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Concept Paper

On Fractional Power Inequalities In b-Metric-Type Spaces

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Abstract

By using a generalization of the binomial theorem, we give some bounds for the distance in b-metric type spaces. In particular, we prove that the fractional power inequalities in the results of Dragomir and Gosa [S.S. Dragomir and A.C. Gosa, An inequality in metric spaces, Journal of the Indonesia Mathematical Society, vol. 11, no. 1(2005), 33-38] and Karapinar and Noorwali [Dragomir and Gosa Type Inequalities on b-metric spaces, Journal of Inequalities and Applications, vol. 2019, 1-7].

Keywords: metric space; upper bound; Norm-space; b-metric space; inequalities

MSC: 32A30; 46B20; 54E35; 54E50

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1. Introduction and Preliminaries

We first recall some basic definitions in literature.

Definition 1.1. *A metric space is defined as a pair (X, ψ), where X is a non empty set and $\psi : X \times X \to \mathbb{R}$ is a function. This function ψ satisfies the following properties for any elements ξ, φ, ϱ in X:*

- $\psi(\xi, \varphi) \geq 0$ *and* $\rho(\xi, \varphi) = 0$ *if and only if* $\xi = \varphi$ *(Non-negative);*
- $\psi(\xi, \varphi) = \psi(\varphi, \xi)$ *(symmetry);*
- $\psi(\xi, \varphi) \leq \psi(\xi, \varrho) + \psi(\varrho, \varphi)$ *(Triangle inequality).*

The study of metric space involving distance function provides a powerful tool in mathematics and other science such as fixed point theory, topology and operator theory, see [1,3,11,13]. In 1998, Czerwik [6] (see also [4]) constructed a lemma to obtain some generalizations of the well known Banach's inequality contraction in b-meric spaces using α-relaxed triangle inequality as follows:

$$\psi(\xi, \varphi) \leq \alpha[\psi(\xi, \varrho) + \psi(\varrho, \varphi)], \tag{1.1}$$

where $\alpha \geq 1$. We note that in the case where $\alpha = 1$, every b-metric space is a metric space. Some examples of b-metric are given below:

Example 1.2. *Let $X = [0, 1]$ and $\psi : X \times X \to [0, \infty]$ is defined by $\psi(\varsigma, \varphi) = (\varsigma - \varphi)^2$, for all $\varsigma, \varphi \in X$. Clearly, (X, ψ) is a b-metric space with $k = 2$.*

Example 1.3. *The set $l_p(R)$ with $0 < p < 1$, where $l_p(R) := \{(\varsigma_n) \subset R| \sum_{n=1}^{\infty} |\varsigma_n|^p, \infty\}$, is defined by the function $\psi : l_p(R) X l_p(R) \to R$,*

$$\psi(\varsigma, \varphi) = \left(\sum_{n=1}^{\infty} |\varsigma_n - \varphi_n|^p \right)^{\frac{1}{p}},$$

where $\varsigma = \varsigma_n$, $\varphi = \varphi_n \in l_p(R)$. Then $(l_p(R), \psi)$is a b-metric space such that $\psi(\varsigma, \varrho) \leq 2^{\frac{1}{p}}[\psi(\varsigma, \varphi) + \psi(\varphi, \varrho)]$.

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Definition 1.4. *Let X be a vector space over a field $\mathbb{K}$ and let $s \geq 1$ be a constant. A function $\|\cdot\|_b$ defined by $\|\cdot\|_b : X \to [0,\infty]$ is said to be a b-norm space if the following conditions are satisfied for all $\varsigma, \varphi \in X$:*

- *(bN1)* $\|\varsigma\|_b \geq 0$;
- *(bN2)*$\|\varsigma\|_b = 0$ *if and only if* $\varsigma = 0$;
- *(bN3)* $\|K\varsigma\|_b = |K|^{\log_2 s+1}\|\varsigma\|_b$;
- *(bN4)*$\|\varsigma + \varphi\|_b \leq s[\|\varsigma\|_b + \|\varphi\|_b]$.

In this case $(X, \|\cdot\|)$ is called a b-normed space with constant s.

