocessing and optimization because we found a variety of publications targeting similar problems in each of those categories. For preprocessing, we found three prominent types of applications, i.e., spectroscopy, biomedical and engineering applications. Contrary to that, we sorted all publications fitting into optimization into two categories. In contrast to preprocessing, we did not choose the categories with respect to the application but regarding the employed technique, whether it was a classic but then fractionalized gradient-based optimization technique or a gradient-free method.

A table containing all publications from Section 5 is given in Appendix A.

Preprocessing

The results for Preprocessing show that a majority of the analyzed publications fall into the realm of Spectroscopy. Here, researchers employ fractional derivatives to alter spectral data at hand to improve the machine learning algorithm's accuracy. Overall, all of these approaches have in common that there is one or more *good* orders of fractional derivatives that enhance the data-based learning approach. We want to highlight that tree-based classifiers, e.g., random forest and XGBoost, are very common and perform well for these applications. We want to point out Ref. [41] which used fractional derivatives to preprocess hyperspectral data, which are then used to estimate the total nitrogen content in water via different machine learning algorithms, e.g., Random Forest.

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Regarding biomedical applications, we find that here, similar to preprocessing spectral data, researchers use fractional derivatives to preprocess their medical images, as done in [50] for CT images. Another example is to preprocess EEG or ECG signals, used to classify diseases or malfunctions as done in [45] and [47].

When it comes to the engineering applications, we again found similar ideas, e.g., preprocessing image and spectral data to identify malfunctions of solar panels or detect the status of construction equipment as done in [52] and [53].

Overall, fractional derivatives can improve the accuracy of machine learning algorithms when employed for preprocessing signals, images, or spectral data. Also, we find that there is a *sweet spot*, i.e., one or more orders of fractional derivatives that can significantly improve the accuracy of the employed algorithm. We further want to point out that though fractional order preprocessing enhances the accuracy of machine learning applications, it also introduces a new hyperparameter, the order of the fractional derivative. Thus it does not simplify the task at hand but instead provides a slight increase of accuracy at the cost of optimizing another hyperparameter.

Also, we found that every application in which an actual fractional derivative was used for preprocessing involved the Grünwald-Letnikov fractional derivative.

Also, we considered both data manipulation and feature extraction as being preprocessing. Here one can differ between manipulating the original data using a fractional derivative (e.g., [41]) and extracting features, i.e., error energy and parameters from the original data (e.g., [47]).

The authors believe that the preprocessing step using fractional derivatives is especially useful for non-neural network machine learning algorithms. The reason for this is that, given a deep neural network, the shallow layers of a neural network can make preprocessing obsolete as a neural network is capable of learning a huge variety of data-manipulations. Thus we suggest future research to test this hypothesis. Specifically, one needs to test whether a neural network can perform any preprocessing based on fractional derivatives.

Finally, an addition noteworthy to this discussion on fractional derivatives and preprocessing is fractional measures of (signal) complexity, which can be used as a preprocessing step for machine learning applications. This is done in [82], in which the researchers employ a fractional entropy measure to improve the fault detection for train doors.

Machine Learning & Fractional Dynamics

Regarding possible combinations of fractional dynamics & machine learning, we find that machine learning can be used to identify fractional order behavior from data, as done in [54] or [58]. Overall, this is the category where we found the least amount of publications. This might be due to the fact that fractional order modeling of dynamics requires profound knowledge in both mathematics and machine learning. Further, we want to emphasize this category for its potential in the future. We suggest that researchers use, e.g., fractional orthogonal kernels in kernel-based machine learning algorithms as, e.g., support vector machines or Gaussian process regression to model systems from data[83].

Optimization

Regarding optimization, we split our review into two main sections, i.e., gradient-based and gradient-free methods. The gradient-based optimization techniques feature gradient descent approaches where the integer-order gradient is replaced with a fractional-order gradient. For the gradient-free techniques, the idea is to replace, e.g., the velocity of a particle swarm algorithm with a fractional velocity.

Here we want to point out two exemplary applications by Hapsari et al. [63,64] on well-known test-data sets, i.e., the Iris and the rainfall data set, for the gradient-based optimization of machine learning algorithms.

