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Article

There Are Infinitely Many Mersenne Primes

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Abstract

This paper explores Mersenne primes of the form $2^p - 1$ where, p is a prime. By extension, the paper also explores Perfect numbers. An insight into these numbers is explored using novel methods that involve the trigonometric functions with integer factorable arguments. Rational functions play a part in the behavior of many functions including regular primes, Mersenne Primes, and Perfect numbers. The paper first determines relationships for primes, and then procedes to show how Perfect number relations can be derived from trigonometric relations. The relationships of trigomentric functions involving the sum of divisors, provide a novel approach to prove that that the analytic structure of cot(x), when split into Mersenne and non-Mersenne classes through the Bernoulli framework, forces a coupling between the two infinite subsets of integers and the contradiction (negative ratio despite all positive terms) is a proof of necessity for infinite balance between both classes.

Keywords: Mersenne primes; perfect numbers; abondant numbers; deficient numbers; trigonometric functions; primes; cot; trigonometry; sums of divisors; invariance

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1. Introduction

The search for a general formula to determine the n^{th} Mersenne prime is an ongoing challenge in mathematics. Mersenne primes are of the form $M_p = 2^p - 1$, where p is a prime number, and M_p is also a prime number. Not all primes p, can generate a Mersenne prime M_p. For example, the primes, 11, 23, 29, are examples that do not generate Mersenne Primes, M_p, they generate what I refer to as Mersenne Numbers M_n, that have the Mersenne form $M_n = 2^p - 1$, where p is a non-generating prime, and M_n is not. It is extremely difficult to find the Mersenne primes, M_p, without tedious factorization, since the known set of Mersenne primes M_p are separated by long distances of non-primes, M_n.

Perfect numbers, N_p, are numbers defined by the product $N_p = (2^p - 1)2^{p-1}$, where, p is a prime that generates a Mersenne prime, . They have the Sum of Divisors relation, $\sigma(N_p) = 2N_p$. These numbers are related to Mersenne primes, $M_p = 2^p - 1$, by the relation, $N_p = (2^{p-1} - 1)M_p$. Hence the search for Mersenne primes, M_p, is also the search for Perfect numbers, N_p. It is not known in current art if there are infinitely many Perfect Numbers, N_p and also if there is infinitely many Mersenne primes, M_p. So far, all N_p are even numbers, and it is still not yet determined if there are *any* odd N_p.The approach used in this paper on Mersenne Primes, M_p and Perfect numbers, N_p is so far as I know, has not yet been used by researchers.

The Gamma-function, denoted as $\Gamma(s)$, was first introduced by Swiss mathematician Leonhard Euler [1] 1729. Euler's deep insights into Γ-function led to numerous results that provide key insights into many fields of mathematics including Probability theory and Statistics. Other major contributions to the development of the Γ-function used in this paper were developed by Carl Freidman Gauss [2]. Gauss's work led to the famous reflection formula of the ζ-function. A key insight into the Γ-function is its multiplicative nature. New results will be presented in this paper resulting from the properties of the Γ-function . So far, there has been little development in the additive representation of the Γ-function as a series of simple terms. The form of the Γ-function [3], p.895:

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$$\Gamma(s) \sim z^{s-\frac{1}{2}} e^{-s} \sqrt{2\pi} \left\{1 + \frac{1}{12z} + \frac{1}{288s^2} - \frac{139}{51840s^3} - \frac{571}{2488320s^4} + O(s^{-5})\right\}, [|\arg s| < \pi] \quad (1.)$$

for s real and positive is well known. Here, the remainder of the series (1) is less than the last term that is retained.

Similar series exists for $\ln \Gamma(s)$. It will be significant if other forms of these series can be found.

The product-form of the Γ-function due to Gauss, provides further insights into many relations that will be developed in this paper. The product form is given by, [4], p. 896:

$$\Gamma(y \cdot n) = (2\pi)^{\frac{1-y}{2}} y^{(n \cdot y) - \frac{1}{2}} \prod_{k=0}^{y-1} \Gamma\left(n + \frac{k}{y}\right) \quad (2.)$$

Certain invariant relations of the product Γ-function will be developed in this paper to show the connections of the Γ-function to other functions, particularly the Riemann-Zeta function, denoted by $\zeta(s)$. The ζ-function, is defined by the additive series:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s}, \mathbb{R}(s) > 1 \quad (3.)$$

The importance of the ζ-function is its relation to the distribution of primes and the Riemann hypothesis. There is a one-on-one correspondence between the non-trivial roots of the function and the primes. The ζ-function also has a product relation for primes p, given by [4], p. 1037;

$$\zeta(s) = \prod_p \left(\frac{1}{1 - p^{-s}}\right), \quad \mathbb{R}(s) > 1 \quad (4.)$$

Both the ζ-function, and the Γ-function are factorable. These two functions are related by the ζ-function reflection formula developed by Gauss given by [4], p.1038:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{\frac{s-1}{2}} \zeta(1-s) \quad (5.)$$

These relations are well studied, and they provide a wealth of information in Number theory and many disciplines in Mathematics. In this article, I show new relations that govern Mersenne primes and twin primes. All these special integer relations are connected in precious way by powers of 2π.

2. Mersenne Numbers

Mersenne primes were named after the French philosopher and number theorist, Marin Mersenne (1588-1648). Marin Mersenne was also a monk and a theologian, and he had an important influence on many academics such as Fermat, Pascal, Huygens, Descartes and Galileo. He also inspired the invention of the pendulum clock.

Only a few Mersenne primes, M_p are known to exists. It is an ardous task to determine whether a Mersenne number, M_n is either a Mersenne prime, M_p prime or a Mersenne number M_n, since the computation of factors of large Mersenne numbers, M_n is very difficult. When p is a prime, not all $M_n = 2^p - 1$ are Mersenne primes, and it is not known whether there are infinitely many Mersenne primes, M_p. The Great Internet Mersenne Prime Search (GIMPS) has discovered a new Mersenne prime number, M_p = 282,589,933 - 1. The first few Mersenne primes are $M_p \in 3, 7, 31, 127, 8191, 131071, 524287, 2147483647, \ldots$ (Online Encyclopedia of Integer Sequences, (OEIS) #A000668), corresponding to indices $n \in$ 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161… (OEIS A000043).

It is conjectured that there exist an infinite number of Mersenne primes. In Wolfram, we find the best fit line through the origin to the asymptotic number of Mersenne primes M_p with $p \leq \ln x$, for the first 51 known Mersenne primes. The best-fit line gives $C(x) = 2.51763 \ln x$. This fit is illustrated

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below in Figures 1 and 2. It has been conjectured without any particularly strong evidence, that the constant is given by $e^{\lambda}\sqrt{2} = 2.518..$, where λ is the Euler-Mascheroni constant.

In this paper, I will give strong relations for this constant.

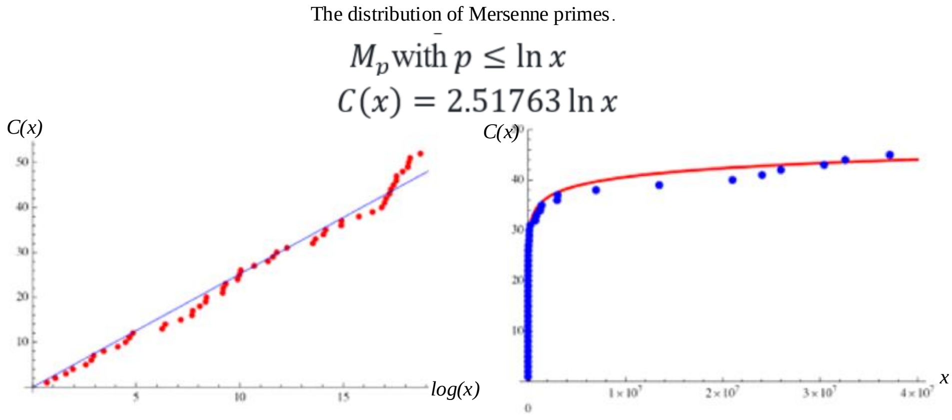

FIGURE 1

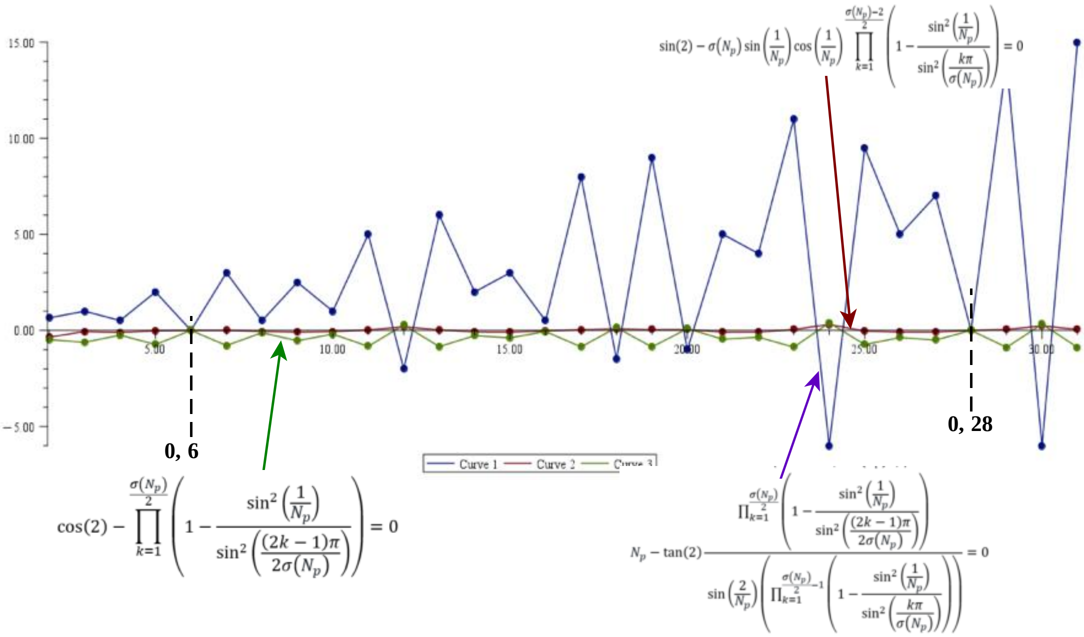

FIGURE 2

Literature on Mersenne primes is mainly dedicated to the search for new Mersenne primes, and very few attempts have made progress on the actual theoretical work. In [8], Zhaodong Cai, Matthew Faust, A.J. Hildebrand, Junxian Li, and Yuan Zhang studied theleading digits of the Mersenne primes. They attempted to show that leading digits of Mersenne numbers behave in many respects more regularly than some sequences of powers of logs of 2. Further information on Mersenne primes can be found in [8–11]. In [12] J. Aust yield bounds on the sums of exponents of Mersenne primes.

