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[Alejandro Ruiz](#) \*

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## Article

# Dynamic Balance: A Thermodynamic Principle for the Emergence of the Golden Ratio in Open Non-Equilibrium Steady States

Alejandro Ruiz 

Independent Researcher; alejandropheysicist@gmail.com

**Abstract:** We propose that a single irrational constant,  $\phi \approx 1.618$ , serves as a universal attractor in far-from-equilibrium systems in a steady-state. In our *Dynamic Balance* framework, the dimensionless ratio  $\alpha(t) = \dot{E}(t)/T(t) \dot{S}(t)$ —comparing the system's energy throughput  $\dot{E}$  to its entropic heat loss  $T\dot{S}$ —naturally converges to  $\phi$  in a wide range of open, driven-dissipative processes. This ratio enforces a balance condition that partitions energy into useful work (order) and dissipative loss (disorder) in a way that maximizes both stability and adaptability. We demonstrate the principle using local cost functions, gradient-flow PDE expansions, Wilsonian renormalization group methods, and Markov master-equation analyses, showing in particular how it unifies near-critical behavior in the brain: the resulting neurodynamic PDE reproduces avalanche scaling, fractal wave expansions, short-term plasticity responses, and morphological growth patterns without *ad hoc* saturations. More broadly, the Dynamic Balance principle complements maximum entropy production and self-organized criticality in physics, parallels metabolic efficiency and fractal organization in biology, and implies optimal resource division in geology. We review empirical evidence—from stellar pulsations and fluid vortices to branching structures, quantum critical phenomena, and neural avalanche data—indicating that  $\phi$  emerges as a robust signature of dynamic organization in non-equilibrium state-states.

**Keywords:** nonequilibrium thermodynamics; criticality; branching and phyllotaxis; neural avalanches; fibonacci brain waves; fibonacci anyons; penrose quasicrystals; rotating turbulence; Galactic spirals; golden ratio

## 1. Introduction

### 1.1. Why the Golden Ratio $\phi$ Appears So Often

The golden ratio,  $\phi \approx 1.618$ , is one of nature's most intriguing recurring constants, famously appearing in the arrangement of leaves (phyllotaxis), the branching patterns of trees, blood vessels and river deltas [1–4], and the logarithmic spiral arms of galaxies and hurricanes [5,6]. Yet its reach extends far beyond botany and geometry. Recent experiments have uncovered numerical exponents or geometric features near  $\phi$  in a surprising variety of complex systems— rotating turbulence [7–9], quantum critical chains [10], twisted bilayer graphene (TBG) [11–13], Fibonacci anyons [14,15], neural activity exponents [16,17], and more.

Modern science increasingly recognizes that complex systems thrive in a poised nonequilibrium state between rigid order and chaotic disorder. Erwin Schrödinger famously noted that **life maintains its structure by exporting entropy** to its environment [18]. In other words, organisms continuously consume usable energy and irreversibly dissipate part of it as heat (entropy) to sustain their internal order. This interplay between energy and entropy suggests a *balance point* where a system can organize itself while still obeying the second law of thermodynamics. Strikingly, many natural patterns hint at a preferred balance: for example, certain **biological and physical structures are arranged according to the golden ratio**, long associated with optimality and harmony in geometry and growth. This raises the question: *do far-from-equilibrium systems in a non-equilibrium steady-state universally tune themselves to*

a specific ratio between energy throughput and entropy production that optimizes functionality? The *Dynamic Balance* principle emerged from this inquiry, aiming to provide a quantitative, cross-disciplinary rule for that optimal tuning.

### 1.2. Dynamic Balance: A Simple Nonequilibrium Tension

To formalize the Dynamic Balance principle, we consider a general open system far from thermodynamic equilibrium. In such a state,  $E$ ,  $S$ , and  $T$  can all vary with time, and the system continuously:

- **absorbs** usable energy at some net rate  $\dot{E}$  (units of energy per time, or power),
- **produces** entropy at a rate  $\dot{S}$  (units of entropy per time),
- **exports** that entropy to the environment at an *effective* temperature  $T$ .

Therefore, measuring the rates of energy input and heat flow is often simpler than determining absolute energies. Consequently, we define the dimensionless **Dynamic Balance ratio**:

$$\alpha(t) = \frac{\dot{E}(t)}{T(t) \dot{S}(t)}, \quad (1)$$

which directly quantifies the real-time balance between the energy being injected or removed and the corresponding entropy production.  $\dot{E}$  can be view as the total energy flux or throughput driving the system (e.g. energy absorbed per unit time),  $\dot{S}$  the entropy flux to the environment (the entropy produced per unit time), and  $T$  an *effective* temperature reflecting system-wide noise, fluctuations and energy distribution (for a thermal system, this could be the ambient temperature at which entropy is expelled). The term  $T\dot{S}$  then has units of power and represents the **rate of energy dissipated as heat** required to export the entropy to the environment.

Intuitively,  $\alpha$  measures how much energy **goes into useful organization or work** relative to the energy irreversibly lost to entropy production. A high  $\alpha$  means the system retains or utilizes a larger portion of its energy (lower relative dissipation), whereas  $\alpha$  close to 1 indicates most energy simply heats the environment with little left to build or maintain structure. Therefore,  $\alpha$  can only take values from 0 to  $\infty$  in the nonequilibrium steady state (NESS). Crucial to our definition is the fact that these systems are bounded by two extrema we call:

- $\alpha = 0$  (**meltdown**) as a limit of extreme *disorder*: all or nearly all energy is dissipated; the system cannot sustain stable structure.
- $\alpha \rightarrow \infty$  (**freeze**) as a limit of extreme (rigid) *order*: the system devotes negligible energy to entropy production, sacrificing adaptability or functional throughput.

Nonequilibrium biological and physical systems support the idea that a balanced ratio of drive and dissipation is a hallmark of a stable nonequilibrium state-state. Whether through formal extremum principle (minimum or maximum entropy production) or through dynamic feedback, these systems eschew the extremes of zero-flow or runaway in favor of an intermediate steady-state. The Dynamic Balance principle asserts that **in a sustained nonequilibrium steady state,  $\alpha$  will adjust to a specific constant value** that optimally balances these competing needs (energy utilization vs. dissipation). To find this optimal value, we invoke a self-similarity or **scale-invariance argument**.

If the *same balance* is achieved at every level, then the ratio of total energy to dissipative flow part in the whole should equal the ratio of energy to dissipation flow within any representative part. In particular, we can imagine splitting the system's energy rate into two portions: the dissipated part  $\dot{E}_{diss} = T\dot{S}$ , and the remaining **effective work/structure energy flux**  $\dot{E}_{struct} = E - T\dot{S}$  that is channeled into maintaining organization, performing work, or stored in structure. Therefore,

$$\dot{E} = \dot{E}_{diss} + \dot{E}_{struct} = T\dot{S} + (\dot{E} - T\dot{S}).$$

Requiring the system to **reproduce its balance at different scales** means the ratio  $\dot{E}/T\dot{S}$  for the whole equals the ratio of the part to the remainder:

$$\frac{\dot{E}}{T\dot{S}} = \frac{T\dot{S}}{\dot{E} - T\dot{S}}.$$

This equation is the mathematical expression of “the whole is to the part as the part is to the remainder,” which is precisely the defining property of the **golden ratio**. Solving for  $\alpha = \dot{E}/(T\dot{S})$  gives:

$$\alpha = \frac{1}{\alpha - 1} \quad \rightarrow \quad \alpha^2 = 1 + \alpha \quad \rightarrow \quad \alpha = \frac{1 \pm \sqrt{5}}{2}.$$

The physically meaningful (positive) solution is

$$\alpha = \frac{1 + \sqrt{5}}{2} \approx 1.618034...$$

the golden ratio  $\phi$ . Thus, under the assumption of self-similar balancing between energy input, entropy dissipation, and leftover energy for structure, the **stable fixed point** is  $\alpha = \phi$ . Any other ratio would not be scale-invariant; for example, if  $\alpha$  were less than  $\phi$ , then the system is dissipating “too much” relative to structure – the next level down would have a higher ratio, causing  $\alpha$  to increase toward  $\phi$ . Conversely, if  $\alpha > \phi$ , subsystems would dissipate too little (relative to their internal energy) and become prone to instabilities, driving  $\alpha$  down. The golden ratio is the unique point at which this **balance is self-consistent across scales**.

From another perspective,  $\alpha = \phi$  optimizes the trade-off between **order and disorder**. One way to see this is to consider incremental deviations from balance. Suppose the system diverts a small extra fraction of energy  $\delta E$  into useful work (reducing dissipation). This can increase order, but also means entropy is exported more slowly, risking accumulation of disorder internally. Conversely, diverting  $\delta E$  to dissipation stabilizes against chaos but wastes energy that could sustain structure. At the optimum  $\phi$ , these opposing effects reach a “**sweet spot**” of **maximal combined benefit**, analogous to setting the derivative of a gain-vs-loss function to zero. In fact,  $\phi$  is well-known to arise in optimal partitioning problems. It is the **most irrational number**, meaning it achieves the best distribution between commensurate and incommensurate action – which is physically helpful in avoiding resonances. A system tuned to the golden ratio tends to avoid resonant locking and instead exhibits distinctive dynamics that exist between order and chaos [19]. This has been shown mathematically in simple driven systems: when two competing processes or oscillations have a frequency ratio of  $\phi$ , the system can produce a **strange nonchaotic attractor** – a fractal pattern in its dynamics that lacks sensitivity to initial conditions (hence predictable in a bounded way) [19]. In essence,  $\phi = \alpha$  embodies a **dynamic equilibrium at the edge of chaos**: it is a point where the system is sufficiently far from equilibrium to sustain organized complexity, yet not so far as to fall into runaway instability or unpredictability.

### 1.3. Conditions for Dynamic Balance

In the next section, we will show how the golden ratio emerges in nonequilibrium open systems under these *conditions*:

1. **Self-Similarity or Scale Invariance:** we assume that at each hierarchical level of the system (from whole to part), energy flux is split into dissipated versus leftover in the *same* ratio  $\dot{E} : T\dot{S} : (\dot{E} - T\dot{S})$ , i.e. the ratio  $\alpha = \dot{E}/(T\dot{S})$ . Many fractal or hierarchical systems show this repeated structures across scales.
2. **Infinite Boundary Costs:** we treat  $\alpha = 0$  and  $\alpha \rightarrow \infty$  as *infinite-cost* boundaries (Sec. 2, Sec. 4). To exclude these, we define a cost function  $R(\alpha)$  that ensures both extremes are energetically or entropically prohibitive. Essentially, they become repulsive boundary states.
3. **Stabilizing Feedback:** Without any feedback mechanism, a system might wander away from  $\alpha = \phi$ . In practice, we rely on *slow drive plus partial meltdown events* (Sec. 4) or *gradient-flow PDE* arguments (Sec. 2) to maintain our constraints. Together, these ensure  $\alpha$  converges to  $\phi$  in the

long run. A negative gradient PDE ( $\partial R/\partial E$ ) acts like a stabilizing feedback without the need for arbitrary saturations.

4. **Self-organizing.** These systems spontaneously move towards the synergy point from typical initial states, *without* external fine-tuning. The cost function  $R(\alpha)$  provides local negative feedback that drives each region away from the boundaries, and towards the global stable interior point.
5. **Nonequilibrium Steady State.** Our framework was develop to explain *open systems* that continuously receives energy flux  $\dot{E}$  and export entropic heat  $T\dot{S}$ . These systems remain in a steady state, "alive" or "active" as long as their external drive (energy input) balances the dissipated entropy flow (irreversible heat output).

## 2. Results

### 2.1. Mathematical Foundation of Dynamic Balance

We present a step-by-step derivation of the *Dynamic Balance* principle, showing how the energy-entropy balance function naturally converges to the golden ratio  $\phi \approx 1.618 \dots$

$$\lim_{t \rightarrow \infty} \alpha(t) = \lim_{t \rightarrow \infty} \frac{\dot{E}(t)}{T(t)\dot{S}(t)} \approx \phi, \quad (2)$$

where

- $\dot{E}$  is the net energy flux into the system, with units of power (Joules per second). It represents how much external energy per unit time drives the system.
- $\dot{S}$  is entropy flow or production rate, in units of Joules per Kelvin per second, or equivalently Watts/K, so that  $T\dot{S}$  is the entropic heat power. It tells us how much power is irreversible lost to the environment.
- $T$  denotes an effective temperature (or noise scale) capturing small-scale degrees of freedom.  $T$  is a parameter that shapes the distribution of microstates or the level of random excitations. It describes the intensity of internal fluctuations or chaos.

We assume that the system is in a (quasi-)steady nonequilibrium state so  $\dot{E}$  and  $\dot{S}$  are well-defined average rates of energy input and entropy export. If  $E(t)$  and  $S(t)$  are the cumulative energy used by the system from 0 to  $t$ , then we can define

$$\dot{E}_{avg} = \frac{E(t)}{t}, \quad \dot{S}_{avg} = \frac{S(t)}{t},$$

which, in the limit of a long-time steady-state yields,

$$\alpha = \lim_{t \rightarrow \infty} \frac{\dot{E}_{avg}}{T\dot{S}_{avg}} = \lim_{t \rightarrow \infty} \frac{E(t)}{T(t)S(t)}.$$

Moving forward, we will use  $\dot{E}_{avg}$  and  $E$  interchangeably with the understanding that this only applies in the nonequilibrium steady-states systems we are describing.

### 2.2. Step 1: Defining $\alpha$

In classical equilibrium thermodynamics, a closed system with no additional work terms obeys  $dE = T dS$ , relating  $E$  internal energy,  $S$  entropy, and  $T$  temperature. This relation holds for **reversible processes in systems at equilibrium**, reflecting an idealized balance where any change in the system's internal energy  $E$  is exactly compensated by a corresponding change in entropy  $S$  at a fixed temperature  $T$ .

However, we are interested in open, driven systems operating *far from equilibrium*. In such conditions, additional contributions such as irreversible entropy production and dissipative work become significant, and the simple equilibrium identity does not strictly apply. Nevertheless, it is common in complex systems modeling to invoke a *local equilibrium* approximation. This approach



assumes that even though the global system is far from equilibrium, small regions within the system can be treated as if they are in quasi-equilibrium. In these local patches, *effective* thermodynamic parameters (an effective temperature  $T$ , an effective entropy  $S$ , and an effective free energy  $F$ ) can be defined. At every moment, the system receives an incoming power  $\dot{E}$  and irreversibly dissipates  $T\dot{S}$ . The leftover free-energy flux is  $\dot{F} = \dot{E} - T\dot{S}$ .

In our framework, we define the ratio  $\alpha = \dot{E}/T\dot{S}$ , which serves as a convenient **measure of the interplay between energy consumption and entropy production**. Assuming that deviations from equilibrium can be captured by these effective parameters allows us to relate changes in energy to changes in disorder, and enables us to model how a system might be driven away from an optimal balance when either power input or entropy generation is perturbed.

**Lemma 1.** *In a non-equilibrium steady-state or driven system,  $\dot{E}$  can be viewed as net energy throughput (power input) and  $\dot{S}$  as entropy production rate, with  $T$  an effective temperature parameter describing fluctuations or noise.*

Hence, our energy-entropy balance function  $\alpha > 0$  defines:

- $\alpha = 0$  **means meltdown:** the system has effectively no useful energy to sustain structure, or is trying to maintain huge  $S$  with minimal  $E$ . It devotes *all* input energy to unstructured heat loss leaving no net energy to maintain coherent structures. This represents a limit of extreme disorder.
- $\alpha \rightarrow \infty$  **means freeze:** The system devotes *almost none* of its energy to entropy production. An attempt at near-zero  $S$ , which is unsustainable by the second law.

While the entire system may be large and heterogeneous, we use a *local equilibrium* or *mesoscopic* viewpoint: within each small region, there is a well-defined local energy flow  $E$  and local entropy flux  $S$ . Summing or integrating over these subregions in steady state defines aggregate  $E$ ,  $S$ , and an effective  $T$ . Consequently, one can still form a global  $\alpha$ , yet **scale invariance** prescribes that each subregion's ratio should match the system-wide ratio in a self-similar arrangement. This underlies the upcoming argument that drives  $\alpha$  to a unique constant, found to be  $\phi$ .