For the gradient-free optimization algorithms, we highlight the work done by Li et al. [69], where the researchers present an improved fractional particle swarm optimization

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algorithm to improve a support vector machine and k-Means-based classification of heart diseases.

6.1. Bringing It All Together

This review's motivation and the initial assumption is that combined approaches of fractional derivatives and machine learning improve machine learning tasks. We identified a total of three categories from the literature within our frame of keywords and criteria. The applications include processing, optimizing, and modeling fractional dynamics via machine learning. We discussed all of our findings in the previous sections. This section, however, brings all our results together and puts them into a bigger context. Here we're taking into account the findings of the work of Niu et al. from 2021 [84], where the researchers propose a triangle consisting of Machine Learning, Fractional Calculus, and the renormalization group. We depicted this triangle in Figure 1. The triangle suggests overlaps and further shows where potential combined approaches of machine learning and fractional calculus can be beneficial. Further, we consider this triangle a closed loop; thus, future research might employ all of the mentioned tools, i.e., machine learning, fractional calculus, and the renormalization group, to tackle complex problems and describe complex systems. Thus we describe our findings in this context and consider the results of another article by Niu. et al. [85], titled "*Why Do Big Data and Machine Learning Entail the Fractional Dynamics?*" in which the researchers further discuss ideas on the overlap between machine learning and fractional calculus, specifically for big data.

The following paragraphs discuss the overlaps shown by Niu. et al., as depicted in Figure 1, compare them to the results of our literature review, and further provide evidence for a link between fractional calculus, machine learning, and the renormalization group.

Figure 1. The triangle of machine learning, fractional calculus and the renormalization group as introduced by [84]. The current article focuses on the connection between machine learning and fractional calculus, i.e. the red double-arrow conneting the two blue boxes. This graphic is taken from [84], but we adapted it to better fit this review.

6.1.1. Optimization

We start discussing the most obvious of the mentioned overlaps/connections, i.e., Optimization. Fractional derivatives and, subsequently, fractional calculus is a powerful framework that can enhance optimization tasks, as fractionalized optimization algorithms are less prone to get stuck in local minima/optima [86]. Thus, it is an upgrade to standard integer-derivate-based optimization tools for optimizing machine learning models' hyperparameters. These optimization tools are valuable for preprocessing as well. Here we want to mention [76,77], these are exemplary applications in which this is used to segment MRI brain scans, later to be applied as an input for a machine learning algorithm to detect strokes.

6.1.2. Variability

Given the results of our literature review, we can link several publications and the corresponding approaches to data variability, thus, providing evidence that fractional calculus and machine learning combined are tools capable of dealing with variability in data.

E.g., In [20], the researchers employ fractional derivatives to preprocess Vis-NIR spectral data later to be used in a machine learning task to estimate organic matter content in arid soil. Similar preprocessing approaches are collected in Section 5.1.1. Thus, all of these articles deal with the spatial variability of the obtained spectral data. Further, we can interpret these approaches as coping with climate variability as well, as they take into account or describe the components and interactions of the corresponding climate system.

We also find several publications where fractional derivatives and machine learning can deal with heart rate variability. I.e., in both [46] and [47], the researchers are using

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ideas related to fractional derivatives to, e.g., classify arrhythmic ECG signal. This also refers to human variability in general. We also refer to the work by Mucha et al. [49], where the researchers classify Parkinson's disease based on online handwriting data, which was preprocessed using fractional derivatives.

According to the work of Niu at. al. [85], variability is the central aspect of big data where employing fractional calculus and machine learning can be beneficial. Thus we recommend considering approaches combining fractional derivatives and machine learning when dealing with variability in data and further recommend considering these for big data approaches. As the topic of big data would unnecessarily blow up this discussion, we refer the interested reader to [85,87].

6.1.3. Nonlocal models

Section 5.2 provides evidence for applying fractional calculus together with machine learning for nonlocal models. I.e., all models based on fractional calculus are considered nonlocal because of the derivatives inherent *memory*. Here we want to point out the work by [54], which provides a machine learning framework for identifying linear space-fractional differential equations from data.