Most of this research is related to the present work only in an attempt to categorize properties that Mersenne primes may have found to have, however, the present paper does not rely on any of the current work known on Mersenne primes, but starts a new trend in expoloring the properties of Mersenne primes. To begin, let us explore the concepts that lead to the final proof.

3. The Invariance of the GAMMA Function to Substitution $\sigma(m) \to \sigma(m+j)$

I first want to introduce the curious fact that any function with a relational product $\{n \cdot y\}$, can be represented by the Sums of Divisor function, $\sigma(m)$. Here is a simple example:

$$\log(n \cdot y) = \log n + \log y, \tag{6.}$$

Then, if $n \cdot y = m$, we can put $n = \sigma(m), y = \frac{m}{\sigma(m)}$, and so,

$$\log(m) = \log \sigma(m) + \log \frac{m}{\sigma(m)} \tag{7.}$$

Then, if $n \cdot y = N_p$, we can put $n = \sigma(N_p), y = \frac{N_p}{\sigma(N_p)}$, then, a Perfect number N_p, has the relation:

$$\log(N_p) = \log\left(\sigma(N_p)\right) + \log\left(\frac{N_p}{\sigma(N_p)}\right) \tag{8.}$$

$$\log(N_p) = \log\left(\sigma(N_p)\right) + \log\left(\frac{1}{2}\right) \tag{9.}$$

Here is another example:
If $n \cdot y = m$, we can put $n = \sigma(m), y = \frac{m}{\sigma(m)}$, and so, applied to the formula [3], p.41:

$$\begin{aligned} \sin(n \cdot x) &= n \sin(x)\cos(x) \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{k\pi}{n}\right)}\right) \\ \cos(n \cdot x) &= \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{(2k-1)\pi}{2n}\right)}\right) \end{aligned}, \quad [n \text{ is even}] \tag{10.}$$

$$\begin{aligned} \sin(n \cdot x) &= n \sin(x) \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{k\pi}{n}\right)}\right) \\ \cos(n \cdot x) &= \cos(x) \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{(2k-1)\pi}{2n}\right)}\right) \end{aligned}, \quad [n \text{ is odd}] \tag{11.}$$

Interestingly, $(10) \in even$, and $(11) \in odd$, differentiate between odd and even values of n. Since primes have $\sigma(p) = p + 1$, an even number, and $p + 1$ is always even except for the prime 2, the relations $(11) \in odd$ *anddoes not apply to primes*! Since $\sigma(2) = 3$. For example,

$$\cos(2) = \cos\left(\frac{2}{3}\right) \prod_{k=1}^{1} \left(1 - \frac{\sin^2\left(\frac{2}{3}\right)}{\sin^2 \frac{(2k-1)\pi}{6}}\right), \quad [\sigma(2) \text{ is odd}] \tag{12.}$$

$$-0.4161468365\ldots = 0.7858872608..\left(1 - \frac{0.3823812134\ldots}{0.2500000000}\right) = -0.4161468365\ldots \tag{13.}$$

By using the sum of divisor function, for Perfect numbers, N_p, the even trigonometric relations $[(10),(11)] \in even$, apply, but the relations, $[(12),(13)] \in odd$ do not apply, so we can put, $\sigma(N_p) = 2N_p$. The fact that the sum of divisor function $\sigma(m)$, can be manipulated this way leads to some interesting formulas that can produce significant and unexpected results.

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4. Application of the Trigonometric Function to Perfect Numbers

A Perfect Number N_p, is defined as a number for which $\sigma(N_p) = 2N_p$. A list of some known Perfect numbers is

$N_p \in \{6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2658455991569831744654692615953842176, \ldots\}$

Hence for, example, in (10), putting $n = \sigma(j), (n\ even),\ x = \frac{1}{j}$: then, we have

$$\sin\left(\frac{\sigma(j)}{j}\right) = \sigma(j)\sin\left(\frac{1}{j}\right)\cos\left(\frac{1}{j}\right)\prod_{k=1}^{\frac{\sigma(j)-2}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{k\pi}{\sigma(j)}\right)}\right),\quad \cos\left(\frac{\sigma(j)}{j}\right) = \prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right) \quad [\sigma(j)\text{ is even}] \quad (14.)$$

$$\tan\left(\frac{\sigma(j)}{j}\right) = \frac{\sigma(j)\sin\left(\frac{1}{j}\right)\cos\left(\frac{1}{j}\right)\prod_{k=1}^{\frac{\sigma(j)-2}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{k\pi}{\sigma(j)}\right)}\right)}{\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)} \quad [\sigma(j)\text{ is even}] \quad (15.)$$

LEMMA 1: The rational trigonometric functions $\sin\left(\frac{\sigma(j)}{j}\right), \cos\left(\frac{\sigma(j)}{j}\right)$ determine $Perfect\ Numbers$.

Proof:

$$\sigma(j) = \left[\frac{\tan\left(\frac{\sigma(j)}{j}\right)\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{\sin\left(\frac{1}{j}\right)\cos\left(\frac{1}{j}\right)\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}\right] \quad (16.)$$

$$\sigma(j) = 2\left[\frac{\tan\left(\frac{\sigma(j)}{j}\right)\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{2\sin\left(\frac{1}{j}\right)\cos\left(\frac{1}{j}\right)\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}\right] \quad (17.)$$

$$\sigma(j) = 2\left[\frac{\tan\left(\frac{\sigma(j)}{j}\right)\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{\sin\left(\frac{2}{j}\right)\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}\right] \quad (18.)$$

If $j = N_p$ is a Perfect number, then, the equality applies only when.

$$N_p = \frac{\tan\left(\frac{\sigma(N_p)}{N_p}\right)\prod_{k=1}^{\frac{\sigma(N_p)}{2}}\left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1}\left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)\right)} \tag{19.}$$

Taking the limits:

$$\lim_{N_p\to\infty} N_p = \lim_{N_p\to\infty}\left(\tan(2)\left\{\frac{\prod_{k=1}^{\frac{\sigma(N_p)}{2}}\left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1}\left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)\right)}\right\}\right) \tag{20.}$$

Now, for large values of y, $\sin\left(\frac{1}{y}\right) \to \frac{1}{y}$, and so we can approximate the product for large values of N_p as follows:

$$\lim_{N_p\to\infty} N_p = \lim_{N_p\to\infty}\left(\frac{\tan(2)}{\sin\left(\frac{2}{N_p}\right)}\left(1 - \left(\frac{2\sigma(N_p)}{N_p(\sigma(N_p)-1)\pi}\right)^2\right)\left\{\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1}\frac{\left(1-\left(\frac{2\sigma(N_p)}{N_p(2k-1)\pi}\right)^2\right)}{\left(1-\left(\frac{\sigma(N_p)}{N_p k\pi}\right)^2\right)}\right\}\right) \tag{21.}$$

$$\lim_{N_p\to\infty} N_p = \lim_{N_p\to\infty}\left(N_p\frac{\tan(2)}{2}\left\{\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1}\frac{\left(1-\left(\frac{2\sigma(N_p)}{N_p(2k-1)\pi}\right)^2\right)}{\left(1-\left(\frac{\sigma(N_p)}{N_p k\pi}\right)^2\right)}\right\}\right) \tag{22.}$$

Put $\frac{\sigma(N_p)}{N_p} = x = 2$,

$$1 = \frac{\tan(2)}{2}\left\{\prod_{k=1}^{\infty}\frac{\left(1-\frac{4(x)^2}{(2k-1)^2\pi^2}\right)}{\left(1-\frac{(x)^2}{k^2\pi^2}\right)}\right\} \tag{23.}$$

For the infinite product we have,

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty}\left(1-\left(\frac{x}{k\pi}\right)^2\right), \quad \cos(x) = \prod_{k=0}^{\infty}\left(1-\frac{4(x)^2}{(2k-1)^2\pi^2}\right) \tag{24.}$$

$$1 = \frac{\tan(2)}{2}\left\{\frac{2\ \cos(2)}{\sin(2)}\right\} = 1 \tag{25.}$$

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$$\sin\left(\frac{\sigma(N_p)}{N_p}\right) = \sigma(N_p)\sin\left(\frac{1}{N_p}\right)\cos\left(\frac{1}{N_p}\right)\prod_{k=1}^{\frac{\sigma(N_p)-2}{2}}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right), \quad [\sigma(N_p)\text{ is even}] \quad (26.)$$

$$\cos\left(\frac{\sigma(N_p)}{N_p}\right) = \prod_{k=1}^{\frac{\sigma(N_p)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)$$

It is clear that there if there exists a continued set of infinitely large Perfect Numbers then,

$$\left.\begin{array}{r}\sin(2)-\sigma(N_p)\sin\left(\frac{1}{N_p}\right)\cos\left(\frac{1}{N_p}\right)\prod_{k=1}^{\frac{\sigma(N_p)-2}{2}}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)\\ \cos(2)-\prod_{k=1}^{\frac{\sigma(N_p)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)\\ N_p-\tan(2)\frac{\prod_{k=1}^{\frac{\sigma(N_p)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)\right)}\end{array}\right\} = 0, \begin{array}{c}[\sigma(N_p)\text{ is even}], (\vDash)\ for\ n\in N_p\\ otherwise\ for\ n\notin N_p\end{array} \quad (27.)$$