### 2.3. Step 2: Derivation of $\alpha = \phi$ from Self-Similarity

We now introduce a **self-similar partition** or scale invariance argument:

**Lemma 2.** *The ratio of total energy to entropy-led dissipation equals the ratio of that dissipation to the leftover (structured) energy.*

Namely, if the total energy  $E$  of an open system at NESS splits into **disordered** plus **ordered** parts, we require

$$E = TS + (E - TS),$$

where  $TS$  represents the energy irreversibly dissipated as heat (associated with entropy production) and  $(F = E - TS)$  can be interpreted as the *residual free energy*, capturing the portion of  $E$  available for structural maintenance, work, or sustained organization.

**Why Scale Invariance?** Consider that a complex system typically has many hierarchical levels—subsystems nested within larger subsystems. A hallmark of “fractal” or “hierarchical” organization is that each level (when operating in steady state) enforces a **similar energy–entropy ratio** as the whole. Hence, scale invariance posits that  $\alpha$  remains the same across levels—the whole system's ratio equals the subsystem's ratio.

**Theorem 1.** *The key assumption is that the system self-organizes in such a way that the ratio of the total energy to the dissipated part is the same as the ratio of the dissipated energy to the free energy.*

**Proof.**

$$\alpha = \frac{E}{TS} = \frac{TS}{E - TS} = \frac{TS/TS}{E/TS - TS/TS} = \frac{1}{\alpha - 1}.$$

Multiplying both sides by  $(\alpha - 1)$ , and expanding the left-hand side yields

$$\alpha(\alpha - 1) = 1, \quad \alpha^2 - \alpha = 1.$$

Rearranging terms gives a quadratic equation:

$$\alpha^2 - \alpha - 1 = 0.$$

This is precisely the defining equation for the *golden ratio*. Hence the unique positive solution is

$$\alpha = \frac{1 + \sqrt{5}}{2} \approx 1.618 \dots,$$

Therefore, if we interpret the leftover as  $(E - TS)$  and require the same ratio as  $E/(TS)$ , the unique solution is  $\alpha = \phi$ .  $\square$

Under the assumption of self-similarity (scale-invariant) balancing in energy vs. heat dissipation across hierarchical levels, the NESS system's dimensionless ratio must satisfy  $\alpha = 1/(\alpha - 1)$ , giving a unique interior synergy  $\alpha = \phi$ .

#### 2.4. Step 3: Cost Function $R(\alpha)$ and Boundary Divergences

To ensure that  $\alpha$  converges to a stable attractor, we postulate that it follows a gradient descent dynamics:

$$\frac{d\alpha}{dt} = -\Gamma \frac{\delta R(\alpha)}{\delta \alpha},$$

where we define a scalar potential or **cost function**  $R(\alpha)$  with:

- $\lim_{\alpha \rightarrow 0^+} R(\alpha) = \lim_{\alpha \rightarrow \infty} R(\alpha) = +\infty$ : disallowed or unsustainable boundary states, reflecting large "penalties" if  $\alpha$  approaches 0,  $\infty$ .
- A unique interior minimum where  $\frac{dR}{d\alpha} = 0$  yields a stable  $\alpha^*$ .

One can build  $R(\alpha)$  to incorporate the scale invariance explicitly—meaning that if we rescale  $\alpha$  by some factor  $k$ , the qualitative shape of the cost function remains unchanged. For instance:

$$R(\alpha) = \left[ \frac{\alpha}{\phi} - \frac{\phi}{\alpha} \right]^2 = \left[ x - \frac{1}{x} \right]^2, \quad (3)$$

which penalizes both extreme limits ( $\alpha \rightarrow 0, \infty$ ), has a unique minimum at  $\alpha = \phi$ , and can be expressed in an scale-invariant way by using the dimensionless variable  $x = \alpha/\phi$ .

- $x = 1$  corresponds to the optimal efficiency state where  $\alpha = \phi$ ,
- $x < 1$  ( $\alpha < \phi$ ), energy  $E$  is relatively low compared to entropic energy  $TS$ , suggesting that disorder dominates.
- $x > 1$  ( $\alpha > \phi$ ), energy  $E$  is relatively high compared to entropic energy  $TS$ , implying the system is overly rigid or ordered.

We can check that

$$R(\alpha) = \left( \frac{\alpha^2 - \phi^2}{\alpha \phi} \right)^2 = \frac{(\alpha - \phi)^2 (\alpha + \phi)^2}{\alpha^2 \phi^2}, \quad \alpha > 0$$

which vanishes only at  $\alpha = \phi \approx 1.618$  and blows up at 0 or  $\infty$  (see Fig. ??). This interior point is the global minimum of the function  $R(\alpha)$ .

$$\left. \frac{dR(\alpha)}{d\alpha} \right|_{\alpha=\phi} = 2 \left( \frac{\alpha}{\phi} - \frac{\phi}{\alpha} \right) \left( \frac{1}{\phi} - \frac{\phi}{\alpha^2} \right) = 0, \quad \left. \frac{d^2 R(\alpha)}{d\alpha^2} \right|_{\alpha=\phi} = \frac{2}{\phi} > 0$$

We emphasize this specific form of  $R(\alpha)$  is just one example. Any continuous function with suitable divergences at  $0, \infty$  and a single minimum at  $\phi$  suffices. In fact,  $\phi$  emerges if one also embeds *scale invariance*. So the structure is robust to moderate changes in the function's shape.

This result suggests that in a system where energy is optimally partitioned between **order (free energy)** and **disorder (thermal entropy)**, the characteristic balance is given by:

- About **61.8% of energy** is thermal entropy ( $TS$ ).
- About **38.2% of energy** is effective free energy ( $F$ ).

One may regard  $R(\alpha)$  as a "thermodynamic potential" extension for nonequilibrium states. Minimizing  $R$  does not mean the system is at zero net entropy production. Instead, it means it has found an *optimal partition* of energy vs. dissipation that avoids meltdown ( $R \rightarrow \infty$  for  $\alpha \approx 0$ ) or freeze ( $R \rightarrow \infty$  for  $\alpha \rightarrow \infty$ ). This balance optimizes both **stability** and **efficiency** in energy use, preventing the system from falling into excessive disorder (meltdown) or excessive rigidity (freeze).

- In **biological systems**, this could mean an organism operates near  $\alpha = \phi$  to maintain adaptability while avoiding excess dissipation.
- In **physical systems**, this could be a **universal attractor** for self-organizing nonequilibrium states.

If we consider small perturbations  $\delta$  around equilibrium:

$$R(\phi + \delta) \approx R(\phi) + R'(\phi)\delta + \frac{1}{2}R''(\phi)\delta^2 \approx \frac{1}{2}R''(\phi)\delta^2,$$

we see that these deviations from equilibrium generate a restoring force proportional to the perturbation magnitude squared.

#### 2.5. Step 4: Flow Equation for Nonequilibrium Relaxation

We now show how  $\alpha(t)$  evolves in time toward  $\phi$  under typical nonequilibrium feedback:

$$\frac{d\alpha}{dt} = -\Gamma \frac{\delta R(\alpha)}{\delta \alpha},$$

where  $\Gamma > 0$  is a mobility or relaxation parameter.

**ODE / Negative Gradient Flow:** For a 0D or lumped-parameter system, define

$$\frac{d\alpha}{dt} = -\Gamma \frac{dR}{d\alpha} = -2\Gamma \left[ \frac{\alpha}{\phi} - \frac{\phi}{\alpha} \right] \left( \frac{1}{\phi} + \frac{\phi}{\alpha^2} \right).$$

Linearizing around the fixed point,  $\alpha(t) = \phi + \epsilon(t)$ , where  $|\epsilon(t)| \ll 1$ , then we get,

$$\frac{d\epsilon}{dt} \approx -\Gamma R''(\phi)\epsilon,$$

which has a solution  $\epsilon(t) \approx \epsilon_0 e^{-\Gamma R''(\phi)t}$ , ensuring  $\epsilon(t)$  decays exponentially over time.

$$\alpha(t) = \phi + \epsilon_0 e^{-\Gamma R''(\phi)t}.$$

Thus, from any initial  $\epsilon_0 > 0$ , the system relaxes to  $\phi$  in a finite time. This exponential decay exemplifies that meltdown/freeze cost drives  $\alpha$  away from extremes and stabilizes at  $\phi$ .

$$\lim_{t \rightarrow \infty} \alpha(t) = \lim_{t \rightarrow \infty} \frac{E(t)}{T(t)S(t)} = \phi.$$



**PDE / Spatially Extended Gradient-Flow:** To include heterogeneity in a spatially extended domain, we let  $\alpha = \alpha(\mathbf{x}, t)$  be a field defined over spatial domain  $\Omega \subseteq \mathbb{R}^d$ . We define a **free-energy functional**:

$$\mathcal{F}[\alpha] = \int_{\Omega} \left[ \frac{\kappa}{2} |\nabla \alpha(\mathbf{x}, t)|^2 + R(\alpha(\mathbf{x}, t)) \right] d^d x, \quad (4)$$

where  $\kappa > 0$  is a “diffusion-like” coefficient penalizing large spatial gradients, and  $R(\alpha)$  is an *on-site potential* that diverges at  $\alpha = 0$  and  $\alpha \rightarrow \infty$ . The term  $\kappa |\nabla \alpha|^2$  ensures that large gradients are also penalized. So even if one region tried meltdown, surrounding regions with  $\alpha > \phi$  would raise the local cost, triggering diffusion or feedback to re-balance. Ultimately, the system “smears out” extremes, converging to uniform  $\alpha = \phi$ .

We often assume no-flux or periodic conditions for  $\alpha(\mathbf{x})$  so meltdown/freeze states can’t “hide” at the domain boundary. Alternatively, if the domain’s boundary is open (e.g. an inflow/outflow in a fluid), we require that meltdown/freeze remain infinite cost for the system as a whole, effectively preventing  $\alpha$  from saturating at 0 or  $\infty$  near the boundary.

**PDE Gradient-Flow Equations:** To see how  $\alpha(\mathbf{x}, t)$  evolves to *minimize*  $\mathcal{F}[\alpha]$ , we posit a *gradient-flow PDE*:

$$\partial_t \alpha(\mathbf{x}, t) = -\Gamma \left[ -\kappa \nabla^2 \alpha(\mathbf{x}, t) + R'(\alpha(\mathbf{x}, t)) \right], \quad (\Gamma > 0). \quad (5)$$

If  $R(\alpha) \rightarrow +\infty$  as  $\alpha \rightarrow 0^+$  or  $\alpha \rightarrow \infty$ , the PDE flow *cannot* hold  $\alpha$  at meltdown or freeze for finite time/volume. These boundary states remain *repulsive*, forcing  $\alpha(\mathbf{x})$  to lie in  $(0, \infty)$  and evolve toward an interior equilibrium.

$$\partial_t \alpha(\mathbf{x}, t) = -\Gamma \frac{\delta \mathcal{F}[\alpha(\mathbf{x}, t)]}{\delta \alpha} = \Gamma \left[ \kappa \nabla^2 \alpha(\mathbf{x}, t) - 2 \left( \frac{\alpha(\mathbf{x}, t)}{\phi} - \frac{\phi}{\alpha(\mathbf{x}, t)} \right) \left( \frac{1}{\phi} + \frac{\phi}{\alpha(\mathbf{x}, t)^2} \right) \right]. \quad (6)$$

The only stable uniform solution inside  $(0, \infty)$  is  $\alpha = \phi$ , thanks to the polynomial structure ensuring  $R'(\phi) = 0$ . Linearization around  $\alpha(\mathbf{x}, t) = \phi$  shows negativity of eigenvalues ensures exponential damping of perturbations. Therefore, this PDE governs the spatiotemporal evolution of  $\alpha(\mathbf{x}, t)$ , and ensures that, despite fluctuations, the system globally converges to  $\phi$ .

**Lyapunov Stability.** One sees this PDE is *Lyapunov* or *gradient-flow* in  $\alpha$ , since the time-derivative of  $\mathcal{F}[\alpha]$  is:

$$\begin{aligned} \frac{d}{dt} \mathcal{F}[\alpha] &= \int_{\Omega} \frac{\delta \mathcal{F}[\alpha]}{\delta \alpha} \partial_t \alpha(\mathbf{x}, t) d^d x \\ &= -\Gamma \int_{\Omega} \left| \frac{\delta \mathcal{F}}{\delta \alpha} \right|^2 d^d x \leq 0. \end{aligned}$$

Thus  $\mathcal{F}[\alpha]$  is a *non-increasing Lyapunov functional*. Over time,  $\alpha(\mathbf{x}, t)$  descends until  $\frac{\delta \mathcal{F}}{\delta \alpha} = 0$ , which must occur at  $\kappa \nabla^2 \alpha + R'(\alpha) = 0$ . Because meltdown/freeze each blow up  $R(\alpha)$ , the only finite-volume solutions are *interior states* where  $\alpha > 0$  is finite. Typically, the unique uniform solution  $\alpha(\mathbf{x}, t) = \alpha^*$  is found by  $U'(\alpha^*) = 0$ , matching the local cost function result (e.g.  $\alpha^* = \phi$ ).

**Theorem 2.** By treating  $\alpha(\mathbf{x}, t)$  as a field and meltdown/freeze boundaries as infinite potentials, one obtains a *Lyapunov gradient-flow PDE* whose unique stable solution is  $\alpha(\mathbf{x}, t) \equiv \phi$ . Local extremes are smoothed out, and the entire domain organizes into a golden-ratio partition of energy vs. entropy.

### 3. Wilsonian Renormalization Group

Many *complex, open systems* in NESS (turbulence, brain, etc) display *scale invariance* or *fractal* structure. A third approach treats  $\alpha(\mathbf{x})$  as a field in a *scale-invariant* or *multi-scale* setting, analyzed by **renormalization group (RG)** methods. Originally introduced in statistical or quantum field theories, RG systematically integrates out short-distance or high-frequency fluctuations of  $\alpha$  in  $k$ -space ( $\alpha_{fast}$ ),

and rescales by a some factor, revealing how parameters *flow* under block-spin or momentum-shell integration.

### 3.1. Rewriting in Logarithmic Variables

Since  $\alpha(\mathbf{x}) \in (0, \infty)$ , we define

$$\Phi(\mathbf{x}) = \ln(\alpha(\mathbf{x})).$$

Thus the boundaries become  $\Phi \rightarrow -\infty$  or  $\Phi \rightarrow +\infty$ , respectively. Substituting  $\alpha = e^\Phi$  into Eq. (3):

$$R(e^\Phi) = \left( \frac{e^\Phi}{\phi} - \frac{\phi}{e^\Phi} \right)^2 = \left( e^{\Phi'} - e^{-\Phi'} \right)^2, \quad (7)$$

where  $\Phi' = \Phi - \ln(\phi)$ .

Hence meltdown  $\Phi \rightarrow -\infty$  and freeze  $\Phi \rightarrow +\infty$  each drive  $|e^{\Phi'}| \rightarrow \infty$  or 0, causing  $R(e^\Phi) \rightarrow +\infty$ . In simpler terms,

$$R(e^\Phi) = 2 \cosh(2\Phi') - 2,$$

which is 0 only at  $\Phi' = 0$ , i.e.  $\Phi = \ln \phi$ . This well-defined function is strongly divergent at  $\Phi' \rightarrow \pm\infty$ . This ensures meltdown/freeze remain infinite-cost boundaries.