Remark 1.5. *Clearly, when $s = 1$, we recover the definition of a norm linear space, see [6] .*

Example 1.6. *Let $X = \mathbb{R}$ and define $\|\cdot\|_b : X \to [0,\infty]$ by $\|\xi\|_b = |\xi|^p$ where $p \in (1,\infty)$ then, using the relation $(\xi + \varphi)^p \leq 2^{p-1}(\xi^p + \varphi^p)$, we can easily deduce that $(X, \|\cdot\|)$ is a b-normed space with constant $s = 2^{p-1}$ for all $\xi, \varphi \in X$.*

One of the important properties of a (classical) distance function on any abstract set X is the triangle inequality, i.e., $\psi(\xi,\varphi) \leq \psi(\xi,\varrho) + \psi(\varrho,\varphi)$. Several generalizations and refinements of the concept of a distance have been achieved by relaxing the triangle inequality, see [10].
Dragomir and Gosa [7] established the polygonal inequality in the metric space setting by obtaining the following result:

Theorem 1.7. *Let (X,ψ) be a metric space and $\xi_i \in X$, $\rho_i \geq 0$, $i \in \{1,...,N\}$ with $\sum_{i=1}^N \rho_i = 1$. Then we have the inequality*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \rho_i\rho_j\psi(\xi_i,\xi_j) \leq \inf_{\xi\in X}\left[\sum_{i=1}^{N}\rho_i\psi(\xi_i,\xi)\right]. \tag{1.2}$$

In a recent paper, Karapinar and Noorwali [10] gave an improved version of Dragomir and Gosa's result as follows:

Theorem 1.8. *Let (X,ψ) be a b-metric space with constant $m \geq 1$ and $\xi_i \in X$, $\rho_i \geq 0$, $i \in \{1,2,...,N\}$ with $\sum_{i=1}^N \rho_i = 1$. Then we have the inequality*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \rho_i\rho_j\psi(\xi_i,\xi_j) \leq \frac{N}{m}\inf_{\xi\in X}\left[\sum_{i=1}^{N}\rho_i\psi(\xi_i,\xi)\right]. \tag{1.3}$$

The aim of this paper is to obtain some upper bounds for the distance on b-metric spaces. Thus, our results are generalizations of [2,7,8,10]. Before we give our main result, we can state the following result that is regarded as a generalization of the binomial theorem.

Theorem 1.9. *(Neo-classical inequality; Theorem 1.2 in [9]) For $k \in \mathbb{N}$ and $0 < s < 1$, we have*

$$s\sum_{k=0}^{m}\binom{sm}{sk}x^{sk}y^{s(m-k)} \leq (x+y)^{sm}, \quad x,y \geq 0. \tag{1.4}$$

Remark 1.10. *When $s = 1$, the equality holds in (1.4), which is just the conventional binomial theorem.*

2. Main Results

Now, we first discuss the following new concept:

Definition 2.1. *Let X be a non empty set and $\alpha, \beta \in \mathbb{N}$ be a given real number. A mapping $\psi_b : X \times X \to \mathbb{R}^+$ is said to be a variant of b-metric if for all $\varsigma, \varrho, \varphi$ in X, the following conditions are satisfied:*

- *(b1):* $\psi_b(\varsigma,\varphi) = 0$ *if and only if* $\varsigma = \varphi$;

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- *(b2):* $\psi_b(\varsigma,\varphi) = \psi_b(\varphi,\varsigma)$ *(symmetry);*
- *(b3):* $\psi_b(\varsigma,\varphi) \leq \alpha[\psi_b(\varsigma,\varrho) + \beta\psi_b(\varrho,\varphi)]$ *(Triangle inequality).*

It is easy to see that when $\beta = 1$ then it is b-metric space [6] which in turn is a generalization of the standard metric space [13]. It may be of interest to study this new variant of b-metric space for the case $\beta \geq 1$
The following example may be stated to support Definition 2.1.

Example 2.2. *Consider the set X of all continuous functions $\varsigma : [0,1] \to \mathbb{R}$ defined by the distance function $\psi_b : X \times X \to \mathbb{R}^+$ as:*

$$\psi_b(\varsigma,\varphi) = \int_0^1 |\varsigma(t) - \varphi(t)|^2 dt.$$

This is a metric space that satisfies the condition $\psi_b(\varsigma,\varphi) \leq \alpha[\psi_b(\varsigma,\varrho) + \beta\psi_b(\varrho,\varphi)]$ with $\alpha = 2$ and $\beta = 2$, since

- *(b1):* $\psi_b(\varsigma,\varphi) = 0$ *if and only if* $\varsigma = \varphi$*;*
- *(b2):* $\psi_b(\varsigma,\varphi) = \psi_b(\varphi,\varsigma)$ *(symmetry);*
- *(b3):* $\psi_b(\varsigma,\varphi) \leq 2\int_0^1 |\varsigma(t) - \varrho(t)|^2 dt + 4\int_0^1 |\varrho(t) - \varphi(t)|^2 dt = 2[\psi_b(\varsigma,\varrho) + 2\psi_b(\varrho,\varphi)].$