Each of these three relations is only true when N_p is a Perfect number.
Figure 2 shows the correletion of the relation (27) with Perfect Numbers.
From symmetry, and considering the form for the divisor function:

$$N_p = \frac{\tan(2)\prod_{k=1}^{N_p}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)} \quad (28.)$$

Since $N_p = (2^p-1)2^{p-1}$, where p is a prime, we can factor the perfect number N_p, as follows:

$N_p = (2^p-1)2^{p-1} = (2P-1)P$, where $P = 2^{p-1}$. This factorization leads to the following results:

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$$F(P) = P - \frac{\tan\left(\frac{\sigma(P)}{P}\right)\prod_{k=1}^{\frac{\sigma(P)}{2}}\left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(P)}\right)}\right)}{\sin\left(\frac{2}{P}\right)\left(\prod_{k=1}^{\frac{\sigma(P)}{2}-1}\left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{k\pi}{\sigma(P)}\right)}\right)\right)}$$

$$G(P) = 2P - 1 - \frac{\tan\left(\frac{\sigma(2P-1)}{2P-1}\right)\prod_{k=1}^{\frac{\sigma(2P-1)}{2}}\left(1 - \frac{\sin^2\left(\frac{1}{2P-1}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2P-1)}\right)}\right)}{\sin\left(\frac{2}{2P-1}\right)\left(\prod_{k=1}^{\frac{\sigma(2P-1)}{2}-1}\left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2P-1)}\right)}\right)\right)} \quad (29.)$$

It is clear that the there is a direct correspondence between the Perfect Number N_p, and P. The graphs of the two functions is shown in Figure 3.

$$F(P) = P - \frac{\tan\left(\frac{\sigma(P)}{P}\right)\prod_{k=1}^{\frac{\sigma(P)}{2}}\left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(P)}\right)}\right)}{\sin\left(\frac{2}{P}\right)\left(\prod_{k=1}^{\frac{\sigma(P)}{2}-1}\left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{k\pi}{\sigma(P)}\right)}\right)\right)} \quad (30.)$$

Graphs of the two functions, showing that the zeroes are strictly on the correspondence *Perfect Numbers* ✎ *P*.

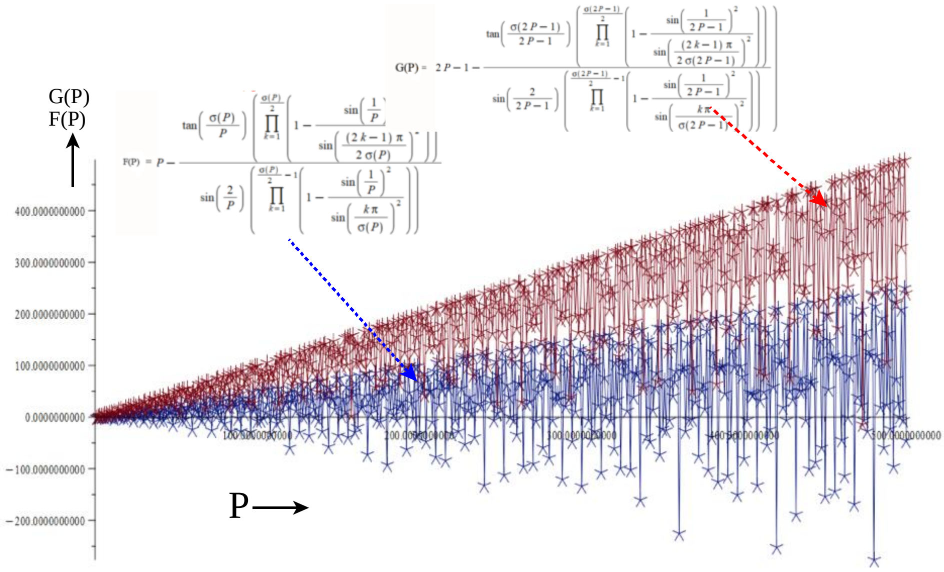

FIGURE 3

Figure 4 shows the correspondence $F(P) \rightarrow N_p$.

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Graphs of the two functions, showing that the zeroes of F(P) are strictly on the correspondence *Perfect Numbers* ✐ *P*.

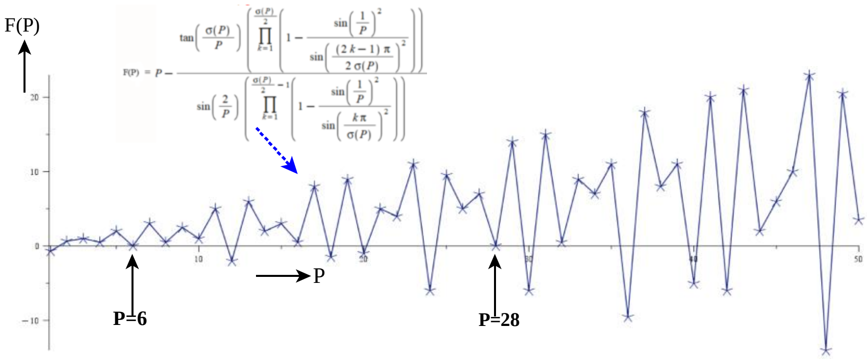

FIGURE 4

FIGURE 5 shows the symmetry of the odd and even product expressions.

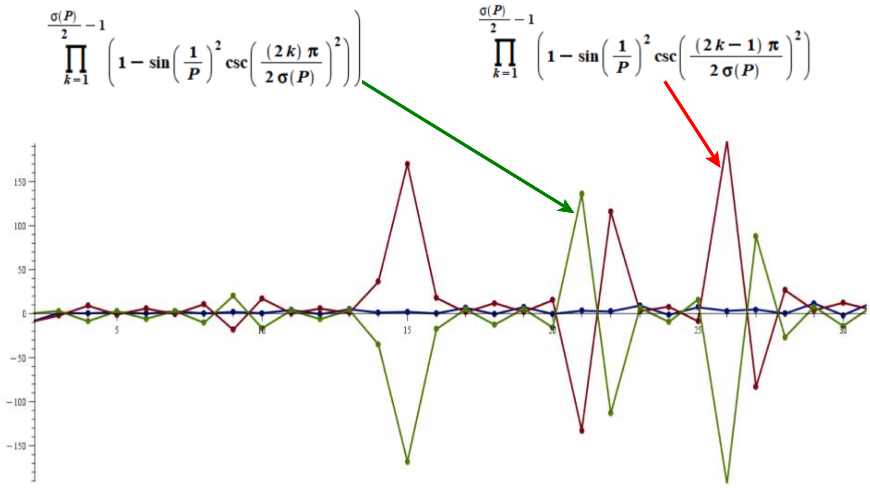

FIGURE 5

The relations (19) hold for all Perfect Numbers. The right hand side of (19) does not depend on implicit rational relationships between $\sigma(N_p)$ and N_p. It is clear that the basic rational trigonometric functions capture the properties of integers. We now explore the general forms of trinometric and exponential forms that capture Perfect numbers, Abondant numbers and deficient numbers in one relation.

5. The General Relation That Captures the Behavior of Abondant Numbers, Perfect Numbers and Deficient Numbers

Definition 1: *An Abundant number is a positive integer for which the sum of its proper divisors excluding itself is greater than the number itself.*

Definition 2: *A Perfect number is a number for which the sums of all divisors is equal to twice the number.*

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Definition 3: *A Deficient number is a number for which the sums of all divisors is less than twice the number.*

LEMMA: If n is a Perfect number, then,

$$\frac{\cos\left(\frac{2\pi n}{\sigma(n)}\right)}{\sin\left(\frac{\pi n}{\sigma(n)}\right)} = -1 \tag{31.}$$

Proof: for a Perfect number, $\sigma(n) = 2n$. Hence,

$$\frac{\cos(\pi)}{\sin\left(\frac{\pi}{2}\right)} = -1 \tag{32.}$$

The distribution of **perfect numbers, abondant numbers** and **deficient numbers** is captured by the general relation:

$$\cos\left(\frac{2n\pi}{\sigma(n)}\right) + \sin\left(\frac{n\pi}{\sigma(n)}\right) = 0 \tag{33.}$$

a. For perfect numbers, $\frac{2n}{\sigma(n)} = 1$, and the relation (33) vanishes.

b. For **abondant numbers**, $\frac{2n}{\sigma(n)} < 1$, and the relation does not vanish but generates negetaive imaginary values for $n \in$ **abondant numbers.**

c. For **deficient numbers**, $\frac{2n}{\sigma(n)} < 1$, and the relation does not vanish but generates positive imaginary values for $n \in$ **deficient numbers.**

To see this, put the relation (33) in the form:

$$\frac{\cos\left(\frac{2n\pi}{\sigma(n)}\right)}{\sin\left(\frac{n\pi}{\sigma(n)}\right)} = -1 \qquad \sin\left(\frac{n\pi}{\sigma(n)}\right) \neq 0, \tag{34.}$$

Obviously, the zeros of the function (34) occur at the Perfect numbers. However, for clarity we convert this relation to the exponential form:

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} \tag{35.}$$

Figure 6 shows the complex map of the function $F(n)$, over the range $n = 0..20{,}000$.