#### 3.1.1. Defining the Effective Hamiltonian and Partition Function

We next embed  $R(e^\Phi)$  in a continuum *Hamiltonian* or *effective free-energy* functional (or *action* in statistical field-theory language):

$$\mathcal{H}[\Phi] = \int_{\Omega} \left\{ \frac{\kappa}{2} |\nabla \Phi(\mathbf{x})|^2 + 2 \cosh(2(\Phi(\mathbf{x}) - \ln \phi)) - 2 \right\} d^d x, \quad (8)$$

where  $\kappa > 0$  is a gradient stiffness term (analogous to  $\kappa |\nabla \alpha|^2$  in the local PDE approach), and  $-2$  is a constant offset chosen so that  $R(e^\Phi) = 0$  at  $\Phi = \ln \phi$ . The partition function is thus

$$Z = \int \mathcal{D}\Phi(\mathbf{x}) \exp[-\mathcal{H}[\Phi]], \quad (9)$$

where the domain of integration for  $\Phi(\mathbf{x})$  is now *all real values*  $\Phi : \Omega \rightarrow (-\infty, +\infty)$ , reflecting  $\alpha(\mathbf{x}) > 0$ .

**Interpretation of Boundaries:** Meltdown or freeze appear if  $\Phi(\mathbf{x}) \rightarrow \pm\infty$  anywhere in space, which yields a *locally infinite* cost in  $2 \cosh(2(\Phi - \ln \phi))$ . Hence such configurations are heavily suppressed in  $Z$ , effectively forbidding meltdown/freeze at finite volume.

#### 3.1.2. Momentum-Shell Decomposition for a Wilsonian RG

Following the standard Wilsonian RG approach, we split:

$$\Phi(\mathbf{x}) = \Phi_{<}(\mathbf{x}) + \Phi_{>}(\mathbf{x}),$$

where  $\Phi_{>}$  contains fast (short-wavelength) modes in the momentum shell  $\Lambda/b < |\mathbf{k}| < \Lambda$  and  $\Phi_{<}$  contains slow modes in  $|\mathbf{k}| \leq \Lambda/b$ . Here  $\Lambda$  is an initial UV cutoff (e.g. lattice or Kolmogorov scale), and  $b > 1$  is the RG scaling factor. Then:

$$Z = \int \mathcal{D}\Phi_{<} \mathcal{D}\Phi_{>} \exp[-\mathcal{H}(\Phi_{<} + \Phi_{>})]. \quad (10)$$

We **integrate out**  $\Phi_{>}(\mathbf{x})$  to obtain an *effective Hamiltonian*  $\mathcal{H}'[\Phi_{<}]$ , so that

$$Z = \int \mathcal{D}\Phi_{<} \exp[-\mathcal{H}'(\Phi_{<})], \quad \text{where} \quad \exp[-\mathcal{H}'(\Phi_{<})] = \int \mathcal{D}\Phi_{>} \exp[-\mathcal{H}(\Phi_{<} + \Phi_{>})]. \quad (11)$$

**Why Nontrivial?** The original potential term  $2 \cosh(2(\Phi - \ln \phi)) - 2$  is *exponential* in  $\Phi$ . Expanding around small  $\Phi_>$  can be done if  $\Phi_<$  is close to  $\ln \phi$ , but meltdown/freeze correspond to  $\Phi \rightarrow \pm\infty$ , making the potential blow up. Thus, meltdown/freeze remain strongly suppressed in the path integral, but capturing them *exactly* is non-perturbative.

### 3.2. Local Expansion Around the Interior Minimum

To proceed in a quasi-perturbative sense, we **assume** that in typical configurations, the slow field  $\Phi_<(\mathbf{x})$  is near the stable point  $\ln \phi$ . Then define

$$\Phi_<(\mathbf{x}) = \ln \phi + \varphi_<(\mathbf{x}), \quad \Phi_>(\mathbf{x}) = \varphi_>(\mathbf{x}).$$

Hence  $\Phi(\mathbf{x}) - \ln \phi \equiv \varphi(\mathbf{x})$ . We expand the potential about  $\varphi = 0$ . From Eq. (7),

$$2 \cosh(2 \varphi(\mathbf{x})) - 2 = 2 \left[ \cosh(2\varphi) - 1 \right].$$

A standard cosh expansion yields

$$\cosh(2\varphi) = 1 + 2\varphi^2 + \frac{(2\varphi)^4}{4!} + \dots = 1 + 2\varphi^2 + \frac{2^4 \varphi^4}{4!} + \dots$$

Thus

$$2 \cosh(2\varphi) - 2 = 2 \left[ 1 + 2\varphi^2 + \frac{(2\varphi)^4}{4!} + \dots - 1 \right] = 4\varphi^2 + \frac{16}{4!} \varphi^4 + \dots \quad (12)$$

leading to even-power terms in  $\varphi$ . That is reminiscent of a  $\varphi^2, \varphi^4, \dots$  expansion around  $\varphi = 0$ , except with specific coefficients from the cosh series.

#### Implications:

1. The  $\varphi^2$  term sets a positive curvature at  $\varphi = 0$ .
2. Higher-order  $\varphi^{2n}$  terms appear, ensuring the potential grows exponentially for large  $|\varphi|$ , i.e. meltdown ( $\varphi \rightarrow -\infty$ ) or freeze ( $\varphi \rightarrow +\infty$ ).

Hence the “bulk” of typical fluctuations near  $\alpha = \phi$  can be treated with a polynomial-like expansion. Meltdown/freeze remain in the far tails  $\varphi \rightarrow \pm\infty$ , each imposing infinite cost.

### 3.3. Running Couplings and Beta Functions

We can write the Hamiltonian in momentum space:

$$\mathcal{H}[\varphi] = \int_{|\mathbf{k}| \leq \Lambda} \left[ \frac{\kappa}{2} |\mathbf{k}|^2 |\varphi(\mathbf{k})|^2 + V_{\text{int}}(\varphi(\mathbf{x})) \right] d^d k,$$

with  $V_{\text{int}}(\varphi)$  capturing the expansions in  $\varphi^2, \varphi^4, \dots$ . The meltdown/freeze boundaries appear as  $\varphi \rightarrow \pm\infty$ . In a Wilsonian step from  $\Lambda$  down to  $\Lambda/b$ , we decompose  $\varphi = \varphi_< + \varphi_>$ . We integrate out  $\varphi_>(\mathbf{k})$  for  $\Lambda/b < |\mathbf{k}| < \Lambda$ . The resulting effective Hamiltonian  $H'(\varphi_<)$  inherits new interactions from that integration, giving us **flow equations** ( $\beta$ -functions) for each coupling.

- **Relevant, marginal, or irrelevant** couplings are classified in standard field-theory ways. The meltdown/freeze phenomenon is realized by the exponential shape of  $V_{\text{int}}(\varphi)$  at large  $|\varphi|$ , not a typical polynomial  $\varphi^4$  form.
- **Interior stable fixed point** Because meltdown/freeze remain non-perturbative extremes, the only viable fixed point for the rescaled potential is an *interior* finite  $\Phi$ . By matching the self-similar partition condition and the fact that  $U(\Phi)$  is minimal at  $\Phi = \ln(\phi)$  so  $\alpha = e^\Phi = \phi$ . This is a stable interior fixed point under iteration of the RG transformation.

**Physical Meaning.** Under RG coarse-graining, any local region that tries to meltdown ( $\varphi \rightarrow -\infty$ ) or freeze ( $\varphi \rightarrow +\infty$ ) gains huge cost in  $V_{\text{int}}$ . Consequently, the integrated-out fast modes cannot

push the slow field to meltdown or freeze states. Repeated block-integration steps continue to find meltdown/freeze repulsive, leaving the *interior synergy*  $\alpha = \phi$  as the stable large-scale limit.

### 3.4. Discussion of Boundary Divergences

One must note that  $\phi \rightarrow \pm\infty$  is a *non-perturbative* region. A purely polynomial expansion around  $\phi = 0$  does not capture the entire boundary behavior. However, the extremely steep potential  $2 \cosh(2\phi) - 2$  ensures the measure for large  $|\phi|$  is exponentially suppressed. Thus meltdown/freeze do not significantly affect typical fluctuations near  $\phi = 0$  at each RG stage. The net effect is that meltdown/freeze remain effectively excluded at finite volume, confirming they are *repulsive boundary fixed points*.

Because the meltdown/freeze boundary states remain high cost at every length scale, the Wilsonian RG flow never flows “outward” to  $\pm\infty$ . Instead, it flows to an interior attractor,  $\phi = 0$  or equivalently  $\alpha = \phi$ . This matches our PDE-level gradient-flow result and the local cost function approach. Therefore, from an RG standpoint, meltdown/freeze each act as repulsive boundary fixed points, guaranteeing a *unique stable interior synergy*  $\alpha = \phi$ . Hence, in a *full Wilsonian RG* sense, we confirm that

**Lemma 3.** *Meltdown/freeze boundaries correspond to non-perturbative extremes at  $\Phi \rightarrow \pm\infty$  with infinite cost, while the interior synergy  $\alpha = \phi$  remains the unique stable fixed point.*

This is precisely the more rigorous, field-theoretic restatement of our Dynamic Balance principle using the self-similar cost function  $R(\alpha) = \left(\frac{\alpha}{\phi} - \frac{\phi}{\alpha}\right)^2$ .

## 4. Markov Master Equation

In this section, we provide a purely *discrete* and *nonequilibrium* demonstration of Dynamic Balance, constructing a Markov chain with forbidden boundary states at  $\alpha = 0$  (meltdown) and  $\alpha \rightarrow \infty$  (freeze). We show that the system’s nonequilibrium steady-state distribution concentrates in the interior at  $\alpha \approx \phi$ . This approach complements the continuous ODE/PDE + RG arguments (Sec. 2,3) by explicitly encoding open-system transitions and partial meltdown “avalanches.”

### 4.1. Step 1: Discretizing the $\alpha$ Variable

We replace the continuum  $\alpha \in (0, \infty)$  with a finite or countably infinite set of levels:

$$\{\alpha_1, \alpha_2, \dots, \alpha_N\},$$

where

$$0 < \alpha_1 < \alpha_2 < \dots < \alpha_N < \infty.$$

If needed, one can imagine letting  $\alpha_1 \approx 0$  and  $\alpha_N$  be very large to approximate  $\infty$ . Later,  $N \rightarrow \infty$  can recover a dense spectrum.

At time  $t$ , the system “occupies” exactly one level  $\alpha_i$ . Let

$$P_i(t) = \text{Prob}(\alpha(t) = \alpha_i).$$

Hence  $\sum_{i=1}^N P_i(t) = 1$ . The system evolves according to a Markov chain or master equation, capturing *open nonequilibrium* transitions.

### 4.2. Step 2: Forbidding Boundary States

To impose Dynamic Balance, we define  $\alpha_1 \approx 0$  and  $\alpha_N \gg 1$  as “boundary states” with *infinitely large cost*. In Markov terms, we can either:

- **Disallow transitions** into  $\alpha_1$  or  $\alpha_N$  from any interior state,

- or assign **exponentially small** rates going to  $\alpha_1$  or  $\alpha_N$  so that their steady-state occupation is effectively zero.

Thus, meltdown ( $\alpha_1$ ) and freeze ( $\alpha_N$ ) remain effectively unoccupied in the long-term distribution. No matter how the chain evolves internally, it *cannot remain* in meltdown or freeze at steady state. This mimics the boundary divergences in the PDE cost function approach.

#### 4.3. Step 3: Transition Rates for an Open-Driven System

Define a transition-rate matrix

$$W_{i \rightarrow j} \geq 0, \quad (i \neq j), \quad \text{with} \quad \sum_{j \neq i} W_{i \rightarrow j} < \infty.$$

The master equation is:

$$\frac{dP_i}{dt} = \sum_{j=1}^N [W_{j \rightarrow i} P_j - W_{i \rightarrow j} P_i], \quad \sum_{i=1}^N P_i = 1. \quad (13)$$

We specify these rates to reflect *slow external drive* (pushing  $\alpha$  upward) and *partial meltdown topplings* (pushing  $\alpha$  downward if  $\alpha$  becomes too large).

##### Slow Drive.

We let

$$W_{i \rightarrow i+1} = \epsilon > 0, \quad i = 1, \dots, N-1,$$

representing a small *injection* that moves  $\alpha_i$  one step upward. This “drive” ensures the chain does not collapse at lower levels. For  $i = N$ , we disallow  $i \rightarrow i+1$  transitions as that would correspond to freeze  $\alpha_{N+1} = \infty$ . Physically, this models new energy input raising the system’s ratio from  $\alpha_i$  to  $\alpha_{i+1}$ .

##### Threshold Meltdown (Downward Jumps).

When  $\alpha_i$  is above a threshold index  $i_{\text{thres}}$ , meltdown “avalanches” occur, where the system can drop multiple level:

$$W_{i \rightarrow i-m} = \nu_m > 0, \quad \text{for } i > i_{\text{thres}}.$$

This can be single-step ( $m = 1$ ) or multi-step topplings. If  $i > i_{\text{thres}}$ ,  $\alpha_i > \alpha_{i_{\text{thres}}}$ , the chain partially “melts down” to  $\alpha_{i-m}$ , reflecting local threshold feedback that counters freeze.

##### Forbidding Boundaries.

We finalize by setting

$$W_{k \rightarrow 1} = 0, \quad W_{k \rightarrow N} = 0, \quad \text{for all } k = 2, \dots, N-1,$$

ensuring no transitions into  $\alpha_1$  or  $\alpha_N$ .

The slow upward  $\epsilon$  ensures  $\alpha$  doesn’t get stuck at low levels, while meltdown rates  $\nu_m$  keep it from drifting too high. This interplay typically yields a unique interior peak in  $P_i(\infty)$ .

#### 4.4. Step 4: Steady-State Distribution and $\alpha \approx \phi$

Solving  $\dot{P}_i = 0$ .

The chain’s steady state satisfies

$$\frac{dP_i}{dt} = 0, \quad \text{for all } i = 2, \dots, N-1,$$

with  $P_1(\infty) = P_N(\infty) = 0$ . Balanced flow among interior states enforces

$$\sum_{j \neq i} j (W_{j \rightarrow i} P_j^\infty - W_{i \rightarrow j} P_i^\infty) = 0.$$



Depending on the meltdown rates  $\nu_m$  and drive  $\epsilon$ , one typically finds a single maximal probability at some  $i$ , i.e.  $P_i^\infty > \max_{j \neq i^*} P_j^\infty$ .

### Unique Interior Peak.

Since boundary transitions are disallowed,  $P_1(\infty) = P_N(\infty) = 0$ , the chain must dwell in  $\alpha_2, \dots, \alpha_{N-1}$ . Balancing upward slow drive vs. meltdown topplings typically yields a single peak  $i^*$ , with  $\alpha_{i^*}$  the most probable interior state. If we design the rates  $W_{i \rightarrow i+1}$  and  $W_{i \rightarrow i-m}$  to reflect scale invariance (or the partition condition from Sec. 2), it can be shown that  $\alpha_{i^*} \approx \phi$ .

### Scale-Invariant Rates for $\alpha = \phi$ .

To ensure  $\alpha_i \approx \phi$ , we can design the meltdown/drive rates so that the ratio  $E/(TS)$  is partitioned in a golden-ratio sense. Concretely, one might require

$$\frac{W_{i \rightarrow i+1}}{W_{i+1 \rightarrow i}} \approx \phi.$$

- $W_{i \rightarrow i+1} = \epsilon$  for  $i < N$ ,
- $W_{i \rightarrow i-1} = \nu > 0$  for  $i > i_{\text{thres}}$ ,

and choose  $i_{\text{thres}}$  near  $\alpha_{\text{thres}} \approx \phi$ . This ensures that if  $\alpha_i$  exceeds  $\phi$ , meltdown rates dominate, driving it downward; if  $\alpha_i < \phi$ , the slow drive pushes it upward. The Markov chain's stationary distribution  $P_i(\infty)$  clusters around  $i^*$  with  $\alpha_{i^*} \approx \phi$ . More sophisticated meltdown rules (e.g. multiple steps  $m > 1$ ) can yield fractal avalanche bursts but preserve  $\langle \alpha \rangle \approx \phi$ .

Because meltdown/freeze states are effectively unreachable,  $P_1^\infty = P_N^\infty = 0$ . The "bulk" of  $P_i^\infty$  localizes around a single interior index  $i$  with  $\alpha_i \simeq \phi$ . Minor fluctuations might see small probabilities for  $i < i^*$  or  $i > i^*$ , but the strong meltdown/drive feedback ensures that deviations from  $i^*$  remain transient.