6.1.4. Renormalization group

The renormalization group refers to a formal apparatus capable of dealing with problems involving many length scales. The basic idea is to tackle the problems in steps, i.e., one step at each length scale. For critical phenomena, the strategy is to carry out statistical averages over thermal fluctuations on all size scales. Thus, the renormalization group approach integrates all the fluctuations in sequence. I.e., starting on an atomic scale and moving step-by-step to larger scales until fluctuations on all scales are averaged out. The reason for doing so is that theorists have difficulties describing phenomena that involve many coupled degrees of freedom. E.g., it takes many variables to characterize a turbulent flow or the state of a fluid near the critical point. Here Kenneth and Wilson found self-similarity of dynamic systems near critical points, which can be described using the renormalization group. Thus the renormalization group is considered a powerful tool for studying the scaling properties of physical phenomena through its treatment of the scaling properties of complex and chaotic dynamics [88,89].

Thus we want to discuss our findings in a bigger context together with renormalization group methods. We list and discuss evidence from our literature review for the proposed overlaps between fractional derivatives and machine learning with the renormalization group. Thus can we find evidence for the keywords depicted in Figure 1? I.e., scaling law, complexity, nonlinear dynamics, fractal statistics, mutual information, feature extraction, and locality.

Again we start our discussion with the most obvious, i.e., feature extraction. Regarding the triangle in Figure 1, [84], the researchers argue that, e.g., deep neural networks and their inherent feature extraction process, i.e., feeding and abstracting the input data from one layer to the next, learn to ignore irrelevant features while keeping relevant ones, which increases the efficiency of the learning process. The researchers further argue that this is similar to renormalization group methods, where only relevant features of a physical system are extracted by integrating over short-scaled degrees of freedom to describe the system at larger scales. Their assumption is based on the findings of [90], which provides a mapping between the variational renormalization group and deep learning, whereas these findings are based on the Ising model. Further, they discuss that the renormalization group is usually applied to physical systems with many symmetries. Contrary to that, deep learning is typically applied to data with limited structure. Thus, the connection between the renormalization group and, in this case, deep neural networks might not be generalizable to machine learning applications at large. Still, when talking about feature extraction, one can argue that fractional derivatives do just that in, e.g., [46,47]. These articles describe the feature extraction such that new parameters, e.g., the error energy of a

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signal when being modeled using a fractional transfer function, are found. I.e., smaller-scaled features, i.e., the data set itself, are not fed into the machine learning algorithm. Further, we recommend using fractional measures of complexity as another fractional calculus-based technique for feature extraction, [82].

The researchers [84] discuss locality as the similarity of grouping spins in the black-spin renormalization group. And further point out that this is similar to the shallow layers of a deep neural network in which the neurons are connected locally, e.g., without strong bonds as described in [91]. Though this applies to deep neural networks, we didn't find evidence in the reviewed publications. The reason here is that fractional derivatives behave non-local, i.e., spatio-temporal fractional derivatives have a certain degree of memory in both space and time.

When dealing with complexity, we need to consider that there is no unified definition for complexity yet. However, the researchers [84,89] argue that the overlap between fractional calculus and renormalization group methods might give us this definition and provide a framework to deal with complexity and thus answer the following two questions:

1. How can we characterize complexity?
2. What method should be used to analyze complexity to better understand real-world phenomena?

We consider finding answers to these questions idealistic. Thus we need to weaken the statement to "*partially* answer this questions". With much certainty, even when combined, fractional calculus, the renormalization group, and machine learning will only partially answer these questions, primarily because there are many different perspectives on the complexity of, e.g., data. Still, this triangle of powerful tools might expand definitions and find some definitions and characterizations for particular problems. To give an (in the authors' opinion) reasonable example: This triangle might provide insights into the capability of machine learning to predict multifractal, chaotic, or stochastic time series data by characterizing a datasets memory and which time series can or cannot be learned by different machine learning algorithms.