The map of the first 20,000 integers showing the origin as the point for which *n* is a Perfect Number.
Using the function:

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}$$

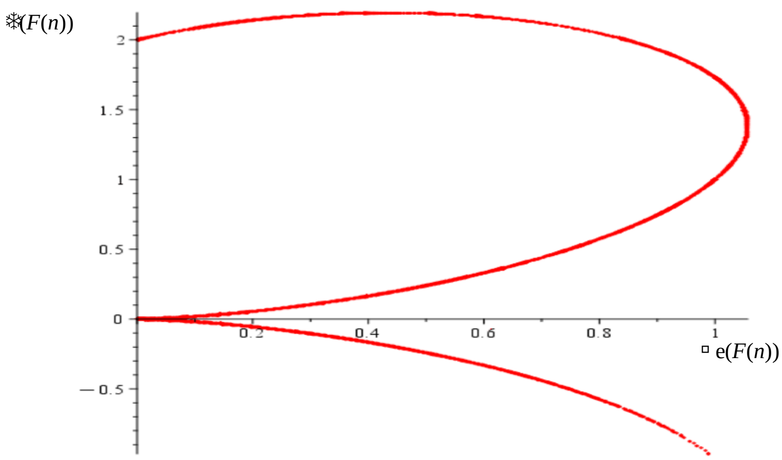

FIGURE 6

The zeros of the function $F(n)$, occur at the values 6, 28, 496, 8124….
NOTE*: The **Mersenne primes** and the perfect numbers can only exist on the upper right quadrant corrsponding to **deficient numbers**. **Perfect numbers** are the **zeros** of the function $F(n)$.

The general locations of primes and Mersenne primes are shown in Figure 7. As can be seen, the oprimes do not generate negative imaginary values, and are located on the top-right quadrant of the complex plane.

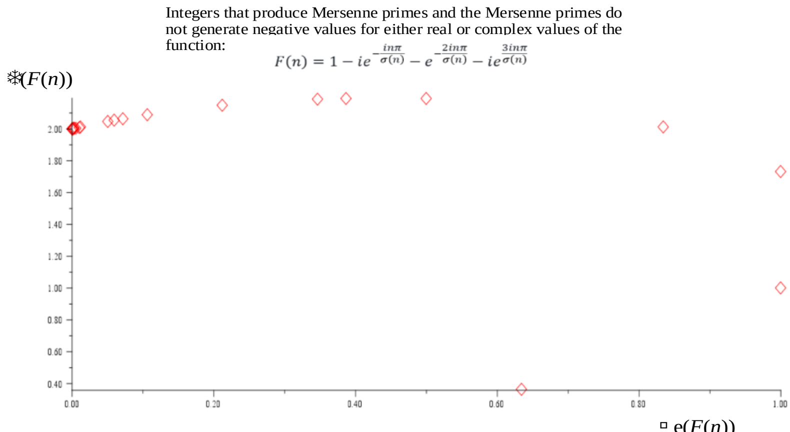

FIGURE 7

Hence, $\sigma(n) > 2n$. It is clear that the sequence of **abundant numbers**,
[12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, 104, 108, 112, 114, 120, 126, 132, 138, 140, 144, 150, 156, 160, 162, 168, 174, 176, 180, 186, 192, 196, 198, 200, 204, 208, 210, 216, 220, 222, 224, 228, 234, 240, 246, 252, 258, 260, 264, 270, 272, 276, 280, 282, 288, 294, 300, 304, 306, 308, 312, 318, 320, 324, 330, 336, 340, 342, 348, 350, 352, 354, 360, 364, 366, 368, 372, 378, 380, 384, 390, 392, 396, 400, 402, 408, 414, 416, 420, 426, 432, 438, 440, 444, 448, 450, 456, 460, 462, 464, 468, 474, 476, 480, 486, 490, 492, 498, 500],

produce values of $F(n)$ in (35) that lie on the lower right quadrant of the complex plane. This distinct observation for the first 500, **abondant numbers** provides a clue as to their distribution.

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The map of the first set of Abondant Numbers to the right lower quadrant of the function:

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}$$

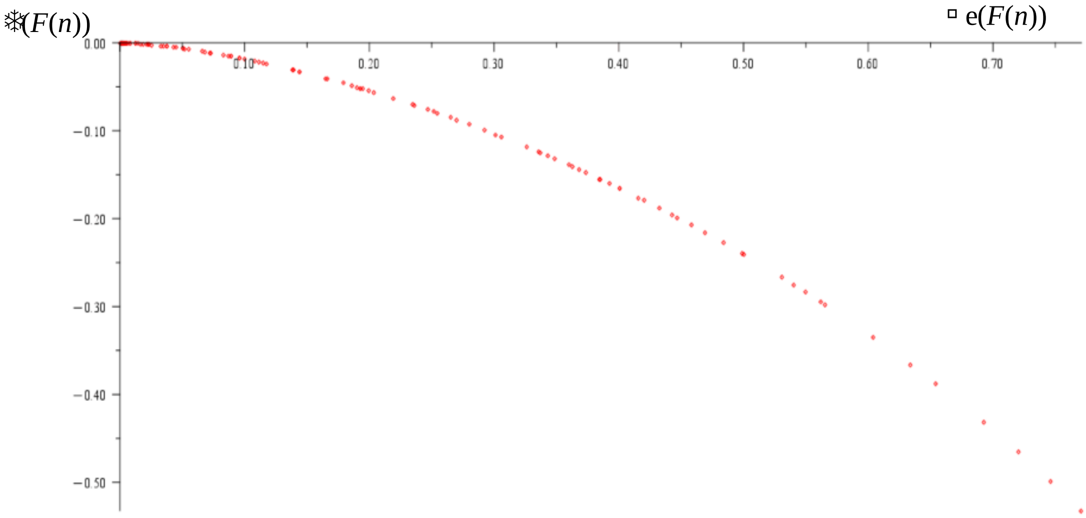

FIGURE 8

It is clear the first numbers between 0 and 500 that generate a sequence of **deficient numbers**: [2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, 29, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59, 61, 62, 63, 64, 65, 67, 68, 69, 71, 73, 74, 75, 76, 77, 79, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 97, 98, 99, 101, 103, 105, 106, 107, 109, 110, 111, 113, 115, 116, 117, 118, 119, 121, 122, 123, 124, 125, 127, 128, 129, 130, 131, 133, 134, 135, 136, 137, 139, 141, 142, 143, 145, 146, 147, 148, 149, 151, 152, 153, 154, 155, 157, 158, 159, 161, 163, 164, 165, 166, 167, 169, 170, 171, 172, 173, 175, 177, 178, 179, 181, 182, 183, 184, 185, 187, 188, 189, 190, 191, 193, 194, 195, 197, 199, 201, 202, 203, 205, 206, 207, 209, 211, 212, 213, 214, 215, 217, 218, 219, 221, 223, 225, 226, 227, 229, 230, 231, 232, 233, 235, 236, 237, 238, 239, 241, 242, 243, 244, 245, 247, 248, 249, 250, 251, 253, 254, 255, 256, 257, 259, 261, 262, 263, 265, 266, 267, 268, 269, 271, 273, 274, 275, 277, 278, 279, 281, 283, 284, 285, 286, 287, 289, 290, 291, 292, 293, 295, 296, 297, 298, 299, 301, 302, 303, 305, 307, 309, 310, 311, 313, 314, 315, 316, 317, 319, 321, 322, 323, 325, 326, 327, 328, 329, 331, 332, 333, 334, 335, 337, 338, 339, 341, 343, 344, 345, 346, 347, 349, 351, 353, 355, 356, 357, 358, 359, 361, 362, 363, 365, 367, 369, 370, 371, 373, 374, 375, 376, 377, 379, 381, 382, 383, 385, 386, 387, 388, 389, 391, 393, 394, 395, 397, 398, 399, 401, 403, 404, 405, 406, 407, 409, 410, 411, 412, 413, 415, 417, 418, 419, 421, 422, 423, 424, 425, 427, 428, 429, 430, 431, 433, 434, 435, 436, 437, 439, 441, 442, 443, 445, 446, 447, 449, 451, 452, 453, 454, 455, 457, 458, 459, 461, 463, 465, 466, 467, 469, 470, 471, 472, 473, 475, 477, 478, 479, 481, 482, 483, 484, 485, 487, 488, 489, 491, 493, 494, 495, 497, 499],

produce values of $F(n)$ that lie on the **upper right quadrant** of the complex plane. This distinct observation for the first 500, **defficient numbers** and **abondant numbers** provides a clue as to their distributions.

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The map of the first set of **Dificient Numbers** , from 0-500, to the right upper quadrant of the function:

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}$$

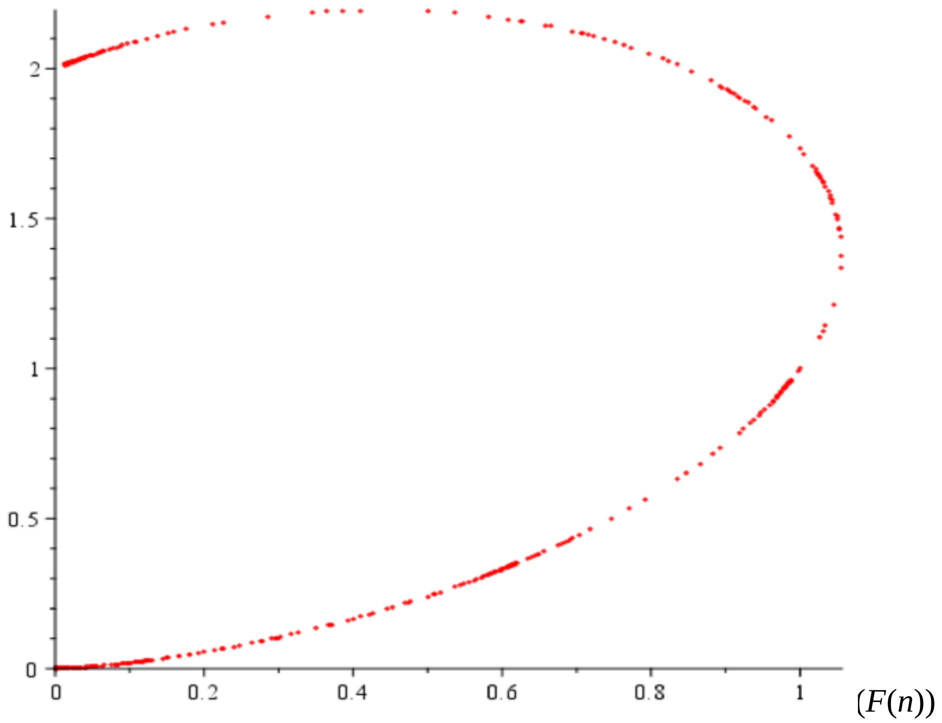

FIGURE 9

Between the **abondant numbers** and the deficient numbers, are the P**erfect Numbers**, [6, 7, 28, 496, 8128, 33550336,….], that generate the zeros of the function:

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} = 0. \tag{36.}$$

Hence, the imaginary part of the function $F(n)$ determines if a number is an **abondant number,** a **perfect number or** a **deficient number.**

$$\Im\left(1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}\right) \begin{matrix} < 0, & n \in abondant\ numbers \\ = 0, & n \in perfect\ numbers \\ > 0, & n \in deficient\ numbers \end{matrix} \tag{37.}$$

The first set of even numbers from 0..500 that lie on the **defient number** curve but are not **abondant numbers** are:
[2, 4, 6, 8, 10, 14, 16, 22, 26, 28, 32, 34, 38, 44, 46, 50, 52, 58, 62, 64, 68, 72, 74, 76, 82, 86, 92, 94, 98, 106, 110, 116, 118, 122, 124, 128, 130, 134, 136, 142, 146, 148, 152, 154, 158, 164, 166, 170, 172, 178, 182, 184, 188, 190, 194, 202, 206, 212, 214, 218, 226, 230, 232, 236, 238, 242, 244, 248, 250, 254, 256, 262, 266, 268, 274, 278, 284, 286, 290, 292, 296, 298, 302, 304, 310, 314, 316, 322, 326, 328, 332, 334, 338, 344, 346, 356, 358, 362, 370, 374, 376, 382, 386, 388, 394, 398, 404, 406, 410, 412, 418, 422, 424, 428, 430, 434, 436, 442, 446, 452, 454, 458, 466, 470, 472, 478, 482, 484, 488, 494, 496].
These numbers are clearly defined by (37).

Figure 10 shows the 2D plot of the function covering both odd and even numbers in the range $n = 0..500$.

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The map of the first set of **odd numbers** (blue) and even numbers (red) . The even numbers that generate points that fall on both the **deficient numbers** and the **abondant numbers**. Hence the graphs are dense with no gaps.

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}$$

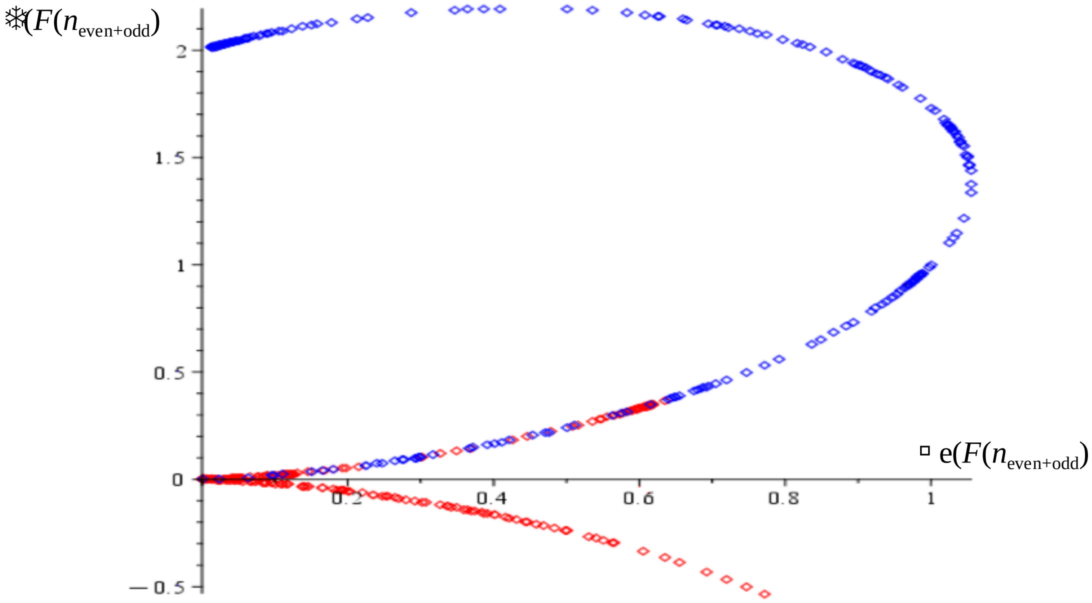

FIGURE 10

It is clear that the even numbers (red points) can fall on both the **deficient number** curve and the **abondant number** curve. The deficient numbers seem to be bounded by the line $1.0562990583978304996\!3 + 1.37659573355141432857i$ and a maximum imaginary value of $0.43293432010231995809 + 2.19494797760015472936i$.

Definition 4: *An Deficient disturbing number , (DDN), is a deficient number which:*

$$\Im\left(1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}\right) > 0 \quad \in \boldsymbol{DDN} \qquad (38.)$$

These are the red points on Figure 10 that intermingle with the blue odd number points.

$DDN \in$

[2,4,8,10,14,16,22,26,32,34,38,44,46,50,52,58,62,64,68,74,76,82,86,92,94,98,106,110,116,118,122,124,128,130,134,136,142,146,148,152,154,158,164,166,170,172,178,182,184,188,190,194,202,206,212,214,218,226,230,232,236,238,242,244,248,250,254,256,262,266,268,274,278,284,286,290,292,296,298,302,310,314,316,322,326,328,332,334,338,344,346,356,358,362,370,374,376,382,386,388,394,398,404,406,410,412,418,422,424,428,430,434,436,442,446,452,454,458,466,470,472,478,482,484,488,494…..].

The map of the first set of **even numbers** (red) . The even numbers generate points that fall on both the **deficient numbers** and the **abondant numbers**.

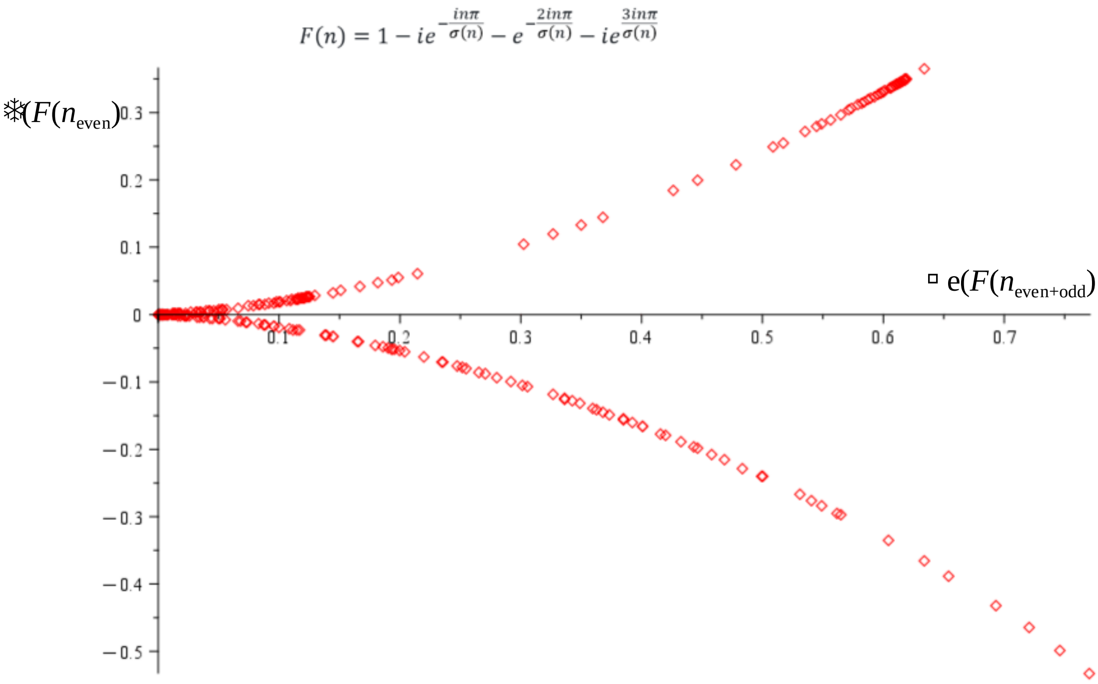

FIGURE 11

The extent to which the even numbers infiltrate the deficient number space for up to $n =$ **150000** seems to be confined to the approximate range,

$$0 \leq \left(1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}\right) \leq 0.98575151303581662431 + 0.36599952081502975396i \text{ , } \boldsymbol{DDN} \leq \mathbf{150000} \qquad (39.)$$

The extent to which the even numbers penetrate the abondant number space is unknown. However it is known that there exists in infinite number of abundant numbers. It has been shown that every multiple $6(n \geq 6)$ is either an abondant number, or taking more multiples of 6 of such numbers leads to an bondant number. Since there is an infinite number of multiples of 6, then there are an infinite number of abondant numbers. Erdos &Graham, 1980, [], showed that even numbers greater than 46 are either abundant numbers or the sum of two abondant numbers.

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The map of the first 8000 odd **numbers** (blue) and the first 8000 odd numbers. The prime density for F(n) increases as F(n) approaches 0+2*i*. The primes also seem to follow the curve in order.

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}$$

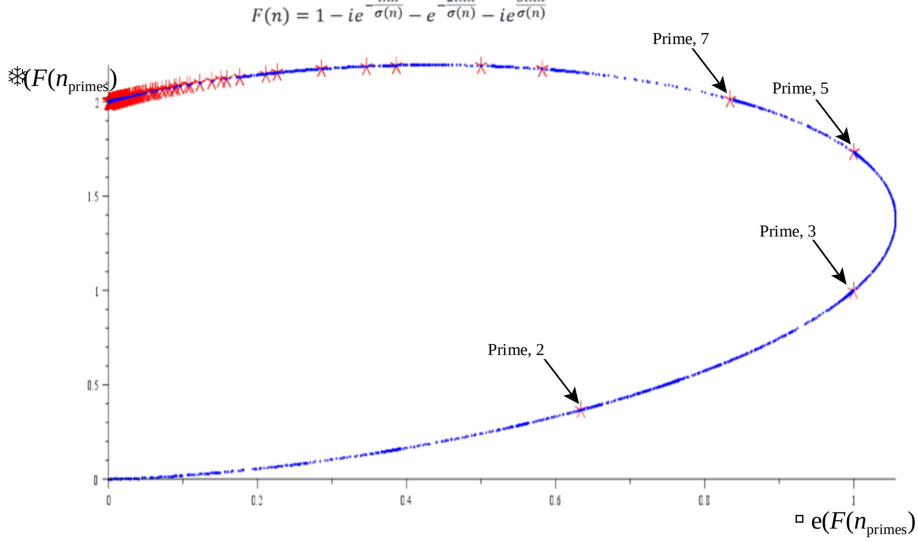

FIGURE 12

Figure 13 shows the distribution of the Mersenne primes with the regular primes.