This Markov approach thus mirrors the meltdown–freeze boundary seen in our continuous PDE, leading to a most-likely ratio  $\alpha \approx \phi$  in the nonequilibrium steady state.

## 4.5. Step 5: Partial Meltdown Events and Self-Organized Criticality

### Avalanche Statistics.

A key virtue of the Markov chain approach is that meltdown events  $i \rightarrow i - m$  can occur spontaneously whenever  $i > i_{\text{thres}}$ . Over many cycles, these downward jumps can be of various magnitudes  $m$ . Provided the system is slowly driven upward and meltdown occurs in threshold bursts, the meltdown sizes can follow a power-law or heavy-tailed distribution—i.e. typical SOC avalanche behavior [20].

### $\alpha \approx \phi$ Despite Fractals.

Notably, even though meltdown events lead to fractal or wide-scale fluctuations, the *global* ratio  $E/(TS)$  in a steady sense hovers around  $\phi$ . The boundaries still remain infinite cost, so the Markov chain cannot drift to  $i = 1$  or  $i = N$ . This *self-organized* partial meltdown stabilizes  $\alpha \approx \phi$  while producing fractal avalanche dynamics in the interior states.

We have explicitly shown how Dynamic Balance is implemented in a Markov chain with:

- forbidden or negligible boundary transitions ( $\alpha_1 \approx 0, \alpha_N \approx \infty$ ),
- slow upward drive from  $\alpha_i \rightarrow \alpha_{i+1}$ ,
- partial meltdown (avalanches) from  $\alpha_i \rightarrow \alpha_{i-m}$  if  $i > i_{\text{thres}}$ .

The resulting steady state does *not* occupy the forbidden boundaries, but resides at an interior  $\alpha_{i^*} \approx \phi$ , with meltdown bursts distributed in a fractal or SOC-like manner. Crucially, the *average* or *most likely*  $\alpha$  remains near  $\phi$ . The system hovers around that ratio, with occasional meltdown bursts that bring it back from partial freeze. For an additional example see Appendix B.

In summary, the discrete Markov chain corroborates the Dynamic Balance principle, showing that  $\alpha \approx \phi$  emerges from a balance of slow drive and threshold resets, even with fractal avalanche statistics.

#### 4.6. Conclusion of Mathematical Proof

We have rigorously shown:

1. **Boundary Perspective:** By discretizing  $\alpha$  into states and forbidding meltdown/freeze boundaries, we replicate the same logic from the PDE and cost function frameworks. There is a unique interior index  $i^*$  where  $P_{i^*}^\infty$  peaks, typically at  $\alpha_{i^*} \approx \phi$ . Setting transition rates to zero (or extremely small) into  $\alpha_1 \approx 0$  and  $\alpha_N \approx \infty$  means meltdown/freeze remain unoccupied at steady state.
2. **SOC Mechanism:** In a purely stable scenario,  $\alpha = \phi$  would be a normal minimum. By adding slow drive and threshold meltdown “avalanches,” we obtain *self-organized criticality*, turning  $\alpha = \phi$  into a *marginal* or near-critical point.
3. **Fractal / Power-Law Phenomena:** The resulting nonequilibrium steady state exhibits fractal avalanches on all scales, consistent with observed scale-free data in biological, turbulent, or quantum systems. Meanwhile, meltdown/freeze remain infinite-cost boundaries in the RG sense, ensuring  $\alpha(\mathbf{x}, t) \in (0, \infty)$  never saturates nor vanishes.

In the following sections (Sec. 5, 6), we turn to empirical examples, each showing exponents or ratios near 1.6 in real nonequilibrium open systems.

### 5. The Brain as an Open NESS Under a *Dynamic Balance* Constraints

The human brain, though only 2% of body mass, consumes almost 20% of resting metabolic energy—roughly 20 W [21,22]. Empirical measurements show a consistent partition: about 60% of this energy supports *active* neural signaling (action potentials, synaptic recycling), while 40% funds *housekeeping* (resting potentials, basic cellular maintenance) [22,23]. This stable balance reflects a deeper thermodynamic tension: the brain must maintain high excitability for complex computation while avoiding excess excitatory activity (unbounded meltdown) or complete inactivity (freeze).

Meanwhile, scale-free or **near-critical properties** pervade cortical organization. Experiments reveal neuronal avalanches with power-law size distributions [24,25], suggestive of self-organized criticality (SOC). These near-critical states confer broad dynamic range, heightened sensitivity, and optimized information processing [26,27]. At the anatomical level, **dendritic** and **cortical connectivity** exhibit fractal or small-world topologies [28,29], supporting efficient long-range communication under metabolic constraints. Lastly, **multi-frequency oscillations** (delta to gamma) underlie cognitive processes [30], with cross-frequency couplings (CFC) sometimes clustering near *incommensurate* ratios—empirically close to the golden ratio  $\phi \approx 1.618$  [31,32]—possibly maximizing coherence across scales [33].

Taken together, these observations hint at an underlying **nonequilibrium principle** balancing energy usage, fractal connectivity, critical avalanches, and multi-frequency rhythms. Below, we embed a *Dynamic Balance* constraint into a neural PDE, offering a thermodynamic rationale for how the brain remains near critical thresholds yet avoids catastrophic extremes.

#### 5.1. PDE for Neural Dynamics

We first recall a prototypical Wilson–Cowan PDE neural-field approach [34], where excitatory ( $E$ ) and inhibitory ( $I$ ) populations evolve via:

$$\frac{\partial E}{\partial t} = D_E \nabla^2 E - \frac{E}{\tau_E} + S_E(w_{EE} E - w_{EI} I + P_E), \quad (14)$$

$$\frac{\partial I}{\partial t} = D_I \nabla^2 I - \frac{I}{\tau_I} + S_I(w_{IE} E - w_{II} I + P_I). \quad (15)$$

Here,  $D_E, D_I > 0$  represent synaptic or lateral diffusion-like connectivity,  $\tau_E, \tau_I$  are decay constants, and  $S_E(\cdot), S_I(\cdot)$  are typical saturating nonlinearities. While (14)–(15) can exhibit waves, bumps, or oscillations, it still relies on **ad hoc** saturations terms to prevent runaway excitations.

To provide a thermodynamic open-system interpretation, we introduce our *dynamic balance* cost function that keeps  $E$  and  $I$  in a *self-similar* partition. Let

$$\alpha(x, t) = \frac{E(x, t)}{I(x, t) + \varepsilon}, \quad \varepsilon \ll 1$$

and impose the golden ratio,

$$\alpha(x, t) \approx \phi \approx 1.618,$$

via a penalty function,

$$R(\alpha) = \left( \frac{\alpha}{\phi} - \frac{\phi}{\alpha} \right)^2.$$

When  $\alpha = 0$  (meltdown, or excitatory collapse) or  $\alpha \rightarrow \infty$  (freeze, unbounded  $E$  with negligible  $I$ ), we get  $R(\alpha) \rightarrow \infty$ , thus disallowing such extremes. We then add the negative gradient of  $R(\alpha)$  to (14)–(15):

$$\text{PDE: } \frac{\partial E}{\partial t} = D_E \nabla^2 E + F_E(E, I) - \Gamma_E \frac{\partial R}{\partial E}, \quad (16)$$

$$\frac{\partial I}{\partial t} = D_I \nabla^2 I + F_I(E, I) - \Gamma_I \frac{\partial R}{\partial I}, \quad (17)$$

$$\text{Boundaries: e.g., periodic or Neumann on } \partial\Omega, \quad (18)$$

$$\text{Initial: } E(x, 0) = E_0^*(x) + \delta E_0(x), \quad I(x, 0) = I_0^*(x) + \delta I_0(x), \quad (19)$$

where  $\Gamma_E, \Gamma_I > 0$  control meltdown–freeze feedback strengths, and  $\mathcal{F}_E(E, I), \mathcal{F}_I(E, I)$  denote local reaction or firing-rate terms. Substituting  $\alpha = E/(I + \varepsilon)$  into

$$\frac{\partial R}{\partial E} = \frac{\partial R}{\partial \alpha} \frac{\partial \alpha}{\partial E}, \quad \frac{\partial R}{\partial I} = \frac{\partial R}{\partial \alpha} \frac{\partial \alpha}{\partial I},$$

produces explicit PDE terms that enforce  $\alpha \approx \phi$ . Note that while  $\mathcal{F}_I, \mathcal{F}_E$  describe *local* interactions,  $R(\alpha)$  sets a *global* thermodynamic constraint, unifying both scales.

**Lemma 4.** *Unlike typical Wilson–Cowan saturations, our argument emerges from a scale-invariance partition argument: “the ratio  $E : I$  at each scale must match the ratio  $I : (E - I)$ .” Solving that self-similarity yields  $\phi$ . Thus,  $\phi$  is not a guess but the unique positive solution to  $\alpha^2 = \alpha + 1$ . This unifies the energy usage (excitatory) with inhibitory (entropic) damping in an open, nonequilibrium framework.*

When  $\Gamma_E = \Gamma_I = 0$ , the model reduces to the classical neural PDE. Thus, dynamic balance extends rather than replaces Wilson–Cowan, embedding a physically motivated ratio constraint to unify  $E - I$  regulation under a thermodynamic principle.

## 5.2. Local Stability and Near-Critical Avalanches

A uniform solution  $(E^*, I^*)$  with  $\alpha^* = E^*/(I^* + \varepsilon) = \phi$  is stable if small perturbations do not diverge. Linearizing (16)–(17) around  $(E, I) = (E^*, I^*) + (\delta E, \delta I)$  yields

$$\frac{\partial}{\partial t} \begin{pmatrix} \delta E \\ \delta I \end{pmatrix} = \underbrace{\begin{pmatrix} D_E \nabla^2 + \frac{1}{\tau_E} \frac{\partial F_E}{\partial E} & \frac{1}{\tau_E} \frac{\partial F_E}{\partial I} \\ \frac{1}{\tau_I} \frac{\partial F_I}{\partial E} & D_I \nabla^2 + \frac{1}{\tau_I} \frac{\partial F_I}{\partial I} \end{pmatrix}}_J \begin{pmatrix} \delta E \\ \delta I \end{pmatrix} + \dots, \quad (20)$$

where  $\mathbf{J}$  is the Jacobian evaluated at  $(E^*, I^*)$ . The derivative portion of  $R$  contributes:

$$-\Gamma_E \left. \frac{\partial^2 R}{\partial E^2} \right|_* \delta E - \Gamma_E \left. \frac{\partial^2 R}{\partial E \partial I} \right|_* \delta I,$$

and similarly for  $I$ . Because  $dR/d\alpha = 0$  at  $\alpha = \phi$ , second derivatives ( $d^2R/d\alpha^2$ ) come into play.

$$\left. \frac{\partial^2 R}{\partial E^2} \right|_{\alpha=\phi} = \left. \frac{d^2 R}{d\alpha^2} \right|_{\alpha=\phi} \left( \frac{\partial \alpha}{\partial E} \right)^2 + \dots$$

Since  $d^2R/d\alpha^2|_{\alpha=\phi} > 0$ , Dynamic Balance indeed imposes a *convex* penalty around  $\alpha = \phi$ . This imposes strong negative feedback against meltdown and freeze.

Detailed matrix analysis reveal negative real parts in  $\mathbf{J}$ 's eigenvalues under typical positivity conditions for  $\Gamma_E, \Gamma_I$ . Hence the ratio  $\alpha$  is locally stable, preventing spontaneous divergence unless external forcing or parameter changes exceed meltdown thresholds.

### Avalanche Onset.

When a parameter  $\nu = w_{EE} - w_{EI}$  crosses a critical value  $\nu = 0$ , one eigenvalue may approach zero, indicating a meltdown boundary [27]. Close to this threshold, the system exhibits **power-law** avalanche expansions with exponent  $\tau \approx 1.5$  and branching ratio  $\alpha_t \approx 2$  appear [24,26]. Crucially, the system can self-organize near meltdown without finely tuned saturations, naturally exhibiting near-critical neural avalanches.

### 5.3. Amplitude Equations and Fractal Wave Expansions

A bifurcation occurs when the real part of an eigenvalue crosses zero, changing stability or leading to emergent dynamics such as patterns or oscillations. If  $\nu \gtrsim 0$  triggers temporal (Hopf) or spatial (Turing) instabilities, standard weakly nonlinear analysis yields a Ginzburg–Landau–type amplitude PDE or real Swift–Hohenberg form. We expand  $E = E^* + \sqrt{\mu} A e^{ik_c x} + \dots$  (Turing) or  $E = E^* + \sqrt{\mu} A e^{i\omega_0 t} + \dots$  (Hopf), collecting terms of order  $\mu^{3/2}$ , etc. The Dynamic Balance ratio derivatives appear at nonlinear orders, saturating amplitude.

$$\frac{\partial A}{\partial T} = \mu A + D \frac{\partial^2 A}{\partial X^2} - \gamma |A|^2 A, \quad (21)$$

which robustly yields universal power-law exponents for avalanche sizes and durations, as well as fractal scaling near criticality. The meltdown–freeze penalty contributes a robust **nonlinear saturation** that keeps wave amplitudes finite. As a result, traveling pulses, waves, or bumps can form “naturally” under physically grounded constraints.

### Turing (Pattern) Instability.

When  $\text{Re}(\lambda(k_c)) = 0$  at some  $k_c \neq 0$  spatial structures emerges (e.g., traveling waves or Turing patterns). Different diffusion rates or strong cross-coupling often trigger such instabilities, which can describe wave-like patterns in cortical tissue.

### Hopf (Oscillatory) Bifurcation.

If a purely imaginary eigenvalue  $\lambda = i\omega$  emerges at  $k = 0$  (or some  $k \neq 0$ ), the system transitions to limit cycle oscillations, reflecting neural rhythms. The boundary penalty modifies the real parts of eigenvalues, preventing meltdown or freeze at large amplitude. Hopf bifurcation explains neural rhythms, and emergent oscillatory phenomena (e.g., alpha, theta, gamma waves).

### Traveling Waves and Fractal Splitting.

In certain neural field models, activity propagates as *traveling waves*—coherent wavefronts that move across the domain at roughly constant shape and speed. Standard Wilson–Cowan or conductance-

based PDEs can exhibit such waves if excitatory and inhibitory interactions are well tuned. When a traveling wave's amplitude grows large enough to risk "meltdown" (unbounded  $E$ ), the feedback penalizes further growth. Instead of forming a single big wave, the wavefront breaks into multiple sub-waves, each with a reduced amplitude ratio near  $\phi^{-1} \approx 0.618$  times the original. Physically, the meltdown-freeze mechanism "pushes down" local peaks that exceed  $\alpha = \phi$ , forcing partial wave splitting rather than a single high-amplitude pulse.

Once a wavefront splits into sub-waves, each sub-wave can itself reach meltdown thresholds further downstream, causing a second-level split, and so on. This repeated *wave-splitting* process is *scale-invariant*, because the same penalty applies at every amplitude scale. Consequently, the wave expansions become *self-similar* across multiple levels of splitting, naturally creating *fractal* wave clusters. Standard geometric arguments show that if each wave amplitude is reduced by a factor near  $\phi^{-1}$  at each "generation," the final pattern often has a fractal dimension  $d_f$  close to 1.6, reflecting the golden ratio. Experimental work in neural systems (and other nonequilibrium media) sometimes measures fractal-like clusters or avalanche expansions with a dimension around 1.5–1.7.

#### 5.4. Morphological Growth and Dendritic Branching

The same principle extends to *structural* fractals—e.g., dendritic trees or vascular branching—if we reinterpret  $(E, I)$  as  $(G, R)$ , i.e. growth extension vs. inhibitory resource. In that view:

1. **Growth Force,  $G$ :** A variable tracking how far neurites or vascular sprouts have extended.
2. **Resource,  $R$ :** A limiting factor (nutrients, metabolic supply) that must not be exhausted.
3.  **$\alpha$  Ratio:** If  $G/(R + \varepsilon)$  tries to exceed  $\phi$ , meltdown feedback stops further extension.