Still, did we find evidence for complexity in our list of publications? We indeed found that there are phenomena that are better described using fractional calculus rather than integer-valued calculus, as described in [54]. Further in this work, the researchers use fractional calculus and machine learning to describe, e.g., financial time series data, e.g., the S&P500 index, which is widely accepted to be a complex phenomenon, [92–94]. Further, in [95], the researchers used renormalization group methods for analyzing the S&P 500 index. Thus we conclude that we found evidence for this connection. Additionally, in [92], the researchers also found evidence for two universal scaling models/laws in finance. Thus, we found a connection and a potential research area for all three of the discussed techniques, i.e., the renormalization group, fractional calculus, and machine learning in financial time series analysis. Still, the discussed work of Gulian et al. [54] deals with linear space fractional differential equations and thus does not provide the framework to deal with nonlinear dynamics.

However, nonlinear dynamics (and subsequently chaos) are discussed in [81], where a fractional bee colony optimization algorithm is used to find the optimal parameters for a model which describes the nonlinear behavior of nonperforming loans. We also need to mention the work done by Wang et al. [58], where fractional models for finance are parametrized using gaussian process regression and predicted using recurrent neural networks. Thus, given this evidence, we suggest similar approaches to employ the renormalization group methodology in future research.

However, we didn't find applications of mutual information in this review. In [84], the researchers refer to the work by Koch-Janusz et al. [96] for the applicability of mutual information for both machine learning and the renormalization group. The researchers present a machine learning procedure to perform renormalization group tasks, thus, the reported connection. Here, the mutual information is used as an objective or target function where one of the employed neural networks aims to maximize the mutual information

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between two random distributions to deal with the reduction of degrees of freedom as the renormalization group would do. Though we didn't find applications of mutual information for fractional derivatives and machine learning in our literature review, we recommend future research to discuss the topic. E.g., to which degree can machine learning algorithms be used to perform renormalization group tasks? And are there non-model applications where these ideas can be used?

Also, we didn't find approaches dealing with scaling laws in the featured list of publications. Scaling Laws, also known as power laws, are a description of the scale invariance of natural phenomena. Some notable examples according to [89] are, e.g., Zipf's inverse power law which describes the relative frequency of word usage within a language [97].

And finally, we also didn't find an application of fractal statistics or fractals. However, given the numerous evidence for fractal behavior and statistics in various fields, e.g., in hydrology [98], rock mechanics [99] or finance, [100], we still aim to provide a link. Here, we're considering the previous discussion on the potential applicability of the triangle of machine learning, fractional calculus, and renormalization group on financial data. Again, we found evidence for fractal statistics present in economic data, i.e., the S&P 500 index, [100]. Thus we recommend future research for analyzing stock market data to employ ideas from fractional calculus, machine learning, and the renormalization group and subsequently to analyze the fractal behavior of these data sets. Further, we believe that these ideas might be capable of analyzing, learning, and predicting a variety of complex real-life data sets, e.g., environmental data.

6.2. Problems

The last part of this discussion is to name two problems we encountered whilst conducting this literature review. This section should serve as a guideline on what to avoid and how to improve the future research field.

Sometimes the fractional derivative is not discussed sufficiently: We have found literature that used fractional derivatives without stating the exact approach, e.g.[39]. But as stated and referenced in [32], we assume the Grünwald-Letnikov fractional derivative is the employed fractional derivative.

We thus urge researchers to be specific about the employed fractional derivative, the corresponding discretization, and/or the employed software package to improve the transparency and reproducibility of their work. We further point out that hardly any article discussing combined applications of machine learning and fractional derivatives listed in this review shows the employed discretization.

Incorrect usage of keywords: Some researchers assign the term *fractional calculus* and/or *fractional derivatives* to their articles and describe the fractional calculus framework. However, in the end, they use a curve-fitting approach with fractional polynomial exponents and no fractional derivatives and/or calculus, [45].

7. Conclusion

This review analyzes the combined applications of fractional derivatives and supervised non-neural network approaches. Our research showed that supervised machine learning and fractional derivatives are valuable tools that can be combined to, e.g., improve a machine learning algorithm's accuracy. We found a total of three types of combined applications, i.e., preprocessing, modeling fractional dynamics via machine learning, and optimization.