Example 2.3. *Let $X = \{1,2,3\}$ be a discrete set and let $\psi_b : X \times X \to \mathbb{R}^+$ be a function defined by*

$$\psi_b(1,1) = \psi_b(2,2) = \psi_b(3,3) = 0$$

$$\psi_b(1,2) = \psi_b(2,1) = \frac{1}{3}$$

$$\psi_b(1,3) = \psi_b(3,1) = 3$$

$$\psi_b(2,3) = \psi_b(3,2) = 4$$

By Definition 2.1, (b1) and (b2) clearly holds. For all $\xi,\varrho,\varphi \in X$ it follows that

$$\psi_b(\xi,\varphi) \leq 2[\psi_b(\xi,\varrho) + 3\psi_b(\varrho,\varphi)]$$

The following result may be stated:

Theorem 2.4. *Let (X,ψ_b) be a metric space and $\alpha, m \in \mathbb{N}$, $\beta \geq 1$, $0 < s < 1$, $v_i \in X$, $\rho_i \geq 0$, $i \in \{1,...,N\}$ with $\sum_{i=1}^N \rho_i = 1$, then*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \rho_i\rho_j\psi_b^{sm}(v_i,v_j)$$
$$\geq \frac{s\alpha^{sm}}{2}\sup_{v\in X}\left[2s\sum_{i=1}^{N}\rho_i\psi_b^{sm}(v_i,v) + \sum_{k=1}^{m-1}\binom{sm}{sk}\left(\sum_{i=1}^{N}\rho_i\psi_b^{sk}(v,v_i)\right)\left(\sum_{i=1}^{N}\rho_i\beta^{sk}\psi_b^{s(m-k)}(v_i,v)\right)\right]. \tag{2.1}$$

Proof. Using the b-triangle inequality in metric space, we have that for any $v \in X$ and $i,j \in \{1,...,N\}$ that

$$\psi_b(v_i,v_j) \leq \alpha\left[\psi_b(v_i,v) + \beta\psi_b(v,v_j)\right]. \tag{2.2}$$

Taking the power sm, where $0 < s < 1$, $m = 1,2,...$ to have

$$\psi_b^{sm}(v_i,v_j) \geq \alpha^{sm}\left[\psi_b(v_i,v) + \beta\psi_b(v,v_j)\right]^{sm}. \tag{2.3}$$

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By expanding the RHS of (2.3) using Proposition 1.13 we have

$$\psi_b^{sm}(v_i, v_j) \geq s\alpha^{sm} \left[\sum_{k=0}^{sm} \binom{sm}{sk} \psi_b^{s(m-k)}(v_i, v) \beta^k \psi_b^k(v, v_j) \right] \tag{2.4}$$

where

$$\binom{sm}{sk} = \frac{m!}{s(m-k)!k!}.$$

Multiplying (2.4) by $\rho_i \rho_j$ and summing over i and j from 1 to N, we get

$$\sum_{1 \leq i,j \leq N} \rho_i \rho_j \psi_b^{sm}(v_i, v_j) \geq s\alpha^{sm} \sum_{1 \leq i,j \leq N} \rho_i \rho_j \left(\sum_{k=0}^{sm} \binom{sm}{sk} \psi_b^{s(m-k)}(v_i, v) \beta^(sk) \psi_b^{sk}(v, v_j) \right). \tag{2.5}$$

i.e.,

$$\begin{aligned} \sum_{1 \leq i,j \leq N} \rho_i \rho_j \psi_b^{sk}(v_i, v_j) &\geq s\alpha^{sm} \sum_{k=0}^{m} \binom{sm}{sk} \sum_{i=1}^{N} \rho_i \psi_b^{s(m-k)}(v_i, v) \sum_{j=1}^{N} \rho_j \beta^{sk} \psi_b^{sk}(v, v_j) \\ &= s\alpha^{sm} \left[\sum_{k=0}^{m} \binom{sm}{sk} \left(\sum_{i=1}^{N} \rho_i \beta^{sk} \psi_b^{s(m-k)}(v_i, v) \right) \left(\sum_{i=1}^{N} \rho_i \psi_b^{s(m-k)}(v, v_i) \right) \right] \\ &= 2s\alpha^{sm} \sum_{i=1}^{N} \rho_i \psi_b^{sm}(v_i, v) + \alpha^{sm} \left[\sum_{k=1}^{m-1} \binom{sm}{sk} \left(\sum_{i=1}^{N} \rho_i \psi_b^{s(m-k)}(v_i, v) \right) \left(\sum_{i=1}^{N} \rho_i \beta^{sk} \psi_b^{sk}(v_i, v) \right) \right]. \end{aligned} \tag{2.6}$$