When  $G$  surpasses a local threshold of roughly  $\phi R$ , the growth front "splits" into two or more sub-branches each with amplitude  $\phi^{-1}$  times the original. Repeating this splitting at every scale over  $n$  generations at scale factor  $L/\phi^n$  yields a *self-similar tree*, whose fractal dimension  $d_f \approx 1.6$ . This direct geometric approach aligns with data showing dendritic arbors or vascular networks often have fractal dimensions in the 1.5–1.7 range. Crucially, *no ad hoc shape constraints or saturations* are needed—only the energy–entropy balance function  $\alpha \approx \phi$ , which emerges from scale-invariance in energy vs. resource partitioning.

#### Summary:

Together, the wave-splitting phenomenon and morphological branching demonstrate how the *energy-entropy ratio*  $\alpha = \phi$  can drive both *functional* (traveling wave) and *structural* (neurite or vascular) fractal expansions. In each case, local feedback stops runaway growth or excitatory meltdown, causing partial splitting at scale factor  $\phi^{-1}$ . Iterating this splitting across multiple scales yields fractal dimensions near 1.6—matching a host of neural and biological fractal measurements, and reinforcing the idea that the golden ratio arises from a universal self-similarity condition rather than arbitrary choice.

#### 5.5. Short-Term Plasticity and Cross-Frequency Coupling

##### Short-Term Synaptic Plasticity (STP).

In neural circuits, excitatory synapses often undergo resource depletion or facilitation, described by additional variables  $x, u$  representing [35,36], for instance, the fraction of available vesicles (release probability) and/or fraction of usable resources. [37]. Embedding these in (16)–(17) modifies  $w_{EE} \rightarrow w_{EE}(x, u)$ , but the dynamic balance constraints remain intact. Multiple-timescale expansions confirm that meltdown is averted even under strong facilitation, with  $\alpha \approx \phi$  continuing to hold on short times. The Dynamic Balance cost function ensures  $\alpha \approx \phi$ , avoiding *ad hoc* upper bounds on  $E$ . STP now complements that feedback by adding realistic resource depletion/facilitation. Typically,  $\tau_{\text{rec}}, \tau_{\text{fac}}$  might be tens or hundreds of milliseconds. The Dynamic Balance PDE still acts on  $\tau_E \sim 10\text{--}20$  ms. Both fast processes can coexist with Dynamic Balance feedback, ensuring stable short-term synaptic and ratio regulation.



Real cortical responses to brief stimuli often show *initial facilitation* followed by *depression*, or vice versa [36]. In Dynamic Balance PDE simulations, these STP variables can produce “band-pass” or “pulse-resonant” transients, while  $\alpha \approx \phi$  ensures no runaways. The combined effect is a *more biologically realistic* spatiotemporal response profile.

### Golden-Ratio Frequencies.

Empirical EEG data reports alpha–beta–gamma bands near  $\phi$  frequency spacing or cross-frequency couplings (CFC) [31,32]. In dynamic balance PDE expansions near Hopf onsets, rational frequency locking is penalized, so multi-band oscillations remain near incommensurate ratios.  $\phi$ —the “most irrational number”—naturally emerges as a stable incommensurate partition, explaining why certain cross-frequency couplings peak at or near  $\phi$ .

### 5.6. Universality of Dynamic Balance PDE

In these example, we used our Dynamic Balance formulation in the context of *rate-based* or *Wilson–Cowan–style* neural fields. However, **similar principles can be embedded into conductance-based (spiking) PDEs**—for example, those governed by Hodgkin–Huxley (HH) or FitzHugh–Nagumo (FHN) equations. These more detailed biophysical models incorporate ion-channel dynamics and membrane conductances, permitting phenomena such as action potentials, bursting, and wave propagation [38]. To incorporate Dynamic Balance, we designate an *inhibitory gating* or “cooling” variable  $V_I$ , and an *excitatory variable*  $V_E$ . By embedding the Dynamic Balance ratio

$$\alpha = \frac{V_E}{V_I + \varepsilon} \quad \text{or} \quad \alpha = \frac{E_{\text{cond}}}{I_{\text{cond}} + \varepsilon},$$

into HH/FHN PDEs, we eliminate the requirement for artificially capping  $V$  or synaptic currents, and refine known wave solutions (e.g. spirals, pulses), that are then stabilized by a thermodynamic boundary condition which prevents runaway.

### 5.7. A Unified Brain PDE

Summarizing, the Dynamic Balance PDE

$$\frac{\partial E}{\partial t} = D_E \nabla^2 E + \mathcal{F}_E(E, I) - \Gamma_E \frac{\partial R(\alpha)}{\partial E}, \quad \frac{\partial I}{\partial t} = D_I \nabla^2 I + \mathcal{F}_I(E, I) - \Gamma_I \frac{\partial R(\alpha)}{\partial I},$$

where  $\alpha = \frac{E}{I + \varepsilon}$  and  $R(\alpha) = \left( \frac{\alpha}{\phi} - \frac{\phi}{\alpha} \right)^2$  unifies:

- **Millisecond- to second-scale** short-term plasticity,
- **Spiking-level** Hodgkin–Huxley or FitzHugh–Nagumo PDEs,
- **Large-scale connectivity** in integral (Amari-style) fields,
- **Morphological fractals** for dendritic/vascular branching,
- **Higher-order synaptic plasticity** (BCM, STDP) for long-term learning,
- **Layered or multi-area brain states** bridging local ratio constraints with global cognition,
- **Renormalization group** expansions clarifying avalanche exponents.

All these claims follow from standard PDE expansions (linear stability, amplitude equations, fractal wave geometry). Thus the dynamic balance PDE can replace ad hoc saturations in neural fields with a physically motivated open-system ratio  $\alpha \approx \phi$ , bridging observed metabolic partition (60:40), critical avalanches, fractal geometry, and multi-frequency couplings in one universal principle.

Although mean-field branching exponents ( $\tau \approx 1.5$ ,  $\alpha_t \approx 2$ ) successfully describe basic avalanche size and duration statistics in the brain, the empirical findings of fractal wave expansions, CFC, and quasi-incommensurate multi-band neural oscillations transcend this narrative, and better align with the Dynamic Balance perspective of an open driven-dissipative system—in which small-scale feedback deters complete runaway excitation or rigid lock.

Moreover, recent findings across a range of neurological conditions– including anxiety, depression, multiple sclerosis, Alzheimer’s disease, Parkinson’s disease– support the view that these **brain dysfunction often involves breakdown in metabolic-thermodynamic balance**. Neuro-degenerative disorders (MS, AD, PD) each display characteristic failures in oxidative phosphorylation or glucose metabolism, which in turns leads to entropy building-up (e.g. oxidative stress, damaged myelin, and amyloid/tau pathology). PET and fMRI studies further reveal that once energy supply cannot keep up with the brain’s demand, network complexity and functional entropy collapse. These observations–ranging from metabolic deficit in demyelinated axons of MS, to "energy-starved" and low-entropy cortex in AD, to dopanime-depleted circuits in PD that squander energy–underscore the need for restoring energy throughput and reduce entropic waste in neurons. Future research could systematically quantify how  $\alpha$  varies in these diseases–before and after treatments such as insulin-sensitizers (for MS), or dopaminergic agents (for PD)– to better understand and potentially harness the brains thermodynamic constraints for clinical benefits [21–23,27,39–43].

5.8. Conclusions of NeuroDynamics Section

The brain is an open, driven, and dissipative system: it constantly receives nutrients (such as glucose and oxygen) and radiates heat. Its hierarchical, modular organization—featuring repeated network motifs and fractal-like vascular branching—can exhibit self-similar or scale-invariant patterns. In such iterative feedback loops, the golden ratio often emerges as a fixed point. Consequently, if energy demands and resource distribution replicate across multiple scales (from subcellular processes to large-scale connectivity), one might observe near-golden-ratio partitions in the system’s metabolic usage.

*Dynamic Balance* provides a self-consistent mechanism for maintaining  $\alpha \approx \phi$ , avoiding the extreme boundary states common in classical neural PDEs without fine-tuning saturations. This approach unifies local excitation–inhibition models with a global thermodynamic principle, yielding fractal or near-critical phenomena that align with real-world neural observations—both in *functional* activity (wave propagation, avalanches) and *structural* growth (branching morphologies).

6. Discussions

We now show how the golden ratio ( $\phi$ ) emerges naturally in multiple other domains, reinforcing its role as an optimal attractor in *open, driven-dissipative systems* in a *nonequilibrium steady-state*. Below we list several (though not all) open systems where observations near  $\phi \approx 1.618$  have been reported. Each item briefly describes how *meltdown* ( $\alpha = 0$ ) and *freeze* ( $\alpha \rightarrow \infty$ ) might appear, and cites references indicating numerical or geometric exponents close to  $\phi$ .

System	Observed Ratio/Exponent	Value	Refs
Phyllotaxis	Divergence angle	$\approx 137.5^\circ = 360^\circ(1 - 1/\phi)$	[1,2,4]
Branching	Power-law exp.	1.5 – 1.6	[44–47]
Neural avalanches	Avalanche exp.	1.5–1.6	[16,17,25,48]
Rotating turb.	Energy spectrum	$-1.6 \pm 0.05$	[7,8,49]
Hurricanes	Pitch angle	$30^\circ\text{--}35^\circ \approx \arctan(1/\phi)$	[6,50]
Galactic spirals	Pitch angle	$30^\circ\text{--}35^\circ \approx \arctan(1/\phi)$	[5,51]
Cosmic Web	Fractal dimension	$D_f \approx 1.6$	[52,53]
KIC 5520878	Thermal mode ratio	$f_2/f_1 \approx 1.618$	[19]
Penrose Tiling	Tiling spacing	$\phi$	[54–58]
Ising $E_8$	Mass ratio	$1.618 \pm 0.01$	[10]
Fibonacci anyons	Quantum dimension	$d_\tau = \phi$	[15,59]

Each of these examples has been discussed in the literature as either *observing* or *theorizing* that  $\phi$  emerges as an *optimal* or *critical-like* ratio. In our framework, this is no coincidence:

This balance optimizes both stability and efficiency in energy use, preventing the NESS system from falling into excessive disorder or excessive rigidity.

### 6.1. Astrophysics – Variable Stars

Recent analyses of Kepler space telescope data revealed that certain **variable stars** (“golden” RR Lyrae stars) pulsate with two dominant frequencies whose ratio approaches  $\phi \approx 1.618$  [19]. One notable case is the star KIC 5520878, which alternates between two thermal oscillation modes in *quasiperiodic* fashion. The ratio  $f_2/f_1 \approx \phi$  ensures that the star’s brightness variations form a **strange nonchaotic attractor**: the amplitude and phase display a fractal structure yet do not exhibit exponential sensitivity to initial conditions (the signature of full chaos) [19]. Consequently, the star retains a long-lived, stable *quasiperiodic* pattern rather than succumbing to pure periodic lock (“freeze”) or disordered chaos (“meltdown”).

If the two pulsation modes were *commensurate* (e.g. integer or rational ratio), the star’s amplitude might freeze into strictly periodic expansions and contractions. Conversely, if the mode coupling were too strong or lacked inhibitory feedback, one mode could amplify uncontrollably, risking chaotic or irregular meltdown. Observationally, KIC 5520878’s ratio  $f_2/f_1 \approx \phi$  evades such lock or blow-up, instead inhabiting a distinctive “middle ground” of *fractally structured yet nonchaotic* pulsations. This resonates directly with the **Dynamic Balance** notion: the star’s internal thermal and gravitational feedback processes appear to steer it away from meltdown and freeze, locking into a *partially fractal* attractor at the golden ratio.

**Laboratory Analog– Magnetoelastic Ribbon:** Intriguingly, a similar  $\phi$ -based phenomenon was demonstrated in a *controlled* lab experiment [60]. Savage *et al.* drove a magnetoelastic ribbon with two incommensurate frequencies set close to  $\phi$ , observing a **strange nonchaotic attractor** resembling the star’s fractal pulsations. *Neither* full chaotic meltdown in amplitude nor single rational lock occurred. Instead, the forced system maintained incommensurability at  $\phi$ , creating fractal *yet* stable motion. Thus, both the stellar pulsations and the magnetoelastic ribbon data emphasize that  $\phi \approx 1.618$  is a *natural pivot* between locked order and chaotic disorder—fully in line with the meltdown–freeze synergy’s claim that  $\phi$  emerges from feedback-based avoidance of trivial extremes.

### 6.2. Fluid Dynamics – Vortices and Flow Transitions

Fluid flows often exhibit multiple competing scales (eddies, waves) and can self-organize into coherent structures. Some empirical and theoretical studies have reported **golden ratio benchmarks in fluid systems**. Mokry (2008) documented two examples [61]: (1) In wind tunnel experiments with perforated walls, the onset of a particular resonance mode was governed by a critical Mach number equal to the **reciprocal golden ratio** ( $\sim 0.618$ ) – below that, acoustic waves refracted through the walls; above it, total reflection occurred, indicating a sharp transition at a  $\phi$ -based threshold. (2) In simulations and observations of **vortex merger** (like merging atmospheric vortices or aircraft wake vortices), there appears to be a **critical separation distance of about 1.618 times the vortex diameter**: if vortices start closer than that, they merge into one; if farther, they remain separate. This distance rule effectively uses  $\phi$  as a dividing line between two flow regimes (coalescence vs stability). In turbulent cascade modeling, a recent development is the concept of “**Fibonacci turbulence**.” Instead of the usual octave-based energy cascade (each eddy spawning smaller ones of roughly 1/2 its size), researchers have explored cascade models where the scale ratio is  $\phi$ . In one such model, wave modes were spaced such that each mode’s frequency was a Fibonacci number, leading to a geometric progression with ratio  $\approx 1.618$  [62]. Interestingly, this  $\phi$ -based cascade produced an **inverse energy cascade** and other turbulence statistics that differ from classical Kolmogorov theory, hinting that a  $\phi$ -structure might maximize mixing or entropy in some contexts. While this is an active research area, it shows that  $\phi$  is being considered as a **natural scaling factor** in turbulence.

### 6.2.1. Rotating Turbulence

Classical (isotropic, homogeneous) turbulence, as described by Kolmogorov's 1941 theory, predicts a velocity power spectrum scaling  $E(k) \sim k^{-5/3}$  in the inertial range. This arises under the assumption of a constant energy flux  $\epsilon$  across scales and no preferred direction or external forcing [49,63]. However, in many real-world flows—e.g., rotating, stratified, or astrophysical/geophysical turbulence—significant deviations can occur [7,8]. Empirical and numerical studies sometimes report exponents closer to  $-1.6$ , prompting speculation that a deeper optimization or self-organization mechanism might be at play [9].

From the perspective of Dynamic Balance, turbulence can “avoid” complete chaos with no coherent structures (meltdown) and overly laminar, suppressed fluctuations (freeze) states by settling near a critical exponent  $x \approx \phi \approx 1.618$ . This conjecture aligns with the idea that rotating or forced turbulence may reorganize energy cascades in a way that benefits both large-scale coherence (vortices, waves) and small-scale dissipation.

**Exponent near  $\phi$ :** Numerical and experimental studies of *rotating* or *stratified* turbulence often find an energy spectrum  $E(k) \propto k^{-x}$  with  $x \approx 1.6$  instead of Kolmogorov's  $5/3 \approx 1.6667$  [7,8].

**Shell-Model Derivation:** In Appendix C, we sketch a simple shell-model approach, where meltdown would correspond to no coherent energy cascade (the system breaks down to noise), and freeze would be an overly rigid cascade with minimal small-scale fluctuations. By imposing mild penalties on these extremes, we find an *interior exponent*  $x \approx 1.618$ . The cost function PDE effectively sets the slope of the inertial-range spectrum, explaining repeated sightings of  $-1.6$  in rotating/stratified flows. The flow would “choose”  $x \approx \phi$  to optimize between insufficient cascade (too steep, meltdown) and overly rigid flow states (too shallow, freeze).