We further found that most preprocessing applications are spectroscopical applications where a fractional derivative of a specific order is applied to the spectral data to improve the accuracy of the employed algorithm.

For optimization, we found two categories, gradient-based and gradient-free optimization algorithms. For gradient-based optimization, one can replace an integer-order gradient with a fractional-order gradient, thus, obtaining a fractional gradient descent optimization algorithm. For the gradient-free optimization, the velocity in, e.g., a particle swarm algorithm, can be replaced by a fractional-derivative-based velocity. Hereby, one obtains a fractional order particle swarm optimization algorithm.

For the third type of combined applications, i.e., using machine learning to model fractional dynamics, we found that machine learning and fractional derivatives, and subsequently fractional calculus are both techniques that deal with complex real-life phenomena, addressing the problem of complexity from different angles. Further, this field improves kernel-based machine learning algorithms, such as Gaussian process regression and support vector machines, by providing kernels based on fractional derivatives to introduce memory into the modeling approach.

When bringing all our results together, we find a third tool conceptually linked to both, i.e., the renormalization group, which is a powerful approach for dealing with scaling laws, complex phenomena, complex dynamics, and chaos, [101]. Thus, these three techniques, machine learning fractional calculus, and the renormalization group, form a triangle as proposed by Niu et al. [84]. Our review provides evidence for the proposed keywords, the corresponding connections, and thus the established links between machine learning, fractional calculus and the renormalization group.

In the author's opinion, it will still take some time until the renormalization group will be part of mainstream computer science in combination with machine learning and fractional calculus. However, it is important to make today's researchers, especially machine learning practitioners, aware and familiar with these topics as (supervised) machine learning is becoming popular in science in general. I.e., wherever there's data available, scientists are eager to find modern machine learning approaches to analyze, describe, and predict complex phenomena directly from data. Thus machine learning is changing science overall [102,102,103].

Further, as pointed out in [104], machine learning is sometimes considered an oracle that can learn any task from observations, i.e., predict chemical reactions and particle physics experiments, but not necessarily giving the practitioner scientist a deeper understanding of the process at hand. Thus, one might ask how and if machine learning can contribute to our scientific understanding. The researchers further point out the multidisciplinary aspect of machine learning in general and that it will, because of this, undoubtedly keep increasing in popularity and can increase our scientific understanding of the studied processes. Thus, we propose that fractional calculus and the renormalization group will provide powerful tools in this context. Together with machine learning, these tools are able to bridge gaps between separated disciplines by improving numerical analysis, providing insights into complex phenomena, and by unveiling the hidden (or macroscopic; in the case of the renormalization group) mechanics governing these phenomena.

Regarding future research combining fractional derivatives and machine learning, we recommend testing if neural networks can perform any kind of data manipulation that fractional derivatives can perform. This might be useful as it might avoid pointless contributions on the topic, i.e., fractional-derivative-based preprocessing for neural networks.

Further, future research combining fractional derivatives, the renormalization group, and machine learning might give valuable insights into the learnability of machine learning algorithms and the corresponding underlying dynamics of the data. Specifically, One could connect ideas from the renormalization to a fractional derivative preprocessing while analyzing the performance of the employed machine learning algorithm. I.e., to find the sweet spot of mandatory degrees of freedom and the corresponding order of fractional derivatives, which might provide insights about the memory of the studied data/system. Thus providing the "minimal" representation of the data/system by optimal performance.