The map of the first 8 Mersenne primes (diamonds) and the fregular primes (dots) .Mersenne primes higher than 127 are concentrated close to 0+2*i*.

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}$$

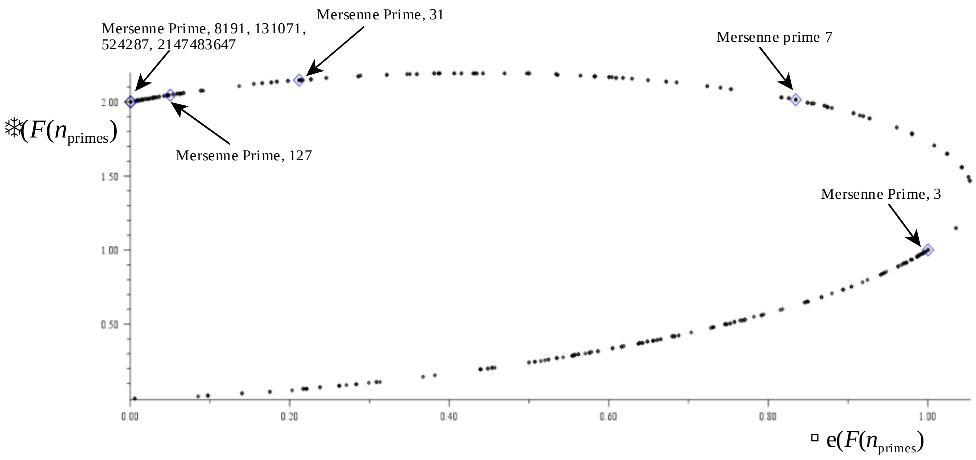

FIGURE 13

6. The Extension of TH Function F(n) to A General Series Form

The function

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} \qquad (40.)$$

behaves like a CyclotomicPolynomial. CyclotomicPolynomialare the minimal polynomials of primitive roots of unity with rational coefficients. The first few CyclotomicPolynomial are shown below:

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$$\Phi_1(x) = x - 1 \quad (5)$$
$$\Phi_2(x) = x + 1 \quad (6)$$
$$\Phi_3(x) = x^2 + x + 1 \quad (7)$$
$$\Phi_4(x) = x^2 + 1 \quad (8)$$
$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1 \quad (9)$$
$$\Phi_6(x) = x^2 - x + 1 \quad (10)$$
$$\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \quad (11)$$
$$\Phi_8(x) = x^4 + 1 \quad (12)$$
$$\Phi_9(x) = x^6 + x^3 + 1 \quad (13)$$
$$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1. \quad (14)$$

An example of a Cyclotomic polynomials are the distribution of the roots of unity on the circle, for *x=50*

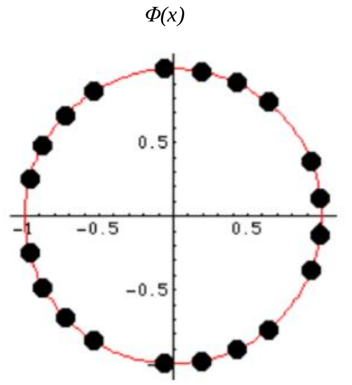

FIGURE 14

A cyclotomic polynomial is of the product form:

$$\Phi_m(x) = \prod_{k=1}^{m}(x - \zeta_m) \quad (41.)$$

where, ζ_m, are the roots of unity in the complex plane, $\mathbb{C}$. In general, the circle, $\zeta_m = e^{\pi i(\omega(x))}$ $where\ \omega(x) = \frac{k}{m}$, and k is taken over integers relative prime to m. It is clear that the function

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} \quad (42.)$$

is composed of functions of cyclotomic polynomials for the the special case of an expansion of some function over the function $\frac{n}{\sigma(n)}$.

Looking at exponential terms with the sequence, $0, -i, 2i, 3i$, we determine the first difference in the powers to be

$$\delta_1 \rightarrow -i, 3i, i, \quad (43.)$$

The second difference gives,

$$\delta_2 \rightarrow 4i, -2i, \tag{44.}$$

The second difference points to the function $F(n)$, following a sequence of powers that is purely linear, but quadratic or alternating in some manner. We assume a quadratic relation, of the form, $Ak^2 + Bk + C$. However, the second differences are not the same constants, and so a recurrence relation of the form, $b_k = f(b_{k-1}, b_{k-2})$ must be used to expand $F(n,m)$ as a series of higher powers for m recurrenses. The sequence of powers in $F(n,m)$, follows the recurrence, with initial conditions,

$$b_k = -b_{k-1} - 7b_{k-2}, \quad b_0 = 0, b_2 = -i \tag{45.}$$

The characteristic equation for the recurrence then yeilds,

$$r^2 + 2r + 7 = 0 \tag{46.}$$

This yields, the two solutions,

$$\begin{aligned} r_1 &= -1 - i\sqrt{6} \\ r_2 &= -1 + i\sqrt{6} \end{aligned} \tag{47.}$$

Since the recurrence (46) follows a second order linear form, the general solution of the recurrence is

$$b_k = C_1 {r_1}^k + C_2 {r_2}^k \ , \qquad C_1, C_2 \ are\ constants. \tag{48.}$$

Solcing for C_1, and C_2, we get:

$$C_1 = -\frac{\sqrt{6}}{84} + \frac{i}{14}, \quad C_2 = \frac{\sqrt{6}}{84} + \frac{i}{14} \tag{49.}$$

Hence we get

$$b_k = \left(-\frac{\sqrt{6}}{84} + \frac{i}{14}\right)\left(-1 - \sqrt{6}\right)^k + \left(\frac{\sqrt{6}}{84} + \frac{i}{14}\right)\left(-1 + \sqrt{6}\right)^k \tag{50.}$$

Hence we have the general form for m terms:

$$F(n,m) = \sum_{k=1}^{m} \left(\frac{\left(-\frac{\sqrt{6}}{84} + \frac{i}{14}\right)\left(-1 - \sqrt{6}\right)^k + \left(\frac{\sqrt{6}}{84} + \frac{i}{14}\right)\left(-1 + \sqrt{6}\right)^k}{k-1} \right)^{k-1} e^{\left(\frac{\pi i n}{\sigma(n)}\right)\left(-\frac{\sqrt{6}}{84} + \frac{i}{14}\right)\left(-1-\sqrt{6}\right)^k + \left(\frac{\sqrt{6}}{84} + \frac{i}{14}\right)\left(-1+\sqrt{6}\right)^k} \tag{51.}$$

This sum produces the first four terms giving the same function:

$$F(n,4) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} \tag{52.}$$

The function,

$$F(n,m) = \sum_{k=1}^{m} \left(\frac{b_k}{k-1}\right)^{k-1} e^{\left(\frac{\pi i n b_k}{\sigma(n)}\right)} \tag{53.}$$

will only have coefficients that are ± 1, or $-i$, for the first 4 terms, $m = 4$. The remaining terms $m > 4$ have large coefficients that blow up quickly. For example for m=7,

$$F(n,7) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} + 625e^{-\frac{20in\pi}{\sigma(n)}} + \frac{2476099}{3125} e^{\frac{19in\pi}{\sigma(n)}} - 24137569e^{\frac{102in\pi}{\sigma(n)}} \tag{54.}$$

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In general, for Perfect numbers,

$$F(n,4) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} = 0,$$

$$F(n=5..\infty) = 625e^{-\frac{20in\pi}{\sigma(n)}} + \frac{2476099}{3125}e^{\frac{19in\pi}{\sigma(n)}} - 24137569e^{\frac{102in\pi}{\sigma(n)}}$$

$$+ \cdots \left(\frac{b_k}{k-1}\right)^{k-1} e^{\left(\frac{\pi i n b_k}{\sigma(n)}\right)} \qquad (55.)$$

In general, we have:

$$F(n,\infty) = \sum_{k=1}^{\infty} e^{i\pi\beta_k}\,, \qquad (56.)$$

k	β_k
1	0
2	$-\frac{1}{2}-\frac{n}{\sigma(n)}$
3	$1+\frac{2n}{\sigma(n)}$
4	$-\frac{1}{2}+\frac{3n}{\sigma(n)}$
5	$-\frac{20n}{\sigma(n)}-\frac{i\log 5^5}{\pi}$
6	$\frac{1}{2}+\frac{19n}{\sigma(n)}-i\left(\frac{\log 19^5+\log 5^5}{\pi}\right)$
7	$1+\frac{102\,n}{\sigma(n)}-i\frac{\log 17^6}{\pi}$
8	$\frac{1}{2}-\frac{337\,n}{\sigma(n)}-\frac{i}{\pi}\log\left(\frac{337}{7}\right)^7$
9	$-\frac{40\,n}{\sigma(n)}-\frac{i}{\pi}\log(5^8)$
10	$\frac{1}{2}+\frac{2439\,n}{\sigma(n)}-\frac{i}{\pi}\log(271)^9$

A 3-d plot of the function, shows that the function $F(n,4)=0$, is the axis of an infinite cylinder ,where the rest of the terms $m>4$ lie.