### 6.2.2. Spiral Galaxies and Hurricanes

Both spiral galaxies and hurricanes often display a logarithmic (log) spiral morphology with a roughly constant pitch angle. In galaxies, differential rotation plus gravitational forces shape the spiral arms [5,51]. Meanwhile, hurricanes (or tropical cyclones) exhibit spiral cloud bands formed by rotating wind systems and moisture convergence around the low-pressure eye [6,50]. Each structure balances opposing tendencies: for galaxies, gravitational pull vs. shear; for hurricanes, inward pressure gradients vs. Coriolis/rotational forces. Interpreted through Dynamic Balance, the systems avoid complete dispersal (no stable vortex or spiral arms) and a rigid or highly collapsed patterns with no swirling pattern, settling into a stable spiral geometry.

## 6.3. Biology – Branching, Fractal Scaling and Phyllotaxis

Living organisms are paragons of dynamic balance, constantly juggling energy intake, utilization for work (growth, maintenance), and entropy export (waste heat, metabolites)[18]. On the level of whole organisms, one could look at **metabolic partitioning**. For instance, humans (and many mammals) expend a portion of energy on basal metabolism (maintaining body temperature, basic organ function) and the rest on activity, growth, reproduction, etc.

### 6.3.1. Fractal Transport Networks

Work by West, Brown, and Enquist [44] and others analyzing vascular or branching structures in biology found scaling exponents near 1.6. River network analyses [47] likewise measure fractal dimensions in 1.5–1.6. These branching networks must partition resources (fluid, nutrients) efficiently while minimizing dissipative costs (pressure drops, conduction losses). If meltdown is unstructured, chaotic branching (excess cost, no coherence), and freeze is rigid, minimal branches (no coverage or adaptability), Dynamic Balance suggests an interior exponent near 1.6.

### 6.3.2. Metabolic Partitioning

A compelling body of literature in microbial, animal, and plant physiology shows that a substantial fraction ( $\sim 60\%$ – $70\%$ ) of energy inflow is inevitably dissipated as *maintenance costs*, with the remaining

~30%–40% channeled into growth, structural buildup, or higher-level functions. This empirical ratio resonates with the *Dynamic Balance*  $\alpha = \phi \approx 1.618$  partition—where  $\frac{TS}{E} \approx 0.618$  and  $\frac{E-TS}{E} \approx 0.382$ —in a real-world biological context.

#### Animal and Brain Studies.

- **Attwell and Laughlin (2001)** [64] provide an energy budget for signaling in the brain, concluding that a majority (on the order of 60%) of glucose/oxygen intake is unavoidably “lost” as heat through essential neuronal processes (action potentials, ion pumps) and baseline *maintenance*, leaving  $\lesssim 40\%$  for higher-order function.
- **Rolfe and Brown (1997)** [65] discuss how mammals allocate their metabolic energy: at least half (often closer to 2/3) goes to protein turnover, ion pumping, and other housekeeping tasks, directly generating heat. They highlight that the “productive” fraction (growth, tissue remodeling) typically remains  $\lesssim 40\%$ .
- **Clarke and Portner (2010)** [66] focus on the evolution of endothermy, noting that warm-blooded animals maintain a high basal metabolic rate, with a large fraction of energy becoming thermal output rather than net structural gain.

#### Microbial Partitioning.

- **Pirt (1965, 1975)** [67,68] introduced the influential “maintenance energy” concept in microbial growth, showing that bacteria or yeast typically devote 50%–70% of substrate energy to basic cellular upkeep (ultimately heat), leaving ~30–50% for biomass production.
- **Herbert (1956)** [69] observed continuous-culture microbes with considerable “maintenance” respiration, again meaning >50% of the input substrate energy converted to heat before net biomass formation.

#### Plant and Crop Productivity.

- **Amthor (1989)** [70] in his treatise on plant respiration found that many crops dissipate well over half of daily photosynthetic gain via maintenance respiration. The leftover fraction ( $\lesssim 40\%$ ) is net gain in biomass or seed yield.
- **Gifford (2003)** [71] similarly emphasizes that maintenance respiration often consumes >50% of the total photosynthate in typical vascular plants, limiting the fraction available for growth or storage organs.

Collectively, these empirical studies from animals, microbes, and plants show compelling evidence that a major share of energy flux (commonly ~60%–70%) is dissipated in “maintenance” (essential functions, eventually manifest as heat), while ~30%–40% remains for structural or organizing processes. Although exact numbers fluctuate across species and conditions, the typical (60:40) ratio *strongly echoes* the fraction of  $\approx 0.618 : 0.382$ .

#### 6.3.3. Phyllotaxis

Many plants arrange their leaves, seeds, or pinecone scales at an angle of  $\approx 137.5^\circ$ , corresponding to

$$\beta = 360^\circ \left(1 - \frac{1}{\phi}\right), \quad (22)$$

This so-called *golden angle* arises naturally in *phyllotaxis* patterns, often associated with Fibonacci sequences, and optimizes sunlight exposure and spatial packing [1,4]. In practice, seed heads (e.g., sunflowers, daisies) and spiral leaf arrangements minimize overlap while maximizing photosynthetic efficiency. In plant phyllotaxis, *meltdown* would be a tangled, overcrowded placement with no recognizable pattern (extreme disorder), while *freeze* might be a rigid alignment that fails to accommodate new growth efficiently (e.g.  $\beta = 0^\circ$  or  $180^\circ$ ).

**Appearance of  $\phi$ :** In Fibonacci-lattice models [1,4], one can show that random angle and overly commensurate angle each lead to suboptimal packing. Including mild repulsion or growth optimiza-



tion enforces an effective polynomial cost with a unique minimum at  $\beta \approx 137.5^\circ$ . These natural systems avoid overcrowding or tangled growth and overly rigid or uniform spacing by converging toward  $\phi$ , thus optimizing both resource acquisition and structural efficiency.

#### 6.4. Condensed Matter Physics

Although many condensed matter systems are traditionally discussed from an *equilibrium* perspective—focusing on ground-state properties, phase diagrams, and thermodynamic potentials—*real* experimental conditions typically impose a degree of *drive* and *dissipation*. Experimental setups often involve gate voltages, applied magnetic fields, or optical pumping, each injecting energy or particles into the system, while Joule heating, electron-phonon scattering, and other dissipative processes provide outflow channels. Consequently, even when researchers speak about “ground states,” the practical regime may still exhibit *effective* non-equilibrium or near-critical behavior.

Moreover, many condensed matter systems of interest lie in proximity to *critical points*, where fluctuations and correlations become large and the system becomes extremely sensitive to small perturbations (and **exhibits power-law correlations, fractal-like bands structures and emergent scale-invariance behavior**). Even within nominally “equilibrium” calculations—such as using linear response for transport properties—one often relies on *effective* temperatures, quasi-particle lifetimes, and emergent regulatory mechanisms to explain observed phenomena. In this sense, the self-consistent or feedback-based interpretation central to non-equilibrium steady states (NESS) can also be adopted, emphasizing that real-world strongly correlated electronic materials often function under continuous energy throughput and unavoidable dissipation.

Extending the *Dynamic Balance* principle into these quantum materials requires one to reinterpret the core ratio  $\alpha = E/TS$ , where  $E$  denotes net energy (band energy, interaction energy),  $T$  can be an *effective* noise level or bath coupling, and  $S$  measures entropy from density-of-states, scattering channels or quantum entanglement. A fully developed approach would likely rely on *Lindblad* master equations or *Keldysh* path integrals to accommodate driving, dissipation, and strong many-body interactions in a nonequilibrium environment. While many properties of strongly correlated systems can be successfully described *near-equilibrium*, there is often deeper layers that standard equilibrium theory does not capture, particularly near phase boundaries where large fluctuations and resource flows can drastically reshape the DOS or emergent orders (See Appendix A).

Ultimately, whether in a biological network or a gated electronic lattice, the unifying idea is that *systems tend to organize into regimes that optimally partition resources*. In an open, driven–dissipative context, this is manifested as a balance of continuous energy inflow and outflow—akin to avoiding meltdown on the one hand and freeze on the other. In strongly correlated or near-critical condensed matter systems, a similar “sweet spot” emerges from the interplay of competing interactions, quantum fluctuations, and emergent symmetries. Indeed, scale invariance, fractal band structures, and self-similar reorganizations are not just byproducts of mathematical idealizations; they reflect a fundamental dynamical tension preventing the system from collapsing into trivial extremes, and instead favoring a robust, partially ordered regime often verging on criticality.

##### 6.4.1. Penrose-tiles (2D) and Quasicrystals (3D)

Quasicrystals are structures with long-range order but no periodic repetition. They often exhibit Penrose-tile-like atomic arrangements (such as fat and thin rhombi) and fivefold or tenfold symmetries, where the golden ratio naturally appears. Experimentally discovered in 1984 by Dan Shechtman [57], icosahedral quasicrystals (e.g. Al-Mn alloys) showed diffraction patterns with forbidden symmetries (5-fold, 10-fold). The mathematics of Penrose tilings explained these findings[54,55]: the ratio of thick to thin tiles in a Penrose tiling is  $\phi$ , and correspondingly the ratio of various interatomic distances in quasicrystals is related to the golden ratio. In other words, if one measures spacings between certain atom clusters in a quasicrystal, they often come in Fibonacci ratios. This structural hallmark has been confirmed by X-ray and electron diffraction analyses of quasicrystal samples, which match the expected Fibonacci scaling of lattice vectors.

Beyond static structure, dynamical experiments have linked quasicrystal behavior to  $\phi$ . A recent neutron scattering study on a quasicrystalline alloy  $\text{Al}_{73}\text{Pd}_{19}\text{Mn}_8$  uncovered an unusual quantization of phonon modes following the golden ratio [72]. The researchers measured the vibrational spectrum of the quasicrystal and found sharp dips in phonon density at energies 0.12, 0.19, 0.31, 0.51... meV, each approximately  $\phi$  times the previous. These energies form a geometric series governed by  $\phi$ . Such a pattern is direct experimental evidence of Fibonacci scaling in quasicrystal vibrations – a signature long theorized to distinguish quasicrystal phonons from those in periodic crystals. It confirms that the non-periodic order (with its inherent  $\phi$ -based scaling) influences physical properties like lattice vibrations. Quasicrystals thus embody the golden ratio both in their atomic structure and in their excitation spectra, matching decades-old theoretical predictions.

Quasicrystals (e.g., Al-Mn-Pd alloys) are *formed under nonequilibrium conditions*, typically rapid cooling from the melt. They exhibit *aperiodic* long-range order, a fascinating middle ground between complete disorder and perfect periodicity [57,58]. If the system solidifies too slowly or disorderedly, meltdown-like states (amorphous or random microstructures) can arise. Meanwhile, if it transitions too rigidly, one obtains a perfect crystalline lattice, i.e. “freeze.” Quasicrystals sit *between* these extremes, stabilizing an incommensurate structure in part because  $\phi \approx 1.618$  fosters the minimal free-energy arrangement that is neither fully periodic nor disordered.

**Nonequilibrium Formation.** During solidification, the system is driven (cooling flux) while exporting heat/entropy. The Dynamic Balance can be interpreted as:

- **meltdown:** large *entropy* production leading to random arrangement,
- **freeze:** negligible *entropy* production but locked periodic order,
- **synergy:** a stable aperiodic arrangement with partial order.

This aperiodic arrangement repeatedly encodes  $\phi$  in tile lengths, angles, or atomic positions. X-ray and electron diffraction confirm  $\phi$  scaling in radial distribution functions or tiling expansions [56]. The structure is robust, indicating the meltdown/freeze boundaries remain effectively infinite cost in the growth process.

### 6.5. $E_8$ Spin Chain and Quantum Criticality in Ising Magnets

The  $E_8$  spectrum in a one-dimensional Ising magnet is a famous theory prediction that involves the golden ratio. In 1989, Zamolodchikov showed that a 1D transverse-field Ising model tuned exactly to its critical point could exhibit an emergent  $E_8$  Lie algebra symmetry [73], yielding eight bosonic modes with specific mass ratios. In particular, the ratio of the two lowest energy excitations is predicted to be  $\phi$  in the  $E_8$  scenario. This remained unverified until Radu Coldea and collaborators’ experiment on cobalt niobate ( $\text{CoNb}_2\text{O}_6$ ), an Ising-like chain compound [10]. In 2010 they applied a transverse magnetic field to drive the system to criticality and probed its spin excitations using neutron scattering.

The experiment observed a series of discrete magnetic resonance modes (spin-flip excitations bound by confinement forces). Strikingly, the lowest two modes had energies in the ratio  $(1.618 \pm 0.003)$ . This is exactly what the  $E_8$  theory predicts for the mass gap ratio. As the field was tuned to the critical value (5.5 T in this material), the ratio converged to  $\phi$ . Coldea et al. reported this as the first observation of the “hidden”  $E_8$  symmetry in a solid. In essence, the complex symmetry of an 8-dimensional lattice ( $E_8$ ) manifested in a simple 1D magnet as a Fibonacci relationship between quasiparticle energies. This experiment provided empirical evidence of a golden-ratio ratio in the excitation spectrum of a quantum magnet, a landmark in quantum critical phenomena. It illustrated how beautifully abstract math (the golden ratio,  $E_8$ ) can emerge in real quantum matter near a critical point.

In the transverse-field Ising chain, the competition between the transverse magnetic field  $h$  (favoring paramagnetism) and the Ising exchange  $J$  (favoring ferromagnetism) can be tuned by varying  $h$  (or doping the chain, or adjusting temperature) so that neither meltdown nor freeze remains stable. Right at (or near) the quantum critical point, the system exhibits an emergent  $E_8$  symmetry, whose first two excitations have energies in the ratio  $m_2/m_1 = \phi$  [10]. While quasicrystal formation

offers a clear classical growth process, *quantum* systems can also be seen to “dynamically” approach a balance state.

- **Meltdown** ( $h \gg J$ ): If  $h$  is extremely large compared to  $J$ , the system is effectively paramagnetic; the spins are fully disordered. Minimal free energy is devoted to ordering, large quantum fluctuations dominating.
- **Freeze** ( $h \ll J$ ): Conversely, if  $h$  is very small, the chain is in a rigid ferromagnetic phase, i.e. freeze. No significant spin excitations or fluctuations remain, so the system is “locked.”
- **Intermediate “Drive”**: One can *dial* the transverse field  $h$  through an intermediate region, which shifts the effective ratio  $h/J$ . If meltdown ( $\alpha = 0$ ) and freeze ( $\alpha \rightarrow \infty$ ) are each “infinite-cost boundaries” in the 1D Ising spins, the chain *self-organizes* near a quantum critical point:  $\alpha \approx \phi$ .

Hence, just as quasicrystals avoid random amorphous and freeze perfect lattice phases by occupying an aperiodic  $\phi$  structure, the quantum Ising chain “avoids” full disorder and full ferromagnetic order at the sweet spot  $h_c$ , unveiling  $E_8$  excitations with a golden-ratio mass. One might view doping or field-sweep as the “drive,” with local quantum fluctuations preventing indefinite freeze or meltdown.

### 6.6. Non-Abelian Fibonacci Anyons

Non-Abelian Fibonacci anyons are quasiparticles whose state space grows as Fibonacci numbers, giving them a quantum dimension equal to  $\phi$  (the golden ratio). They are theoretically pivotal because braiding them can perform universal quantum computation. In nature, the  $\nu = 12/5$  fractional quantum Hall state (a Read-Rezayi  $k = 3$  parafermion state) was proposed to host Fibonacci anyons. To date, however, no direct experimental observation of Fibonacci anyons in a material has been confirmed. For instance, experiments at  $\nu = 12/5$  often see a reentrant integer Hall effect instead of the expected Fibonacci phase, likely due to symmetry-breaking effects. Researchers found via numerical density-matrix renormalization group that an idealized  $\nu = 12/5$  Hall state would indeed realize a Fibonacci anyon phase, but disorder and symmetry breaking complicate its experimental realization.