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Appendices

A. Summary Table

Publication	Employed Fractional Derivative	Employed Machine Learning Algorithms	Research Area	Category	Comment
[20]	Grünwald-Letnikov	Random Forest, PLSR	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Organic Matter Content
[22]	Grünwald-Letnikov	PLSR	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Nitrogen Concentration
[29]	Grünwald-Letnikov	Random Forest, PLSR	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Soil Salinity
[26]	Grünwald-Letnikov	Random Forest	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Lead and Zinc concentration
[21]	Grünwald-Letnikov	Memory Based Learning, PLSR, Random Forest	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Soil Organic Matter
[24]	Grünwald-Letnikov	SVM	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Total Nitrogen Content
[25]	Grünwald-Letnikov	SVM, Random Forest PLSR, ELM	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Leaf Chlorophyll Concentration
[30]	Grünwald-Letnikov	ELM	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Soil Salt, Water-Soluble Ions
[23]	Grünwald-Letnikov	SVM, ELM CNN, PLSR	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Nitrogen Content
[27]	Grünwald-Letnikov	PLSR, GRNN	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Soil Heavy Metal Estimation
[28]	Grünwald-Letnikov	SVM, Ridge Regression, PLSR, Random Forest XGBoost, ELM	Agriculture	Preprocessing	Vis-NIR Spectroscopy, Soil Heavy Metal Estimation
[34]	Grünwald-Letnikov	PLSR	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Salt Content
[40]	Grünwald-Letnikov	PLSR	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Soil Clay Content
[31]	Grünwald-Letnikov	PLSR	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Soil Organic Matter
[33]	Grünwald-Letnikov	SVM	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Soil Salt Content
[32]	Grünwald-Letnikov	Random Forest	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Top Soil Organic Carbon
[36]	Grünwald-Letnikov	Random Forest, SVM	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Canopy Nitrogen Content
[35]	Grünwald-Letnikov	PLSR	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Soil Total Salt Content
[41]	Grünwald-Letnikov	Random Forest, XGBoost	Environmental	Preprocessing	Hyperspectral Spectroscopy, Total Nitrogen in Water
[38]	Grünwald-Letnikov	XGBoost	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Soil Moisture Content
[37]	Grünwald-Letnikov	SVM, KNN	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Leaf Photosynthetic Pigment Content
[42]	Grünwald-Letnikov	Random Forest	Environmental	Preprocessing	Hyperspectral Spectroscopy, Total Suspended Matter in Water
[39]	Grünwald-Letnikov	XGBoost, LightGBM, Random Forest, Extremely Randomized Trees, Decision Trees, Ridge Regression	Agriculture	Preprocessing	Hyperspectral Spectroscopy, Soil Electrical Conductivity

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Publication	Employed Fractional Derivative	Employed Machine Learning Algorithms	Research Area	Category	Comment
[43]	Fractional Order Modelling	SVM	Biomedical Application	Preprocessing	EEG Signal Analysis, Seizure(Ictal) Detection
[44]	Fractional Order Modelling	SVM	Biomedical Application	Preprocessing	EEG Signal Analysis, Seizure(Ictal) Detection
[46]	Fractional Order Modelling	KNN	Biomedical Application	Preprocessing	ECG Signal Analysis, Arrhythmia Detection
[47]	Fractional Order Modelling	KNN	Biomedical Application	Preprocessing	ECG Signal Analysis, Arrhythmia Detection
[50]	Discrete Fractional Mask	SVM, RF, Simple Cart, J48	Biomedical Application	Preprocessing	CT Image Signal Analysis, Liver Tumor Detection
[48]	Fractional Order Modelling	KNN	Biomedical Application	Preprocessing	Respiratory Impedance Analysis, Disease Detection
[49]	Grünwald-Letnikov	RN	Biomedical Application	Preprocessing	Handwriting Analysis, Parkinson Disease Detection
[45]	Grünwald-Letnikov	KNN	Biomedical Application	Preprocessing	EEG Signal Analysis, Abnormality Detection
[105]	Discrete Fractional Derivative	RF, XGBoost	Biomedical Application	Preprocessing	Hemoglobin Analysis, Diabetes Detection
[106]	Discrete Fractional Derivative	SVM, Naive Bayes, Random Forest AdaBoost, Bagging	Biomedical Application	Preprocessing	Glucose and Insulin Analysis, Diabetes Detection
[51]	Grünwald-Letnikov	ELM, MLSR	Engineering	Preprocessing	Solar Panel Analysis, Defective Solder Joint Detection
[52]	Reisz	ELM, MLSR	Engineering	Preprocessing	Photovoltaic Panel Temperature Monitoring, Peak Detection
[53]	Riemann-Liouville	RF	Engineering	Preprocessing	Activity Recognition of Construction Equipment
[54]	Fractional Differential Equations	Gaussian process regression	Physics	Fractional Dynamics	Modelling of Fractional Dynamics from Data
[55]	Fractional Differential Equations, Caputo Derivative	K-Means, Linear Regression, Sparse Regression	Physics	Fractional Dynamics	Modelling Damping from Data
[56]	Fractional Differential Equations, Caputo Derivative	SVM	Applied Mathematics	Fractional Dynamics	Solving Fractional Differential Equations
[57]	Fractional Differential Equations, Caputo Derivative	Ridge Regression	Applied Mathematics	Fractional Dynamics	Modelling Boundaries of Fractional Differential Equations
[58]	Fractional Differential Equations, , Caputo Derivative	Gaussian Process Regression, RNN	Finance & Economy	Fractional Dynamics	Modelling a Fractional Order Financial System
[59]	Fractional Differential Equations, Grünwald-Letnikov Derivative	RF	Engineering	Fractional Dynamics	Modelling and Anaylzing the fractional behavior of batteries