Figure 15 shows the cylinderical form with the axis approaching a line when the cylinder radius approaches infinity. The axis of the cylinder becomes the solutions for Perfect numbers,

$$1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} = 0, \qquad (57.)$$

Now, from (18), for some integer N,

Hence for a Perfect number N, (57) gives:

$$N_p = \frac{\tan(2)\prod_{k=1}^{N_p}\left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)} \qquad (58.)$$

Theorem 1: *There are an infinite number of Mersenne Primes.*

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Distribution of *F(n)* showing distinct regions for the first 4 terms, (red axis), and the cylindral orbit of the remaining terms on the cylinder (blue). **Perfect numbers occur when F(n,4)=0 at the exact axial symmetry of remaining terms higher than the fourth term.**

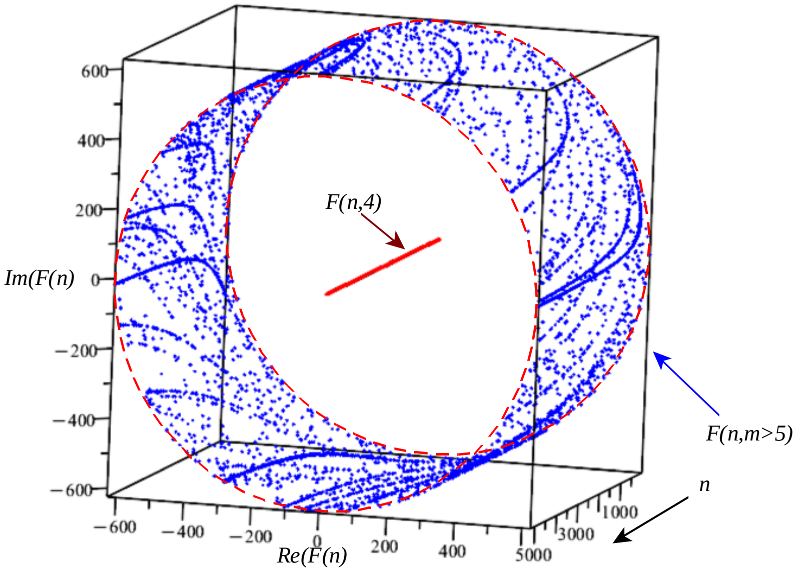

FIGURE 15

Analytic Mersenne Density and Infinitude.

Setup:

Let

$$\cot(x) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1} \quad , \qquad [2^2 < \pi^2] \tag{59.}$$

Fix $x_0 \in (0,\pi)$ *with* $\cot(x_0) \in \mathbb{R}\backslash\{0\}$.

Partition $\mathbb{N}$, into disjoint classes $\wp$, and $\aleph = \mathbb{N}\backslash\wp$, where $\wp$ is the set of Mersenne exponents p, with $(2^{2p} - 1)$ a prime.

Define

$$S_{all(x)} = \sum_{n\geq 1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-1}, \qquad S_{M(x)} = \sum_{p\in\wp}^{\infty} \frac{2^{2p}|B_{2p}|}{(2p)!} x^{2p-1}, \qquad S_{N(x)} = S_{all(x)} - S_{M(x)} \tag{60.}$$

Note $S_{all(x)} = \frac{1}{x} - \cot(x)$ and $S_{N(x)}, S_{M(x)} > 0$ for $x \in (0,\pi)$.

Definition (analytic Mersenne density).

$$\rho_M = \frac{S_{M(x)}}{S_{all(x)}} = \frac{\sum_{p\in\wp}^{\infty} \zeta(2p)\left(\frac{x}{\pi}\right)^{2p}}{\sum_{p\in\wp}^{\infty} \zeta(2n)\left(\frac{x}{\pi}\right)^{2n}} \qquad [0 < \rho_M < 1] \tag{61.}$$

With the set up above, at a fixed $x_0 \in (0,\pi)$, supposew the following holds true:

(H1) (Regularity/positivity of coefficients).

Each summand is positive and satisfies the classical Bernoulli-Zeta representation:

$$|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n), \qquad (62.)$$

Hence, $S_{all(x)} \in (0, \infty)$.

(H2): (Analytic density at x_0). The decomposition of $\cot(x_0)$ through $S_{N(x)}, S_{M(x)}$ yields a normalized quadratic identity in the $\tan(x_0)$ as shown in **LEMMA 2**, only if **LEMMA1** holds.

(H2): (Single valuedness/discriminat collapse). Since $\tan(x_0)$ is single valued, the discriminat of the quadratic in LEMMA 2 vanishes.

Proof Sketch:

Absolute positivity and conditional subtraction.

By (H1), $S_{all(x)} = \sum_{n\geq 1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-1}$, and $S_{M(x)} = \sum_{p\in\wp}^{\infty} \frac{2^{2p}|B_{2p}|}{(2p)!} x^{2p-1}$. The analytic value $\cot(x_0)$ may be negative (e.g. $\cot(x_0 = 2) < 0$), which arises from subtracting the strictly positive $S_{all(x)}$ from $1/x_0$.

Quadratic normalization.

(H2) encodes the partition into a quadratic in $\mathrm{X} = \tan(x_0)$:

$$AX^2 + BX + 1 = 0, \qquad X = \frac{-B \pm \sqrt{B^2 - 4A}}{2A}.$$

Since $\cot(x_0) \neq 0$ and $S_{M(x)} > 0$, we have A and B finite and nonzero.

Discriminant collapse and consistency.

By **(H3),** $B^2 - 4A = 0$. Solve for X: the two roots coincide, so the quadratic exactly reproduces $\mathrm{X} = \tan(x_0)$.

Contradiction from finiteness.

Assume $\wp$, is finite. Then $S_{M(x)} > 0$ is a fixed positive constant, hence A is fixed. Meanwhile $S_{N(x)} = S_{all(x)} - S_{M(x)} > 0$ is also fixed. The identity $B^2 = 4A$ becomes a rational equality among strictly positive finite constants. But this equality must be compatible with the sign of $\cot(x_0)$ (e.g. negative at $X = 2$); when the decomposition is realized by finite sets, the resulting rational combination cannot produce the required analytic sign/phase (it stays on the "algebraic" positive side). This contradicts the actual value of $\cot(x_0)$.

A symmetric argument applies if $\aleph$, is finite: then S_N is fixed and $S_{M(x)} = S_{all(x)} - S_{N(x)}$ must bear the entire analytic burden; again the finite rational identity cannot reproduce the analytic sign at x_0. Therefore, both classes must be infinite.

Interpretation via classical pillars.

Pringsheim (nonnegative coefficients ⇒ real singular control): Positivity of coefficients yields rigid real-axis behavior of generating series; finite truncations cannot emulate the required analytic sign at x_0.

Gap/lacunary theorems (Fabry/Hadamard): Attempting to realize the analytic function from a set with "large gaps" (finite or too-sparse) obstructs continuation/phase needed at x_0; an infinite contribution from both parts is necessary.

Tauberian philosophy (Wiener–Ikehara): Analytic constraints (here, the discriminant identity at a real point) force "density/infinity-type" conclusions for the underlying index sets. Thus both $\mathbb{N}$, and $\wp$ must be infinite.

Corollary A (Intrinsic analytic density)

Under the hypotheses of **Theorem A**, the intrinsic analytic Mersenne density

$$\rho_M(x) = \frac{S_{M(x)}}{S_{all(x)}} = \frac{\sum_{p\in\wp}^{\infty} \zeta(2p)\left(\frac{x}{\pi}\right)^{2p}}{\sum_{p\in\wp}^{\infty} \zeta(2n)\left(\frac{x}{\pi}\right)^{2n}} \qquad [0 < \rho_M < 1]$$

is well-defined with $0 < \rho_M < 1$. In particular, $\rho_M(x_0)$cannot be realized by a finite index set on either side.

THEOREM: *There exists an infinite number of Mersenne Primes.*

Proof:

I start with the relationship between Perfect numbers and their sums of divisors. Let p be a prime number such that $P_p = (2^p - 1)2^{p-1}$ *is a perfect number* $N_{p\in P_p}$. Then the following applies.

Lemma 1: *If* $N_{p\in P_p}$ *is a Perfect number, then,*

$$N_{p\in P_p} = \frac{\tan(2)\prod_{k=1}^{N_p}\left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)} \tag{63.}$$

Proof of LEMMA 1: See equation (19) for Perfect numbers.

Lemma 2: *Let p be a prime that generates a Mersenne prime and a Perfect Number N, then, there exits a unique decomposition of* $\mathbf{cot(x_0)}$ *into a quadratic identity*

$$\boldsymbol{A(x_0)\tan^2(x_0) + B(x_0)\tan(x_0) + 1 = 0} \tag{64.}$$

Proof (LEMMA 1):

Now, from [4], p.42, 1.411 (7) we find an expressions for $\cot(x)$:

$$\cot(x) = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1} \quad , \qquad [x^2 < \pi^2] \tag{65.}$$

Factoring this form into

$$\cot(x) = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{x^{k-1}(2^k - 1)2^{2k}|B_{2k}|}{(2^k - 1)(2k)!} x^k \, , \qquad [x^2 < \pi^2] \tag{66.}$$

We find that by chosing $x_0 = 2$, since (66) holds for $\{2^2 < \pi^2\}$, the expression can be modifed and separated into two class , one over the sum over Mersenne primes to include Perfect numbers, $N_p = 2^{p-1}(2^p - 1)$, when $p \in P_p$, a prime for which $P_p = 2^p - 1$ is a Mersenne prime, and the class of non-Mersenne primes, for $k \notin P_p$

$$\cot(2) = \frac{1}{2} - \sum_{p\in P_p} \frac{2^{p-1}(2^p - 1)2^{2p}|B_{2p}|}{(2^p - 1)(2p)!} 2^p - \sum_{k\notin P_p} \frac{2^{4k-1}|B_{2k}|}{(2k)!} \quad , \qquad [2^2 < \pi^2] \tag{67.}$$

Put $N_p = 2^{p-1}(2^p - 1), p \in P_p$ in (67), then,

$$\cot(2) = \frac{1}{2} - \sum_{p\in P_p} N_p \frac{2^{3p}|B_{2p}|}{(2^p - 1)(2p)!} - \sum_{k\notin P_p} \frac{2^{4k-1}|B_{2k}|}{(2k)!} \quad , \qquad [2^2 < \pi^2] \tag{68.}$$