Emulated evidence for Fibonacci anyons came in quantum simulations. In 2024, Tsinghua University scientists reported creating a Fibonacci topological state on a superconducting qubit processor [14]. They simulated the braiding of Fibonacci anyons for the first time, as published in Nature Physics. This experiment prepared a topologically ordered state via a “string-net” scheme and demonstrated non-Abelian braiding operations. Notably, the extracted anyon properties matched predictions – the engineered excitations had a quantum dimension  $\sim 1.618$ , the golden ratio, consistent with Fibonacci anyon theory. While this was a simulation rather than a naturally occurring anyon, it marks an important step: it showed that the exotic statistics of Fibonacci anyons (related to  $\phi$ ) can be realized and manipulated in practice. This paves the way for topological quantum computing experiments once a physical platform (like a 2D electron system or spin system) can robustly host these anyons.

**Key Property**  $d_\tau = \phi$ : Fibonacci anyons obey the fusion rule  $\tau \times \tau = 1 + \tau$ , implying a quantum dimension  $d_\tau$  satisfying  $d_\tau^2 = 1 + d_\tau$ . Hence  $d_\tau = (1 + \sqrt{5})/2 = \phi$  [15,59]. In *Dynamic Balance* spirit, meltdown would be an exponentially large dimension that kills robust topological gating, while freeze would trivialize the anyons. The Fibonacci dimension  $\phi$  balances these extremes. The idea that topological order can emerge in open driven, dissipative systems, leading to mixed-state topological order that has no parallel in pure quantum states, has also been previously studied [74–76].

### 6.7. The Hofstadter Butterfly in 2D Electron Gases

The Hofstadter butterfly is a fractal pattern in the energy spectrum of electrons on a 2D lattice under a perpendicular magnetic field. Hofstadter’s 1976 [77] theoretical paper predicted that as magnetic flux per plaquette becomes incommensurate (irrational) with the electron lattice, the spectrum splits into self-similar mini-bands and gaps – a recursive butterfly-wing diagram. One of the most incommensurate cases is when the magnetic flux (in units of  $h/e$ ) is the golden ratio  $\phi$  or its reciprocal. This fractal remained a theory for decades because it requires ultra-high fields or perfectly engineered lattices.

In 2013, experiments finally observed Hofstadter's fractal spectrum using graphene-based superlattices [78]. Dean et al. stacked a graphene monolayer on hexagonal boron nitride (hBN) with a small twist, creating a moiré superlattice large enough that moderate laboratory magnetic fields achieved the needed flux ratios. They reported a recursive pattern of quantum Hall conductance gaps as a function of magnetic field and electron density – the hallmark of the Hofstadter butterfly. Zooming in on the spectrum reveals smaller copies of the butterfly repeating, reflecting its fractal self-similarity. Each Landau level subdivides into subbands following the Diophantine equations of the Hofstadter model. This was the first direct experimental confirmation of the fractal quantum Hall effect in a material.

Recent experimental studies (Princeton 2025 [13]) used scanning tunneling spectroscopy on twisted bilayer graphene to visualize the energy spectrum's self-similar gaps in real space, further confirming the fractal pattern. The team also showed that the Hofstadter's butterfly is not a static phenomenon. Instead, it is a dynamic energy landscape that evolves in response to modulations of the electronic environment.

When the gate voltage is varied, the electron density is tuned, and thereby the effective magnetic flux per unit cell is altered. This modulation changes the balance between the lattice potential and the magnetic field, leading to a systematic rearrangement of the fractal sub-bands observed in the spectrum. The fractal pattern is not fixed, it self-adjusts as electrons fill the bands, which is not predicted by Hofstadter's original model [77]. When electrons were added, new gaps opened or closed in the spectrum, likely due to electron-electron interactions causing reordering of the states, as the system responds to maintain some kind of order.

Thus, gating not only adjusts the electron density but effectively serves as a feedback mechanism that regulates the energy distribution in the system. By doing so, the electrons "self-organize" into a state where the energy–entropy ratio  $\alpha$  converges toward the optimal value  $\phi$ . This dynamic self-regulation ensures that the system avoids both an overly chaotic state (meltdown, where  $\alpha \rightarrow 0$ ) and an overly rigid state (freeze, where  $\alpha \rightarrow \infty$ ). The result is a fractal energy spectrum that is robust and self-similar, embodying the principles of dynamic balance.

- **meltdown:** If the magnetic field were too weak relative to the lattice potential, the energy levels would not separate properly, leading to a continuous, featureless spectrum.
- **freeze:** If the magnetic field were too strong, it could force electrons into well-isolated Landau levels (a "freeze" state), suppressing the complex mixing that gives rise to a fractal spectrum.

Hofstadter's own work in *Gödel, Escher, Bach: The Eternal Golden Braid* focused on self-referential structures and feedback loops (strange loops) as powerful engines of complexity to explain how cognition emerges from hidden neurological mechanisms. In his work, he highlights the idea that a system's output at one level becomes input at another, ultimately feeding back to the original level—like a Möbius strip in logical or symbolic form [79].

#### 6.8. Spiral Formations in Nonequilibrium Surfaces

In a variety of driven systems, surfaces or interfaces can develop *spiral* instabilities whenever local growth outpaces mass or heat diffusion. Metallic alloys undergoing rapid solidification often exhibit spiral dendrites or coil-like structures, reflecting nonequilibrium solidification fronts [80,81]. Polymer films cast from solution can similarly form swirling or spiral topographies, driven by competing evaporation and flow [82]. Even oxide layers grown under high electric fields or thermal gradients can reveal spiral waves in their nonequilibrium thickness profiles [83,84]. More generally, spiral instabilities are understood as one of many universal pattern-forming mechanisms in driven-diffusive systems [85].

From the perspective of our Dynamic Balance PDE framework (cf. Sec. 2), these spirals represent a visually striking *universal attractor*, emerging when the global feedback—imposed by the energy–entropy ratio  $\alpha \approx \phi$ —prevents both runaway "freeze" (unbounded growth) and "meltdown" (complete collapse). Under sufficiently strong driving, any small deviation in local growth rate can spiral outward, leading to a persistent coil-like pattern, rather than a flat or featureless interface. Repetitive wave-

splitting events further stabilize the spiral arms, promoting self-similar or fractal-like structures if the system is close to a critical meltdown threshold.

Such nonequilibrium spiral morphologies align with the basic principle that, once diffusion or relaxation can no longer keep up with interface motion, the system settles into a *patterned* steady state enforced by the negative feedback of the meltdown–freeze term. The recurring appearance of spiral instabilities across metals, polymers, and oxide films [80,82,83] reinforces the view that  $\alpha = \phi$  (the golden ratio) serves as a robust attractor in diverse nonequilibrium media.

## 7. Conclusions

We have shown, through multiple mathematical formalisms, that an *open, nonequilibrium system* in a *non-equilibrium steady state* with *forbidden or high cost* boundary conditions universally converges to an *interior synergy*, which under mild self-similarity assumptions takes the unique numerical value  $\alpha = \phi \approx 1.618$ . We called this the *Dynamic Balance* principle to offer a compelling **interdisciplinary insight** for understanding how complex systems maintain a stable organization while remaining flexible and adaptive. It suggests that the golden ratio, a number long revered in art and nature, also emerges as a fundamental constant in the dynamics of complex nonequilibrium systems. By proposing that NESS systems tune the ratio of their energy flow to entropy production to the golden ratio, we explain why this irrational value optimizes the trade-off between useful work and necessary dissipation.

### Key Theoretical Insights

- **Nonequilibrium Attractor:** By penalizing excessive disorder (meltdown) and rigid order (freeze), the *only* viable steady-state ratio in  $(0, \infty)$  is  $\phi$ . This extends “edge-of-chaos” or “critical point” concepts by *quantifying* the exact attractor in open NESS systems.
- **Robust to External Perturbations:** Even if the system is *open*, subject to doping, random forcing, or boundary fluxes, the boundary states remain infinite-cost, maintaining  $\alpha \approx \phi$  unless *catastrophic forcing* takes place.
- **Broad Empirical Validation:** From phyllotaxis and branching to spiral galaxies and rotating turbulence, from neural network and quasicrystals, to one dimensional Ising chains and Fibonacci anyons, a *wide net* of phenomena exhibit signatures near 1.6, all consistent with optimization of energy and entropy production.
- **Independent Derivations:** Polynomial cost function (local), PDE gradient flow (spatial), a Wilsonian RG approach (multi-scale), and Markovian master equation (probability) converge on the same conclusion that extreme boundaries push self-organizing NESS system with feedback toward a unique, stable interior point  $\phi$ .

While many additional parameters (boundary conditions, anisotropies, external forcing) can cause deviations, the *Dynamic Balance* principle shows that  $\phi$  emerges as a robust “middle ground” where the system can organize efficiently without losing adaptability or coherence.

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**Conflicts of Interest:** The authors declare no conflicts of interest.



## Abbreviations

The following abbreviations are used in this manuscript:

NESS	Nonequilibrium Steady-State
PDE	Partial Differential Equation
ODE	Ordinary Differential Equation
RG	Renormalization Group
SOC	Self-Organized Criticality
CFC	Cross-Frequency Couplings
STP	Short Term Plasticity
EEG	Electroencephalogram
fMRI	functional Magnetic Resonance Imaging
PET	Positron Emission Tomography
MS	Multiple Sclerosis
AD	Alzheimer's Disease
PD	Parkinson's Disease

## Appendix A. Thermodynamic Review

### Appendix A.1. The Second Law of Thermodynamics

#### Appendix A.1.1. The Traditional Form of the Second Law

“For an isolated system, the total entropy cannot decrease; it either stays constant or increases.”

In equilibrium thermodynamics, any process is considered quasi-static (infinitesimally slow), so the system remains near equilibrium at each step. In this limit, the Clausius statement  $\Delta S = \int \delta Q/T$  neatly applies, and  $\Delta S \geq 0$ .

#### Appendix A.1.2. Generalized Form of the Second Law

When we drive a system far from equilibrium (fast changes, large gradients of temperature or chemical potential, external forcing, etc.), one cannot necessarily use the simple equilibrium formulas. However, a broader statement still holds:

“The total (system + environment) entropy production is always nonnegative.”

The system's entropy *can* decrease transiently if there is heat or particle flow into its surroundings. However, the environment's entropy increases at least enough to compensate for this local decrease. When you add up all contributions—system entropy change plus environment (reservoirs, heat baths, etc.)—the total entropy change is still  $\geq 0$ .

Nonequilibrium processes involve “irreversible” effects: friction, dissipation, diffusion, chemical reactions, etc. These cause a strictly positive amount of entropy to be generated in real processes, hence the net increase.

$$\frac{dS_{total}}{dt} = \dot{S}_{production} + \dot{S}_{exchange} \geq 0$$

where  $\dot{S}_{production}$  is the intrinsic entropy production rate in the system, and  $\dot{S}_{exchange}$  describes entropy flow between system and surroundings.

#### Appendix A.1.3. Fluctuation Theorems

These theorems generalize the Second Law to small systems or short times, where thermal and quantum fluctuations can temporarily appear to “violate” the law. On average, or in the long-time limit, the inequality still ensures  $\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$  (Jarzynski) and the net entropy production remains nonnegative.

#### Appendix A.1.4. Stochastic Thermodynamics

For systems like colloidal particles, biomolecules, or mesoscopic electronic devices, one can track trajectory-by-trajectory entropy production. Although some trajectories yield negative entropy production, the overall average obeys the Second Law. This approach unifies classical nonequilibrium thermodynamics with probability theory, reaffirming that the mean entropy production is  $\geq 0$ .

#### Appendix A.1.5. Keldysh/Lindblad Formalisms

In open quantum systems described by Lindblad operators, or driven quantum systems analyzed by Keldysh path integrals, you can define an entropy production rate that stems from the mismatch between the system's state and the drive/bath. The second law still manifests as a positive definite production of entropy in the steady state or during transient evolution.

#### Appendix A.2. The Energy-Entropy Balance Function $\alpha(t)$

In nonequilibrium systems,  $E$ ,  $S$ , and  $T$  can all vary with time, so measuring the rates of energy input and heat flow is often simpler than determining absolute energies. Consequently, the ratio

$$\alpha(t) = \frac{\dot{E}(t)}{T(t) \dot{S}(t)}$$

which compares the total energy throughput to the entropic heat loss. For a process connected to a reservoir at temperature  $T$ , a small change in entropy satisfies  $\delta S \geq \delta Q/T$ , implying  $\dot{S}(t) \gtrsim \dot{Q}(t)/T$  over time. Note that  $\dot{E}(t)$  and  $\dot{Q}(t)$  need not coincide, since energy changes may stem from internal interactions rather than heat exchange alone. Thus, tracking  $\alpha(t)$  reveals how closely the second law is “saturated” or “exceeded” in real time, aligning with nonequilibrium extensions of thermodynamics that focus on rates of entropy production rather than static entropy values.

In certain regimes, numerical or experimental evidence suggests that  $\alpha(t)$  converges to  $\phi$ , the golden ratio ( $\approx 1.618$ ). From a thermodynamic perspective,  $\dot{S}(t) \gtrsim \dot{Q}(t)/T$  implies that the ratio  $\alpha(t)$  helps measure how closely the second law is “saturated” at each moment. Thus, discovering that  $\alpha(t)$  hovers near the golden ratio could hint at underlying self-similarity or scaling behavior in the nonequilibrium dynamics.

The appearance of exact  $\phi$  typically indicates a special symmetry or self-similarity in the system. It is not guaranteed for “generic” nonequilibrium processes. Real-world noise, imperfections, and coupling to external baths can blur exact values. Observing  $\phi$  might thus rely on high-precision or well-controlled setups (cold atoms, superconducting qubits, or advanced photonics).

$\alpha \approx \phi$  represents a unique synergy point between energy injection and entropy outflow—suggesting that the system's “irreversibility” or “entropy production” is balanced in a self-similar way.

If the system's driving, feedback loops, and dissipative processes naturally produce a self-similar, iterative scaling of energy vs. entropy changes, Then the ratio  $\alpha(t)$  could stabilize at a universal constant like  $\phi$ .

#### Appendix A.3. Equilibrium versus Nonequilibrium Thermodynamics

##### Appendix A.3.1. Equilibrium

Equilibrium is characterized by a *static*, time-independent Hamiltonian  $H$  whose *thermal* Boltzmann distribution  $\rho$  is a fixed point under the same  $H$ –generated evolution. The equilibrium density matrix is

$$\rho_e = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}$$

with  $\beta = 1/k_B T$ . This is the familiar Gibbs (or Boltzmann) state. Equilibrium implies that there is no net currents or fluxes between states. Observables do not show net flow or growth over time, aside from inconsequential phases.

$$\rho_e(t) = e^{-iHt} \rho(0) e^{+iHt}$$

### Appendix A.3.2. Non-Equilibrium

In an open quantum system, the environment or measurement apparatus alters the evolution via *non-Hamiltonian* terms. The formalism that describes this is a Lindblad master equation

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{L}_{diss}(\rho)$$

where  $\mathcal{L}_{diss}(\rho)$  encodes dissipative and dephasing processes. The resulting steady-state might still have time-dependent density matrix but it may *not* be a simple Boltzmann factor  $e^{-\beta H}$ . Formally, any positive-semidefinite  $\rho$  can be wrtitten as  $\rho = e^{-X} / \text{Tr}(e^{-X})$  for some Hermitian  $X$ . However, if  $X \neq \beta H$ , that  $\rho$  is not the thermal state of the actual Hamiltonian generating the dynamics. The system us then *out-of-equilibrium*. Intead,  $\rho$  reflects net currents or fluxes in the system.

Any situation where the actual time evolution operator  $\hat{U}(t)$  does not coincide with the one implied by  $e^{-\beta H}$  leads to *nonequilibrium phenomena*. This mismatch can arise from time-dependent drive, open-system dissipation, sudden quenches, or forced currents.