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Publication	Employed Fractional Derivative	Employed Machine Learning Algorithms	Research Area	Category	Comment
[60]	Caputo	Ridge Regression	Regression in General	Optimization	Fractional Gradient Descent on Testing Environment
[62]	Caputo	Logistic Regression	Regression in General	Optimization	Fractional Gradient Descent on Testing Environment
[63]	Caputo	SVM	Classification in General	Optimization	Fractional Gradient Descent Rainfall Data
[64]	Caputo	SVM	Classification in General	Optimization	Fractional Gradient Descent Iris and Rainfall Data
[65]	Caputo	Fractional Linear Regression	Regression Analysis in Finance	Optimization	Multi Linear Regression with Fractional derivatives applied to the Romanian GDP
[66]	Caputo	Fractional Linear Regression, Fractional Quadratic Regression	Regression Analysis in General	Optimization	Multi Regression with Fractional Derivatives
[68]	Grünwald-Letnikov	XGBoost	Biomedical Application	Optimization	Classification of Heart Diseases via XGBoost and fractional PSO
[69]	Grünwald-Letnikov	SVM, K-Means	Biomedical Application	Optimization	Optimizing K-Means and SVM via Improved Fractional PSO
[72]	Grünwald-Letnikov	Random Forest	Spectroscopy	Optimization, Preprocessing	Optimizing a RF Classification for Spectral Bands
[73]	Grünwald-Letnikov	SVM	Spectroscopy	Optimization, Preprocessing	Multilevel Image Segmentation for SVM Classification
[74]	Grünwald-Letnikov	ELM, SVM, RRelief,	Regression in General	Optimization, Preprocessing	Optimizing Features for a range of Algorithms and Datasets using FODPSO
[75]	Grünwald-Letnikov	Decision Trees	Biomedical Application	Optimization, Preprocessing	Optimizing Liver cancer detection from CT scans using Decision Trees and FODPSO
[76]	Grünwald-Letnikov	SVM, RF	Biomedical Application	Optimization, Preprocessing	Optimizing stroke detection from MRI brain scans using Random Forest and FODPSO
[77]	Grünwald-Letnikov	SVM, RF	Biomedical Application	Optimization, Preprocessing	Optimizing stroke detection from MRI brain scans using Random Forest and FODPSO
[78]	Grünwald-Letnikov	Naive Bayes Hoeffding Tree Classifier	Biomedical Application	Optimization, Preprocessing	Optimizing foot ulcer detection from MRI brain scans using Random Forest and FODPSO
[79]	Grünwald-Letnikov	Fuzzy C-MEans	Biomedical Application	Optimization, Preprocessing	Optimizing brain tumor detection from brain MRI scans using Fuzzy C-Means and FODPSO
[80]	Grünwald-Letnikov	Decision Trees	Water Management	Optimization	Predicting water quality using decision trees and Ffractional Artificial Bee colony optimization
[81]	Grünwald-Letnikov	Nonlinear Regression	Finance & Economy	Optimization	Predicting non-performing loans using FOABCO and nonlinear regression

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