From (19), LEMMA 1,

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$$N_{p\in P_p} = \frac{\tan(2)\prod_{k=1}^{N_p}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)} \quad (69.)$$

$$\cot(2) = \frac{1}{2} - \tan(2)\sum_{p\in P_p}\left[\left(\frac{2^{3p}|B_{2p}|}{(2^p-1)(2p)!}\right)\frac{\prod_{k=1}^{N_p}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)}\right]$$

$$-\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!} \quad (70.)$$

Divide by $\cot(2)$.

$$1 = \frac{1}{2}\tan(2) - \tan^2(2)\sum_{p\in P_p}\left[\left(\frac{2^{3p}|B_{2p}|}{(2^p-1)(2p)!}\right)\frac{\prod_{k=1}^{N_p}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)}\right] - \tan(2)\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!} \quad (71.)$$

$$1 = \left(\frac{1}{2} - \sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}\right)\tan(2) - \tan^2(2)\sum_{p\in P_p}\left[\left(\frac{2^{3p}|B_{2p}|}{(2^p-1)(2p)!}\right)\frac{\prod_{k=1}^{N_p}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)}\right] \quad (72.)$$

$$\tan^2(2)\sum_{p\in P_p}\left[\left(\frac{2^{3p}|B_{2p}|}{(2^p-1)(2p)!}\right)\frac{\prod_{k=1}^{N_p}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)}\right] + \left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!} - \frac{1}{2}\right)\tan(2) + 1 = 0 \quad (73.)$$

Now, we reduce (73) further with the following identities [[4],page 41]:

$$\cos(nx) = \prod_{k=1}^{\frac{n}{2}}\left(1-\frac{\sin^2(x)}{\sin^2\left(\frac{(2k-1)\pi}{2n}\right)}\right) \qquad n \text{ is even}$$

$$\sin(nx) = n\sin(x)\cos(x)\left(\prod_{k=1}^{\frac{n-2}{2}}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right) \qquad n \text{ is even} \quad (74.)$$

Putting $n = 2N_p, x = \frac{1}{N_p}, p \in P_p,$

$$\cos(2) = \prod_{k=1}^{N_p}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right), \frac{\sin(2)}{2N_p} = \sin\left(\frac{1}{N_p}\right)\cos\left(\frac{1}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right) \quad (75.)$$

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Substitute expressions (74) into (73):

$$\tan^2(2)\sum_{p\in P_p}\left[\left(\frac{2^{3p}|B_{2p}|}{(2^p-1)(2p)!}\right)\frac{\prod_{k=1}^{N_p}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right)\cos\left(\frac{2}{N_p}\right)\left(\prod_{k=1}^{N_p-1}\left(1-\frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)}\right]$$

$$+\left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)\tan(2)+1=0 \quad (76.)$$

$$\tan^2(2)\sum_{p\in P_p}\left[\left(\frac{2^{3p}|B_{2p}|}{(2^p-1)(2p)!}\right)\frac{\cos(2)}{\frac{\sin(2)}{2N_p}}\right]+\left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)\tan(2)+1=0 \quad (77.)$$

$$\tan^2(2)\sum_{p\in P_p}\left[\left(\frac{2^{3p}|B_{2p}|}{(2^p-1)(2p)!}2N_p\right)\cot(2)\right]+\left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)\tan(2)+1=0 \quad (78.)$$

$$\tan^2(2)\sum_{p\in P_p}\left[\left(\frac{2^{3p}|B_{2p}|}{(2^p-1)(2p)!}2(2^p-1)2^{p-1}\right)\cot(2)\right]+\left(\sum_{k\notin p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)\tan(2)+1=0 \quad (79.)$$

$$\left\{\sum_{p\in P_p}\left[\left(\frac{2^{4p}|B_{2p}|}{(2p)!}\right)\cot(2)\right]\right\}\tan^2(2)+\left\{\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right\}\tan(2)+1=0 \quad (80.)$$

Note that the sum for the Perfect Numbers expressed in Mersernne Primes require a modification with a factor $2\cdot 2^{4p-1}$ that is lost in (80) for the original sum defoinition for $p\in P_p$. This is where the result of $\sigma\left(N_{p\in P_p}\right)=2N_{p\in P_p}$ in (19) comes into play.

Put

$$\left.\begin{aligned}X&=\tan 2\\ A&=\left\{\sum_{p\in P_p}\left[\left(\frac{2^{4p}|B_{2p}|}{(2p)!}\right)\cot(2)\right]\right\}\\ B&=\left\{\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right\}\end{aligned}\right] \quad (81.)$$

$$AX^2+BX+1=0 \quad (82.)$$

$$X=\frac{-B\pm\sqrt{B^2-4A}}{2A} \quad (83.)$$

However, by (H3), $\tan(2)$ can only have one value, hence, we get:

$$B^2=4A \quad (84.)$$

Then,

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$$\left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)^2 = 4\sum_{p\in P_p}\left[\left(\frac{2^{4p}|B_{2p}|}{(2p)!}\right)\cot(2)\right] \tag{85.}$$

Inserting the factor of 2 for the Mersenne primes again to make the sums as per the original $\cot(x=2)$ formula,

$$\cot(2) = \frac{\left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)^2}{8\sum_{p\in P_p}\left[\left(\frac{2^{4p-1}|B_{2p}|}{(2p)!}\right)\right]} \tag{86.}$$

$$\frac{\left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)^2}{\sum_{p\in P_p}\left[\left(\frac{2^{4p-1}|B_{2p}|}{(2p)!}\right)\right]} = -3.6612604350 \tag{87.}$$

This is not possible since the sums are all positive quantities. It is clear that the contradiction results in negative sum of the Mersenne primes $p \in P_p$. The reasons are given below.

a) Discriminant condition.

For a single-valued analytic function tan(2), both roots of (83) must coincide, giving the constraint (84), i.e. $B^2 = 4A$.

b) Finite-set contradiction.

Suppose either $\wp$ or $\aleph$ is finite. If $\wp$ is finite, then A is bounded and $\frac{B^2}{4A}$ is strictly positive; hence $\cot(2) > 0$, contradicting the analytic value $\cot(2) \approx -0.4576\dots$.

If $\aleph$ is finite is finite, A diverges, destroying convergence and violating the finite analytic value of $\cot(2)$.

Therefore, both subsets must extend infinitely.

c) Analytic necessity.

The negative finite value of $\cot(2)$ arises from the conditional convergence of the full series. Only infinite, interleaved contributions from both classes can reproduce the correct analytic continuation through the real axis.

Finite truncations cannot yield the required sign reversal because all partial sums are positive.

d) Conclusion.

e) Hence, the equality (84) can hold with finite $\cot(2)$ only if $|\wp| = |\aleph| = \infty$.

Therefore, both the Mersenne-prime and non-Mersenne classes are infinite classes.

Now an estimate of the Mersenne prime sum can be obtained if we consider:

$$X = \tan 2 = -\frac{\left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)}{2\sum_{p\in P_p}\left[\left(\frac{2^{4p}|B_{2p}|}{(2p)!}\right)\cot(2)\right]} = -2.185039863\dots. \tag{88.}$$

Now, $2\cot(2)\tan 2 = 2$, hence,

$$2 = -\frac{\left(\sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!}-\frac{1}{2}\right)}{\sum_{p\in P_p}\left[\left(\frac{2^{4p}|B_{2p}|}{(2p)!}\right)\right]} \tag{89.}$$

$$4\sum_{p\in P_p}\left[\left(\frac{2^{4p-1}|B_{2p}|}{(2p)!}\right)\right] + \sum_{k\notin P_p}\frac{2^{4k-1}|B_{2k}|}{(2k)!} = \frac{1}{2} \tag{90.}$$

$$3\sum_{p\in P_p}\left[\left(\frac{2^{4p-1}\left|B_{2p}\right|}{(2p)!}\right)\right]+\left(\sum_{p\in P_p}\left[\left(\frac{2^{4p-1}\left|B_{2p}\right|}{(2p)!}\right)\right]+\sum_{k\notin P_p}\frac{2^{4k-1}\left|B_{2k}\right|}{(2k)!}-\frac{1}{2}\right)=0 \qquad (91.)$$

$$3\sum_{p\in P_p}\left[\left(\frac{2^{4p-1}\left|B_{2p}\right|}{(2p)!}\right)\right]-\cot(2)=0 \qquad (92.)$$

$$\sum_{p\in P_p}\left[\left(\frac{2^{4p-1}\left|B_{2p}\right|}{(2p)!}\right)\right]=-0.1525525181\ldots \qquad (93.)$$

Again a contradiction.

7. Interpretative Remark

Suppose Equation (87) represents an analytic equilibrium between a sparse harmonic lattice (the Mersenne indices) and the complementary dense continuum (non-Mersenne integers). The finiteness of either subset would destroy the analytic balance and invert the sign of cot(2). Thus, the very existence of a finite negative cotangent value enforces the infinitude of both classes -a remarkable intersection of trigonometric analysis and arithmetic structure.

Remarks and positioning

a) Novelty.

Theorem 1 is not a re-statement of any single classical result; it's a fusion: positivity + analytic identity + discriminant collapse ⇒ infinitude of each class. The closest analogues are Pringsheim (positivity constraints), Fabry/Hadamard (sparsity ↔ analytic behavior), and Tauberian methods (analytic facts ⇒ density/infinitude).

b. The normalization that produces a quadratic in $\tan(x_0)$ encapsulates the single-valuedness of the trigonometric function at x_0; the vanishing discriminant is precisely the statement that the two algebraic branches coincide with the analytic branch. For finite partitions, that coincidence cannot match the true sign/phase unless both classes are infinite.

Robustness. The argument isn't tied to $x_0 = 2$; any $x_0 \in (0,\pi)$ with $\cot(x_0) \neq 0$, yields the same conclusion under **(H1), (H2)** and **(H3).**

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