By recognizing that the generator of dynamics— whether purely Hamiltonian, Lindbladian, or governed by a time-dependent (Floquet) operator—must match the ensemble's exponential form for an equilibrium description, we obtain a crisp dividing line between equilibrium-like and non-equilibrium regimes. When they do coincide, the system can exhibit static thermal properties; when they do no, we inevitable face nonequilibrium phenomena—often involving net flows of energy or particles, drive-dissipative steady states, or intricate time dependencies.

## Appendix B. Additional Markov Chain Example

Here we provide a small state set  $\{1, 2, \dots, N\}$  example of the master equation approach from Sec. 4, with *explicit* rates and an *analytic* steady-state solution.

### Appendix B.1. State Space and Rates

States.

Let the discrete states be  $\alpha_i = i$  for  $i = 1, \dots, N$ . Interpret  $\alpha_1 \approx 0$  (meltdown),  $\alpha_N$  large  $\approx \infty$  (freeze). We define:

$$W_{i \rightarrow i+1} = \epsilon > 0, \quad i = 1, \dots, N-1,$$

for a slow upward drive, but  $W_{N \rightarrow N+1} = 0$  (cannot go beyond freeze). For partial meltdown, define:

$$W_{i \rightarrow i-1} = \nu > 0 \quad \text{if } i > i_{\text{thres}},$$

and zero otherwise. Finally, meltdown/freeze remain forbidden in steady state by:

$$W_{k \rightarrow 1} = 0, \quad W_{k \rightarrow N} = 0, \quad k = 2, \dots, N-1.$$

Hence from  $i = 2, \dots, N-1$ , no transitions into  $i = 1$  or  $i = N$ . States 1 and  $N$  do exist in the chain but are effectively absorbing or high-cost boundaries with zero steady occupation.

## Appendix B.2. Master Equation and Stationary Distribution

Master Equation.

The probability  $P_i(t)$  evolves via

$$\frac{dP_i}{dt} = \sum_{j=1}^N [W_{j \rightarrow i} P_j - W_{i \rightarrow j} P_i], \quad \sum_i P_i = 1.$$

Focus on  $i = 2, \dots, (N-1)$ . At  $\dot{P}_i = 0$ , each equation imposes a local balance of fluxes.

Example of a Small Chain.

Take  $N = 4$ ,  $i_{\text{thres}} = 3$ , so meltdown/freeze are  $i = 1, 4$ . Then:

- $W_{1 \rightarrow 2} = \epsilon$ , but  $W_{2 \rightarrow 1} = W_{3 \rightarrow 1} = W_{4 \rightarrow 1} = 0$ ,
- $W_{2 \rightarrow 3} = \epsilon$ , but  $W_{3 \rightarrow 4} = \epsilon$  *not allowed* if we forbid freeze, or we can set  $W_{3 \rightarrow 4} = 0$ ,
- meltdown for  $i > 2$ :  $W_{3 \rightarrow 2} = \nu$ ,  $W_{4 \rightarrow 3} = \nu$  if 4 existed as an interior state, but we are forbidding it in steady transitions,
- $W_{2 \rightarrow 4} = 0$ ,  $W_{1 \rightarrow 3} = 0, \dots$  ensuring meltdown/freeze remain unoccupied.

Solving  $\dot{P}_2 = 0$  and  $\dot{P}_3 = 0$  yields a ratio  $P_3(\infty)/P_2(\infty)$  determined by  $\epsilon, \nu$ . Typically,  $\alpha_2 = 2$  or  $\alpha_3 = 3$  dominates, whichever better satisfies the Dynamic Balance. One can tweak  $\epsilon$  and  $\nu$  to fix  $3/2 \approx 1.5$ , etc.

## Appendix B.3. General $N$ and Emergence of $\alpha_{i^*} \approx \phi$

One can generalize to  $N \gg 1$ , letting meltdown be  $i = 1$ , freeze  $i = N$ , threshold meltdown  $W_{i \rightarrow i-1} = \nu$  for  $i > i_{\text{thres}}$ , and upward drive  $W_{i \rightarrow i+1} = \epsilon$  for  $1 \leq i < N$ . Solving the linear system  $\dot{P}_i = 0$  with  $P_1 = P_N = 0$  yields a unique stationary solution  $\{P_i(\infty)\}$ . If we design  $\epsilon, \nu$  so that  $P_{i+1}/P_i$  reproduces a near-constant ratio at  $i^*$  in the middle, one obtains  $\alpha_{i^*} \approx \phi$ . Meanwhile, meltdown (state 1) and freeze (state  $N$ ) remain unoccupied.

## Appendix B.4. Interpretation and Conclusion

This small Markov chain example exemplifies how meltdown/freeze synergy forces the chain to settle on an interior state or states near  $\alpha_{i^*} \approx 1.618$ . By adjusting threshold meltdown rates, we can also produce partial meltdown events in batch jumps, leading to fractal avalanche histograms. Hence we see the **practical** steps of specifying a finite set  $\alpha_1, \dots, \alpha_N$ , slow drive, meltdown thresholds, and boundary forbiddance—and verifying the final  $\alpha_{i^*} \approx \phi$  emerges in steady state.

## Appendix C. Forced Shell Turbulence Exponent Shift

### Appendix C.1. Introduction to the Shell Model

The shell model of turbulence [86] approximates the energy cascade in hydrodynamic turbulence by defining complex velocity amplitudes  $u_n$  on *discrete*, logarithmically spaced wavevector shells. Let

$$k_n = k_0 \lambda^n,$$

where  $\lambda > 1$  is a constant scale factor, e.g.  $\lambda = 2$  in classic models. Each shell evolves via a set of coupled ODEs (e.g. the GOY or Sabra model):

$$\frac{du_n}{dt} = i \left( a_n u_{n+1} u_{n+2} - b_n u_{n-1} u_{n+1} \right) - \nu k_n^2 u_n + f_n,$$

where  $(a_n, b_n)$  are interaction coefficients transferring energy between adjacent shells,  $\nu$  is viscosity (dissipating small scales), and  $f_n$  is large-scale forcing. Traditional  $\lambda = 2$  lumps the continuous  $k$ -space into shells of doubling wavevector.

**Golden ratio vs. integer scale.** To investigate whether *incommensurate* scale factors alter the steady-state exponent, we can choose  $\lambda = \phi \approx 1.618$ . If the shell model *self-organizes* to an exponent  $x \approx 1.6$  for  $E_n \propto k_n^{-x}$ , this could be a toy demonstration of how *dynamic balance* might pick an exponent near  $\phi$ .

### Appendix C.2. Energy Transfer and Scaling Laws

In turbulence, one typically measures the *stationary* energy distribution  $E(k) \sim k^{-\zeta}$ , where  $\zeta$  is around  $5/3$  (Kolmogorov) for high-Reynolds isotropic flows. However, when external forcing is *strong* or if the flow experiences feedback constraints (e.g. meltdown–freeze), the exponent can shift.

**Dynamic Balance perspective.** Suppose we impose a self-organization rule so that the *spectral energy* at each shell is bounded away from two extremes:

1. **Meltdown:** overly shallow spectrum,  $x < 1.6$ , injecting too much energy into small scales, risking runaway enstrophy or chaotic blow-up.
2. **Freeze:** overly steep spectrum,  $x \gg 1.6$ , stifling energy at smaller scales so that large-scale forcing dominates and the cascade stalls.

Under the assumption of a nonequilibrium steady state (energy injection at  $k_0$ , dissipation at large  $k$ ), the system might settle on an exponent  $x$  that *balances* these extremes.

**Deriving  $x \approx \phi$ .** While a fully rigorous derivation would require embedding meltdown–freeze cost functionals into the shell-model PDE, a simpler heuristic is: if  $E(k) \propto k^{-x}$  is too shallow, meltdown occurs; if too steep, freeze occurs. Minimizing an *entropy-like* penalty at both boundaries can yield a unique interior synergy. In some meltdown–freeze analyses, that synergy is the golden ratio  $\phi$ , producing  $x = 1.618$  [62]. Thus, we hypothesize:

$$E(k_n) \sim k_n^{-x}, \quad \text{with } x \approx \phi.$$

This is distinct from the classical  $5/3 \approx 1.666\dots$  (Kolmogorov). Notably, some numerical or experimental rotating-stratified turbulence exponents do cluster near  $-1.6$ , hinting a near-golden-ratio slope might naturally emerge if meltdown–freeze feedback is effectively realized.

### Appendix C.3. Toy Simulation Outline

To test the idea:

- **Choose  $\lambda = \phi$ :** define shell wavevectors  $k_n = k_0 \phi^n$ .
- **Implement Sabra/GOY shell couplings** with typical  $a_n, b_n$ , force the first shell(s), and apply viscosity on large  $n$  shells.
- **Add meltdown–freeze penalty:** either as an additional damping term when  $|u_n|^2$  is too large (meltdown) or too small (freeze).
- **Observe the stationary spectrum.** Fit  $E_n = |u_n|^2$  against  $k_n$  to see if exponent  $x \approx 1.618$  emerges.

**Outcome.** If our framework indeed selects  $x \approx \phi$ , we confirm a “golden cascade” scenario [62] wherein partial fractal or incommensurate scale factors *self-tune* away from meltdown or freeze boundaries. While purely illustrative, such toy results may clarify how real turbulence exponents could shift when forced strongly, if constraints are effectively realized.

In summary, the shell-model approach provides a manageable, discrete-stage platform to explore *nonequilibrium* energy cascades. If  $\lambda = \phi$  and meltdown–freeze constraints hold, a stationary exponent  $x \approx 1.6$ – $1.618$  can emerge, explaining occasional turbulent exponents near  $-1.6$ . Though simplifying real turbulence, the toy scenario underscores how dynamic balance at each scale might systematically yield a golden ratio slope across the inertial range.

### Appendix C.4. Remarks

Although real turbulence is more complicated, the shell model highlights the essential meltdown–freeze interplay driving an interior exponent near  $\phi$ , especially under rotation, forcing, or anisotropies that deviate from Kolmogorov's  $5/3$ .

## Appendix D. Self-Regulating Heat Engine

A particularly straightforward example arises in a *self-regulating heat engine*—a toy thermodynamic model often used to illustrate feedback in nonequilibrium open systems. Although simplified, it provides tangible insight into how  $\alpha \approx 1.618$  (the golden ratio) might emerge from balancing two destructive extremes, **meltdown** (engine runaway) and **freeze** (engine stall).

### Setup.

Consider an engine coupled to two reservoirs of different temperatures,  $T_{\text{hot}}$  and  $T_{\text{cold}}$ , driving a continuous flow of heat. Let  $Q_{\text{in}}$  be the rate of heat *absorbed* from the hot reservoir and  $Q_{\text{out}}$  the rate *dumped* to the cold reservoir. By energy conservation, the net *work output* is  $W = Q_{\text{in}} - Q_{\text{out}}$ , so we identify the “engine ratio”

$$\alpha = \frac{Q_{\text{in}}}{T S} \quad \text{where } Q_{\text{out}} = T S, \quad T \approx T_{\text{cold}}, \quad S \text{ is the total entropy production rate.} \quad (\text{D.1})$$

Here we view  $T S = Q_{\text{out}}$  and  $Q_{\text{in}}$  as the *energy fluxes* in an open system, making  $\alpha$  dimensionless.

- **Meltdown:** If the engine attempts to convert nearly all heat to work (maximizing  $W$ ), it runs near the Carnot limit but practically risks unbounded temperature rises or mechanical “overdrive.” Realistically, insufficient cooling can lead to instability or damage.
- **Freeze:** If the engine dissipates too much ( $Q_{\text{out}}$  large) with little net work, it becomes sluggish or stalls, losing the point of running an engine in the first place.

A self-regulating feedback—e.g. a *governor* that adjusts throttle based on engine speed or temperature—acts to avoid meltdown (overheating) and freeze (idle). This feedback pins  $\alpha$  away from 0 or  $\infty$ , steering the engine to an *interior synergy*.

To illustrate *why*  $\alpha \approx 1.618$  might emerge, one can introduce a *cost function*  $R(\alpha)$  diverging at  $\alpha = 0, \infty$ . Minimizing  $R(\alpha)$  yields a unique interior ratio, analogous to the meltdown–freeze PDE approach. The simplest partition argument (splitting  $Q_{\text{in}}$  into *dissipated*  $Q_{\text{out}} = T S$  and *engine structure/free energy*) leads to the defining property of the golden ratio:

$$\alpha = \frac{Q_{\text{in}}}{Q_{\text{out}}} = \phi = 1.618 \dots \implies 61.8\% \text{ of input heat is lost as } Q_{\text{out}}, \quad 38.2\% \text{ is net work.}$$

Though purely a toy derivation, it illuminates how an open-system balance can pin  $\alpha$  at the golden ratio, lying between extrema boundaries. While this minimal engine model does *not* address real engineering details (Carnot cycles, partial expansions, etc.), it *does* capture the essence of *open-system* self-regulation:

- **Drive:** Hot reservoir supplies heat  $Q_{\text{in}}$ ,
- **Dissipation:** Cold reservoir  $Q_{\text{out}} = T S$ ,
- **Feedback:** engine avoids meltdown/freeze.

Hence, it operates as a **nonequilibrium steady state** with continuous influx/outflux, ultimately stabilizing at  $\alpha \approx \phi$ .

Insofar as the engine's feedback enforces a *thermodynamic balance*, one recovers the golden-ratio partition of heat flux, underscoring how  $\phi \approx 1.618$  arises from open-system self-regulation *without* extensive PDE or RG machinery.



## Appendix E. Disclaimer and Outlook

While the *Dynamic Balance* principle presented here is supported by toy-model derivations and fits a range of empirical observations in quasi-steady state non-equilibrium systems, it remains to be tested against more complex, strongly correlated phenomena.

In particular, initial motivations for this work arose from attempts to explain incommensurate orders in frustrated Kitaev magnets (e.g.,  $\text{Li}_2\text{IrO}_3$ ) [87,88], charge density wave (CDW) instabilities in high-temperature superconductors, fractal energy spectra in two-dimensional electron gases (2DEG), and unconventional superconductivity in moiré superlattices [11,12,77,78,89,90]. To date, little direct empirical evidence links those phenomena to  $\phi$  or Fibonacci. Nonetheless, the author believes the framework can naturally extend to each of these contexts, provided future experiments and theoretical developments validate its applicability.

Cuprate superconductivity might be a “middle ground” phase emerging to avoid a “meltdown” limit (a disordered, spin-glass or incoherent Mott phase) and a “freeze” limit (a rigid Fermi liquid or band-insulator). On the underdoped side, cuprates become strongly correlated Mott-like materials with disordered spin or charge order (pseudogap or spin-glass tendencies). One might interpret this as a “meltdown” limit: conduction collapses into localized or incoherent excitations. On the overdoped side, many cuprates revert to a more conventional metallic behavior, eventually forming a Fermi liquid. One can imagine that as a “freeze” limit: standard quasiparticles, minimal emergent complexity, no exotic correlation. In the doping range between these extremes, superconductivity emerges, presumably by harnessing strong correlations (not meltdown) but still retaining coherent conduction (not freeze). This is reminiscent of dynamic balance:  $\alpha \approx \phi$  balancing correlation vs. conduction to find a stable path that is neither Mott meltdown nor trivial Fermi freeze. This could be remanent of how superconductivity emerges in twisted bilayer graphene (TBG) near the “magic angle”. The system sits between two extreme electronic states (a Mott-like insulator on one side and a more conventional metal or band insulator on the other), yet self-organizes into correlated superconductivity at intermediate doping/angle. In the same sense that cuprate superconductivity can be viewed as emerging to “avoid” total meltdown (strongly disordered, incoherent Mott/spin-glass state) and freeze (a rigid Fermi liquid), TBG near 1.1° doping is often interpreted as balancing strong correlations (not meltdown) against conduction (not freeze).

The author acknowledges that some of these extensions may be considered speculative. As a result, they have been omitted from the main discussion due to insufficient empirical support. Future high-resolution neutron scattering, scanning tunneling microscopy, and ARPES studies on candidate systems may help clarify whether this principle indeed underpins these incommensurate orders and correlated states.

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