It is easy to see that

$$\sum_{1 \leq i,j \leq N} \rho_i \rho_j \psi_b^{sm}(v_i, v_j) = 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j \psi_b^{sm}(v_i, v_j). \tag{2.7}$$

So, (2.6) becomes

$$2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j \psi_b(v_i, v_j)$$

$$\geq 2s\alpha^{sm} \sum_{i=1}^{N} \rho_i \psi_b^{sm}(v_i, v) + \alpha^{sm} \left[\sum_{k=1}^{m-1} \binom{sm}{sk} \left(\sum_{i=1}^{N} \rho_i \psi_b^{s(m-k)}(v_i, v_j) \right) \left(\sum_{i=1}^{N} \rho_i \beta^{sk} \psi_b^{sk}(v, v_i) \right) \right].$$

That is,

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j \psi_b^{sm}(v_i, v_j)$$

$$\geq \frac{\alpha^{sm}}{2} \left[2s \sum_{i=1}^{N} \rho_i \psi_b^{sm}(v, v_i) + \sum_{k=1}^{m-1} \binom{sm}{sk} \left(\sum_{i=1}^{N} \rho_i \psi_b^{sk}(v, v_i) \right) \left(\sum_{i=1}^{N} \rho_i \beta^{sk} \psi_b^{s(m-k)}(v_i, v) \right) \right].$$

Hence the result in (2.1) is proved for all $v \in X$. □

Theorem 2.4 has proven to be an extension of the result in ([2,10]) in a more general setting.

Corollary 2.5. *Let (X, ψ_b) be a b-metric space and $\alpha, m \in \mathbb{N}$, $N \geq 2$, $0 < s < 1$, $v_i \in X$ for all $i \in \{1, ..., N\}$ with $\sum_{i=1}^{N} \rho_i = 1$. Then*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j \psi_b^{sm}(v_i, v_j)$$

$$\geq \alpha^{sm} \sup_{v \in X} \left[s \sum_{i=1}^{N} \rho_i \rho_j \psi_b^{sm}(v, v_i) + \frac{1}{2} \sum_{k=1}^{m-1} \binom{sm}{sk} \left(\sum_{i=1}^{N} \rho_i \psi_b^{sk}(v, v_i) \right) \left(\sum_{i=1}^{N} \rho_i \psi_b^{s(m-k)}(v_i, v) \right) \right]. \tag{2.8}$$

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Corollary 2.6. *Let (X, ψ_b) be a b-metric space and $m \in \mathbb{N}$, $N \geq 2$, $0 < s < 1$, $v_i \in X$ and $i \in \{1, ..., N\}$ with $\sum_{i=1}^{N} \rho_i = 1$. Then*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \psi_b^{sm}(v_i, v_j)$$
$$\geq \frac{1}{2}\left[2s \sum_{i=1}^{N} \rho_i \psi_b^{sm}(v, v_i) + \frac{1}{2} \sum_{k=1}^{m-1} \binom{sm}{sk} \left(\sum_{i=1}^{N} \rho_i \psi_b^{sk}(v, v_i)\right)\left(\sum_{i=1}^{N} \rho_i \psi_b^{s(m-k)}(v_i, v)\right)\right].$$

We can state the following result that give an application in b-normed linear spaces using $\psi_b(v_i, v_j) = \|v_i - v_j\|_b$ to give an application in b-normed linear spaces.

Proposition 2.7. *Given that $(X, \|.\|_b)$ is a b-normed linear space and $v \in X$, $\alpha, m \in \mathbb{N}, N \geq 2$, $\beta \geq 1$ $0 < s < 1$. If $\rho_i \geq 0$, $i, j = \{1, ..., N\}$ with*

$$\sum_{i=1}^{N} \rho_i = 1.$$

Indeed, we have by Theorem 2.3 that

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j \|v_i - v_j\|_b^{sm}$$
$$\geq \frac{\alpha^{sm}}{2}\left[2s \sum_{i=1}^{N} \rho_i \|v_i - v\|_b^{sm} + \sum_{k=1}^{m-1} \binom{sm}{sk} \sum_{i=1}^{N} \rho_i \|v_i - v\|_b^{s(m-k)} \sum_{i=1}^{N} \rho_i \beta^{sk} \|v_i - v\|_b^{sk}\right], \quad (3.1)$$

for all $v \in X$.

Proof. It follows from the proof of Theorem 2.2 if we set $\psi_b(v_i, v_j) = \|v_i - v_j\|_b$. □

3. Conclusion

In conclusion, we have provided a fractional power inequality in b-metric-type spaces based on Definition 2.1, condition (b3) for $\beta \geq 1$. Possible consideration of this paper for the case $\beta < 1$ maybe of interest.

Conflicts of Interest: The authors declare no conflict of interest.

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